MODULARED SEQUENCE SPACES

By

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A collection R of sequences of real numbers is called a *modulared sequence spaces*, if a *modular* is defined on R so that R becomes a *modulared semi-ordered linear space.*[1]

The case, when the modular is of unique spectra, was considered by W. ORLICZ,[2] H. NAKANO[3] and I. HALPERIN–H. NAKANO.[4] The purpose of this paper is to generalize some of their results.

§ 1. A *modulared sequence space* is generated by a sequence of non-decreasing convex functions of a real variable:

$$f_1, f_2, \cdots\cdots$$

which satisfies the following properties:

(1) $f_\nu(0) = 0$;

(2) $\lim\limits_{\xi \to \alpha - 0} f_\nu(\xi) = f_\nu(\alpha)$;

(3) $\lim\limits_{\xi \to \infty} f_\nu(\xi) = +\infty$;

(4) there exists a real number $\alpha > 0$ (depending on each f_ν) such that $f_\nu(\alpha) < +\infty$,

for every $\nu = 1, 2, \cdots$. Namely, f_ν $(\nu = 1, 2, \cdots)$ are modulars on the space of real numbers.

For this sequence:

$$f_1, f_2, \cdots\cdots,$$

the set of such sequences of real numbers (ξ_ν) that

$$\sum_{\nu=1}^{\infty} f_\nu(\alpha \xi_\nu) < +\infty$$

for some $\alpha > 0$ is a modulared sequence space, putting its modular

1) H. NAKANO: Modulared semi-ordered linear spaces, Tokyo Mathematical Book Series, Vol. 1 (1950).

2) W. OSLICZ: Ueber konjugierten Exponentenfolgen, Studia Math., III (200–211).

3) H. NAKANO: Modulared sequence spaces, Proc. Japan Acad., 27 (1951), 508–512.

4) I. HALPERIN and H. NAKANO: Generalized l^p spaces and the Schur property, Journal Math. Soc. Japan, 5 (1953), 50–58.

$m(x)$ as

$$m(x) = \sum_{\nu=1}^{\infty} f_\nu(\xi_\nu) \qquad \text{for} \quad x = (\xi_\nu) .$$

This modular is monotone complete. Hence, this space is a Banach space by the norms induced by the modular. This modulared sequence space is denoted by $l(f_\nu)$ in this paper.

When the modular is of unique spectra, namely when

$$f_\nu(\xi) = \xi^{p_\nu} \qquad (\nu=1, 2, \cdots, \xi>0)$$

for a sequence of real numbers $p_\nu \geqq 1$ $(\nu = 1, 2, \cdots)$, this modulared sequence space was denoted by $l(p_1, p_2, \cdots)$ and considered by the authors mentioned above. In this case it is obvious that

$$l \subset l(p_1, p_2, \cdots) \subset m ,$$

where l is a space of summable sequences and m is of bounded sequences, namely, l is a modulared sequence space with

$$f_\nu(\xi) = \xi \qquad (\nu=1, 2, \cdots, \xi>0) ,$$

and m is also a modulared sequence space with, for instance,

$$f_\nu(\xi) = \xi^\nu \qquad (\nu=1, 2, \cdots, \xi>0) .$$

But, in general cases, this fact is not always true.

(1.1) *If* $l \subset l(f_\nu)$ *and* $l(f_\nu)$ *is finite (that is,* $m(x) < +\infty$ *for every* $x \in l(f_\nu)$*), then there exists a real number* $\alpha>0$ *such that* $\sup_{\nu \geqq 1} f_\nu(\alpha) < +\infty$. *Conversely, if* $\sup_{\nu \geqq 1} f_\nu(\alpha) < +\infty$, *then we have* $l \subset l(f_\nu)$.

If $\sup_{\nu \geqq 1} f_\nu(\alpha) = +\infty$ for every $\alpha>0$, then we can select a subsequence ν_μ $(\mu=1, 2, \cdots)$ for which we have

$$f_{\nu_\mu}\left(\frac{1}{2^\mu}\right) > 1 \qquad (\mu = 1, 2, \cdots) .$$

Then, such a sequence (α_ν) that

$$\alpha_\nu = \begin{cases} \dfrac{1}{2^\mu} & \text{for} \quad \nu = \nu_\mu , \\ 0 & \text{for} \quad \nu \neq \nu_\mu , \end{cases}$$

satisfies

$$\sum_{\nu=1}^{\infty} f_\nu(\alpha_\nu) = +\infty \quad \text{and} \quad (\alpha_\nu) \in l ,$$

contradicting the assumption.

Conversely, if there exists $\alpha>0$ such that $\sup_{\nu\geqq 1} f_\nu(\alpha) < +\infty$, then we have

$$f_\nu(\alpha\xi_\nu) \leqq |\xi_\nu| \cdot f_\nu(\alpha) \qquad \text{for almost all } \nu$$

for every $(\xi_\nu)\in l$, so that $(\xi_\nu)\in l(f_\nu)$.

Similarly, we can prove:

(1.2) *If* $l(f_\nu)\subset m$, *then there exists a real number* $\alpha>0$ *such that* $\inf_{\nu\geqq 1} f_\nu(\alpha)>0$. *Conversely, if* $\inf_{\nu\geqq 1} f_\nu(\alpha)>0$ *for some* $\alpha>0$, *then we have* $l(f_\nu)\subset m$.

Now, putting

$$f'_\nu(\xi) = \inf_{\varepsilon>0} \frac{f_\nu(\xi+\varepsilon)-f_\nu(\xi)}{\varepsilon},$$

we have

$$f_\nu(\xi) = \int_0^\xi f'_\nu(\xi)\,d\xi,$$

and there exist non-decreasing functions $g_\nu(\eta)$ $(\nu=1,2,\cdots)$, which satisfy the following properties:

$$g_\nu(\eta-0)\leqq \xi \leqq g_\nu(\eta+0) \qquad \text{if } \eta = f'_\nu(\xi),$$

$$f'_\nu(\xi-0)\leqq \eta \leqq f'_\nu(\xi+0) \qquad \text{if } \xi = g_\nu(\eta).$$

Then the functions:

$$\bar{f}_\nu(\eta) = \int_0^\eta g_\nu(\eta)\,d\eta \qquad (\nu=1,2,\cdots)$$

are also modulars on the real line and we have

$$\xi\eta \leqq f_\nu(\xi)+\bar{f}_\nu(\eta) \qquad \text{for every } \xi,\eta>0,$$

and

$$\alpha\beta = f_\nu(\alpha)+\bar{f}_\nu(\beta)$$

if

$$f'_\nu(\alpha-0)\leqq \beta \leqq f'_\nu(\alpha+0),$$

$$g_\nu(\beta-0)\leqq \alpha \leqq g_\nu(\beta+0).$$

In the sequel, we will denote g_ν by $\bar{f}'_\nu$ $(\nu=1,2,\cdots)$.

Next, we will investigate the relation between $l(f_\nu)$ and $l(\bar{f}_\nu)$. The fact is already known by the general theory of modulared semi-ordered linear spaces. Here, we will use only the calculation of sequences.

(1. 3) *The modular conjugate space of* $l(f_\nu)$ *is* $l(\bar{f}_\nu)$ *and* $\bar{m}$ *is the conjugate modular of* m, *provided that* $\inf_{\nu \geqq 1} f_\nu(\alpha) > 0$ *for some* $\alpha > 0$ *and* f'_ν, $\bar{f}'_\nu$ *are continuous.*

Lemmas (1.3.1)–(1.3.3) constitute the proof.

(1. 3. 1) *For any element* $y = (\eta_\nu) \in l(\bar{f}_\nu)$, *there exists a number* γ *such that*

$$\sum_{\nu=1}^{\infty} \xi_\nu \eta_\nu \leqq \gamma$$

for every $x = (\xi_\nu) \in l(f_\nu)$ *satisfying* $m(x) \leqq 1$.

If we can not find such γ, there exists a sequence of sequences:

$$(\xi_\nu^{(\mu)}) \qquad (\mu = 1, 2, \cdots)$$

such that the following properties are satisfied:

$$\sum_{\nu=1}^{\infty} f_\nu(\xi_\nu^{(\mu)}) \leqq 1 \qquad (\mu = 1, 2, \cdots),$$

$$\sum_{\nu=1}^{\infty} \xi_\nu^{(\mu)} \eta_\nu > 2^\mu \mu \qquad (\mu = 1, 2, \cdots).$$

This sequence is bounded with respect to μ. Because, if there exists a term $\xi_\nu^{(\mu)}$ such that

$$|\xi_\nu^{(\mu)}| > 1,$$

then we have

$$|\xi_\nu^{(\mu)}| f_\nu(\alpha) \leqq f_\nu(\alpha \xi_\nu^{(\mu)}) \leqq 1.$$

Therefore, a sequence:

$$\gamma_\nu = \sum_{\mu=1}^{\infty} \frac{1}{2^\mu} \xi_\nu^{(\mu)} \qquad (\nu = 1, 2, \cdots)$$

is obtained, and

$$\sum_{\nu=1}^{\infty} \gamma_\nu \eta_\nu = \sum_{\nu=1}^{\infty} \eta_\nu \sum_{\mu=1}^{\infty} \frac{1}{2^\mu} \xi_\nu^{(\mu)} \geqq \frac{1}{2^\mu} \sum_{\nu=1}^{\infty} \eta_\nu \xi_\nu^{(\mu)} \geqq \mu.$$

On the other hand we have

$$\sum_{\nu=1}^{\infty} f_\nu(\gamma_\nu) = \sum_{\nu=1}^{\infty} f_\nu\left(\sum_{\mu=1}^{\infty} \frac{1}{2^\mu} \xi_\nu^{(\mu)}\right) \leqq \sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\infty} \frac{1}{2^\mu} f_\nu(\xi_\nu^{(\mu)}) \leqq 1,$$

contradicting the assumption.

(1.3.2) *For a sequence* (η_ν), *if we have*

$$\sum_{\nu=1}^{\infty} \xi_\nu \eta_\nu < +\infty$$

for every element $(\xi_\nu) \in l(f_\nu)$, *then* $(\eta_\nu) \in l(\bar{f}_\nu)$. (*A generalization of* LANDAU*'s theorem.*[5])

For the number $\gamma > 0$ in the previous lemma, when we put

$$\eta_\nu = (\gamma+1)\bar{\eta}_\nu \quad \text{and} \quad \bar{f}'_\nu(\bar{\eta}_\nu) = \xi_\nu \qquad (\nu = 1, 2, \cdots),$$

we have

$$\xi_\nu \bar{\eta}_\nu = \bar{\eta}_\nu \bar{f}'_\nu(\bar{\eta}_\nu), \quad \xi_\nu f'_\nu(\xi_\nu) = \bar{\eta}_\nu \bar{f}'_\nu(\bar{\eta}_\nu).$$

Therefore we have

$$(*) \qquad \sum_{\nu=1}^{\mu} \xi_\nu f'_\nu(\xi_\nu) = \sum_{\nu=1}^{\mu} \bar{\eta}_\nu \bar{f}'_\nu(\bar{\eta}_\nu) \qquad (\mu = 1, 2, \cdots).$$

We will prove that

$$\sum_{\nu=1}^{\infty} \xi_\nu f'_\nu(\xi_\nu) \leqq 1 .$$

If, on the contrary, we have

$$\sum_{\nu=1}^{\infty} \xi_\nu f'_\nu(\xi_\nu) > 1 ,$$

then there exists a number μ_0 such that

$$\rho \equiv \sum_{\nu=1}^{\mu_0} \xi_\nu f'_\nu(\xi_\nu) > 1 .$$

Since

$$\sum_{\nu=1}^{\mu_0} f_\nu\left(\frac{\xi_\nu}{\rho}\right) \leqq \frac{1}{\rho} \sum_{\nu=1}^{\mu_0} f_\nu(\xi_\nu) \leqq 1 ,$$

considering the following sequence:

$$(\xi_1, \xi_2, \cdots, \xi_{\mu_0}, 0, \cdots) \in l(f_\nu) ,$$

we have

$$\sum_{\nu=1}^{\mu_0} \xi_\nu \eta_\nu \leqq \rho\gamma = \gamma \sum_{\nu=1}^{\mu_0} \xi_\nu f'_\nu(\xi_\nu)$$

by the previous lemma, and hence

$$\sum_{\nu=1}^{\mu_0} \xi_\nu \bar{\eta}_\nu \geqq \frac{\gamma}{\gamma+1} \sum_{\nu=1}^{\mu_0} \xi_\nu f'_\nu(\xi_\nu) .$$

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5) E. LANDAU: Ueber einen Konvergenzensatz, Göttingen Nachr., (1907), 25–27.

On the other hand, as $\xi_\nu \bar{\eta}_\nu = \bar{\eta}_\nu \bar{f}'_\nu(\bar{\eta}_\nu)$, we have

$$\sum_{\nu=1}^{\mu_0} \bar{\eta}_\nu \bar{f}'_\nu(\bar{\eta}_\nu) \leqq \frac{\gamma}{\gamma+1} \sum_{\nu=1}^{\mu_0} \xi_\nu f'_\nu(\xi_\nu) .$$

which contradicts the relation (*).

Therefore we have

$$\sum_{\nu=1}^{\infty} \xi_\nu f'_\nu(\xi_\nu) \leqq 1 , \quad \text{that is,} \quad \sum_{\nu=1}^{\infty} \bar{\eta}_\nu \bar{f}'_\nu(\bar{\eta}_\nu) < +\infty ,$$

and, since $\bar{\eta}_\nu \bar{f}'_\nu(\bar{\eta}_\nu) \geqq \bar{f}_\nu(\bar{\eta}_\nu)$,
we have

$$\sum_{\nu=1}^{\infty} \bar{f}_\nu(\bar{\eta}_\nu) < +\infty , \quad \text{that is,} \quad \sum_{\nu=1}^{\infty} \bar{f}_\nu\left(\frac{\gamma}{\gamma+1}\eta_\nu\right) < +\infty ,$$

which means that

$$(\eta_\nu) \in l(\bar{f}_\nu) .$$

As this proof is dual about $l(f_\nu)$ and $l(\bar{f}_\nu)$, we proved that the conjugate space of $l(f_\nu)$ is $l(\bar{f}_\nu)$ and the converse.

(1.3.3) *We have the following relation:*

$$\bar{m}(y) = \sup_{x \in l(f_\nu)} \{(y, x) - m(x)\} ,$$

where

$$(y,x) = \sum_{\nu=1}^{\infty} \xi_\nu \eta_\nu \qquad (x = (\xi_\nu), \quad y = (\eta_\nu)) .$$

Because, we have

$$\sup_{x \in l(f_\nu)} \{(y, x) - m(x)\} = \sup_{x \in l(f_\nu)} \left\{\sum_{\nu=1}^{\infty} \xi_\nu \eta_\nu - \sum_{\nu=1}^{\infty} f_\nu(\xi_\nu)\right\}$$
$$= \sup_{x \in l(f_\nu)} \left\{\sum_{\nu=1}^{\infty} (\xi_\nu \eta_\nu - f_\nu(\xi_\nu))\right\} .$$

(Here we need only consider such element $x \in l(f_\nu)$ that $m(x) < +\infty$).
Now, we will prove that

$$\sup_{x \in l(f_\nu)} \sum_{\nu=1}^{\infty} (\xi_\nu \eta_\nu - f_\nu(\xi_\nu)) \geqq \sum_{\nu=1}^{\infty} \sup_{\xi > 0} (\xi \eta_\nu - f_\nu(\xi)) .$$

For any number $\alpha > 0$ such that

$$\sum_{\nu=1}^{\infty} \sup_{\xi > 0} (\xi \eta_\nu - f_\nu(\xi)) > \alpha ,$$

since $\sup_{\xi > 0} (\xi \eta_\nu - f_\nu(\xi)) \geqq 0$, there exists μ_0 for which, we have

$$\sum_{\nu=1}^{\mu_0} \sup_{\xi>0} (\xi\eta_\nu - f_\nu(\xi)) > \alpha .$$

Hence there exist numbers α_ν $(\nu=1,2,\cdots,\mu_0)$ such that

$$\sup_{\xi>0} (\xi\eta_\nu - f_\nu(\xi)) > \alpha_\nu , \quad \alpha_1+\alpha_2+\cdots+\alpha_{\mu_0} = \alpha .$$

Therefore, we can find ξ_ν $(\nu=1,2,\cdots,\mu_0)$ such that

$$\xi_\nu\eta_\nu - f_\nu(\xi_\nu) > \alpha_\nu .$$

Since the sequence:

$$(\xi_1, \xi_2, \cdots, \xi_{\mu_0}, 0, \cdots)$$

belongs to $l(f_\nu)$, we have

$$\sum_{\nu=1}^{\infty} (\xi_\nu\eta_\nu - f_\nu(\xi_\nu)) = \sum_{\nu=1}^{\mu_0} (\xi_\nu\eta_\nu - f_\nu(\xi_\nu)) > \alpha ,$$

and hence

$$\sup_{x\in l(f_\nu)} \sum_{\nu=1}^{\infty} (\xi_\nu\eta_\nu - f_\nu(\xi_\nu)) > \alpha ,$$

so that the above inequality is obtained.

Since the converse relation:

$$\overline{m}(y) \geqq (y, x) - m(x)$$

is obvious, the relation:

$$\overline{m}(y) = \sup_{x\in l(f_\nu)} \{(y, x) - m(x)\}$$

is obtained. Similarly, we can prove

$$m(x) = \sup_{y\in l(f_\nu)} \{(x,y) - \overline{m}(y)\} ,$$

so that this lemma is completely proved.

§ 2. In this section we consider two modulared sequence spaces $l(f_\nu)$ and $l(g_\nu)$, and suppose that the functions g_ν $(\nu=1,2,\cdots)$ are strictly increasing in order that the inverse functions g_ν^{-1} are uniquely determined.

(2.1) *If the functions $f_\nu g_\nu^{-1}(\xi)$ are convex with respect to ξ and $\sup_{\nu\geqq 1} f_\nu g_\nu^{-1}(\alpha) < +\infty$ for some $\alpha>0$, then we have $l(f_\nu) \supset l(g_\nu)$.*

Let $x=(\xi_\nu)\in l(g_\nu)$. Then there exists $\beta>0$ such that

$$\sum_{\nu=1}^{\infty} g_\nu(\beta\xi_\nu) < +\infty .$$

Since $l \subset l(f_\nu g_\nu^{-1})$[6], there exists $\gamma > 0$ such that

$$\sum_{\nu=1}^{\infty} f_\nu g_\nu^{-1}(\gamma g_\nu(\beta\xi_\nu)) < +\infty .$$

As we may suppose that $\gamma \leqq 1$, we have

$$\sum_{\nu=1}^{\infty} f_\nu(\beta\gamma\xi_\nu) = \sum_{\nu=1}^{\infty} f_\nu g_\nu^{-1} g_\nu(\beta\gamma\xi_\nu) \leqq \sum_{\nu=1}^{\infty} f_\nu g_\nu^{-1}(\gamma g_\nu(\beta\xi_\nu)) < +\infty ,$$

that is, $x \in l(f_\nu)$.

(2.2) *If the functions $f_\nu g_\nu^{-1}$ are convex and $l(f_\nu)$ is finite, in order that*

$$l(f_\nu) \subset l(g_\nu) ,$$

it is necessary and sufficient that

$$\sum_{\nu=1}^{\infty} \overline{f_\nu g_\nu^{-1}}(\alpha) < +\infty$$

for some $\alpha > 0$.

Sufficiency. Let $x = (\xi_\nu) \in l(f_\nu)$. Then there exists $\beta > 0$ such that

$$\sum_{\nu=1}^{\infty} f_\nu(\beta\xi_\nu) < +\infty .$$

Snce we have

$$\alpha g_\nu(\beta\xi_\nu) \leqq f_\nu g_\nu^{-1} g_\nu(\beta\xi_\nu + \overline{f_\nu g_\nu^{-1}}(\alpha) ,$$

by the assumption we have

$$\sum_{\nu=1}^{\infty} g_\nu(\beta\xi_\nu) < +\infty , \quad \text{namely,} \quad x \in l(g_\nu) .$$

Necessity. 1) *In order that $l(f_\nu) \subset l$ it is necessary and sufficient that*

$$\sum_{\nu=1}^{\infty} \overline{f}_\nu(\alpha) < +\infty$$

for some $\alpha > 0$.

Taking a sequence $(1,1,\cdots)$, for any sequence (ξ_ν) such that

$$\sum_{\nu=1}^{\infty} f_\nu(\xi_\nu) < +\infty ,$$

6) The symbol $l(f_\nu q_\nu^{-1})$ is used here for convenience sake. In fact, it not always becomes a modulared sequence space, but the proposition (1.1) is valid for it.

we have

$$\sum_{\nu=1}^{\infty} |\xi_\nu| < +\infty .$$

Therefore, from (1.3) we obtain

$$(1, 1, \cdots) \in l(\bar{f}_\nu) ,$$

namely, there exists $\alpha > 0$ such that

$$\sum_{\nu=1}^{\infty} \bar{f}_\nu(\alpha) < +\infty .$$

2) *If* $l(f_\nu g_\nu^{-1}) \subset l$, *then* $\bar{l}(f_\nu) \subset l(g_\nu)$.
Let $x = (\xi_\nu) \in l(f_\nu)$, then there exists $\alpha > 0$ such that

$$\sum_{\nu=1}^{\infty} f_\nu(\alpha\xi_\nu) < +\infty .$$

Hence, since

$$\sum_{\nu=1}^{\infty} f_\nu g_\nu^{-1} g_\nu(\alpha\xi_\nu) < +\infty ,$$

by the assumption we have

$$\sum_{\nu=1}^{\infty} g_\nu(\alpha\xi_\nu) < +\infty , \quad \text{namely}, \quad x \in l(g_\nu) .$$

3) *If* $l(f_\nu) \subset l(g_\nu)$, *then* $l(f_\nu g_\nu^{-1}) \subset l$.
Let $x = (\xi_\nu) \in l(f_\nu g_\nu^{-1})$, then there exists $\alpha > 0$ such that

$$\sum_{\nu=1}^{\infty} f_\nu g_\nu^{-1}(\alpha\xi_\nu) < +\infty ,$$

so that

$$\sum_{\nu=1}^{\infty} g_\nu g_\nu^{-1}(\alpha\xi_\nu) < +\infty , \quad \text{namely}, \quad x \in l .$$

4) If $l(f_\nu) \subset l(g_\nu)$, from 3) we have

$$l(f_\nu g_\nu^{-1}) \subset l .$$

Hence there exists $\alpha > 0$ such that

$$\sum_{\nu=1}^{\infty} \overline{f_\nu g_\nu^{-1}}(\alpha) < +\infty .$$

§ 3. *SCHUR's lemma*[7] in sequence spaces l was generalized by

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7) J. SCHUR: Ueber lineare Transformation in der Theorie der unendlichen Reihen, Jour. für reine und angew. Math., 151 (1921), 79–111.

H. NAKANO[3] and I. HALPERIN-H. NAKANO[4] to the modulared sequence spaces $l(p_1, p_2, \cdots)$, $\lim_{\nu\to\infty} p_\nu = 1$. In this section, we will establish the generalization in the general modulared sequence spaces. For this purpose, we use the notion of *the exponents of modulars* considered in the previous paper.[8]

The upper (lower) exponent of modular f_ν is the greatest lower (least upper) bound of such number $p \geqq 1$ such that the function of $\xi > 0$:

$$f_\nu(\xi)/\xi^p$$

is non-increasing (non-decreasing) and denoted by χ^{f_ν} (χ_{f_ν}).

Then the following relations are obtained:

$$\chi_{f_\nu} \cdot f_\nu(\xi) \leqq \xi f'_\nu(\xi) \leqq \chi^{f_\nu} \cdot f_\nu(\xi) \qquad (\xi > 0),$$

$$\frac{1}{\chi^{f_\nu}} + \frac{1}{\chi_{\bar{f}_\nu}} = \frac{1}{\chi_{f_\nu}} + \frac{1}{\chi^{\bar{f}_\nu}} = 1 .$$

(3.1) *If, in the modulared sequence space* $l(f_\nu)$, $\lim_{\nu\to\infty} \chi^{f_\nu} = 1$ *and every* f'_ν *is strictly increasing, then the weak convergence and strong convergence coincide in* $l(f_\nu)$.

Let a sequence of elements:

$$x_\mu = (\xi_1^{(\mu)}, \xi_2^{(\mu)}, \cdots) \in l(f_\nu) \qquad (\mu = 1, 2, \cdots)$$

be weak convergent to 0, then we have

$$\lim_{\mu\to\infty} \xi_\nu^{(\mu)} = 0 \qquad (\nu = 1, 2, \cdots),$$

$$\sup_{\mu \geqq 1} \|x_\mu\| < +\infty .$$

Hence, we can suppose that

$$\|x_\mu\| \leqq 1 , \quad \text{namely,} \quad m(x_\mu) \leqq 1 .$$

If there exists a positive number $\varepsilon > 0$ for which

$$m(x_\mu) > \varepsilon ,$$

then we can find a partial sequence $\nu_\mu (\mu = 1, 2, \cdots)$ such that

$$\sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} f_\nu(\xi_\nu^{(\mu)}) > \varepsilon ,$$

$$\chi^{f_\nu} \leqq 1 + \frac{1}{2^\mu}, \quad \text{that is,} \quad \chi_{\bar{f}_\nu} \leqq 1 + 2^\mu \, (\nu \geqq \nu_\mu) .$$

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8) S. YAMAMURO: Exponets of the modulared semi-ordered linear spaces, Jour. Fac. Science, Hokkaidô University, XII (1953), 211-253.

Putting

$$\eta_\nu = 0 \qquad \text{for} \qquad \nu < \nu_1$$

and

$$\eta_\nu = f'_\nu(\xi_\nu^{(\mu)}) \qquad \text{for} \quad \nu_\mu \leqq \nu < \nu^{\mu+1},$$

we have

$$\eta_\nu \bar{f}'_\nu(\eta_\nu) = \xi_\nu^{(\mu)} f'_\nu(\xi_\nu^{(\mu)}) \qquad (\nu_\mu \leqq \nu < \nu_{\mu+1}),$$

and hence

$$\sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} \bar{f}_\nu(\eta_\nu) \leqq \sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} \frac{1}{2^\mu+1} \eta_\nu \bar{f}'_\nu(\eta_\nu) = \sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} \frac{1}{2^\mu+1} \xi_\nu^{(\mu)} f'_\nu(\xi_\nu^{(\mu)})$$
$$= \sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} \frac{1}{2^\mu} \cdot \frac{2^\mu}{2^\mu+1} \xi_\nu^{(\mu)} f'_\nu(\xi_\nu^{(\mu)}) \leqq \sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} \frac{1}{2^\mu} f_\nu(\xi_\nu^{(\mu)}) \leqq \frac{1}{2^\mu},$$

so that we have

$$\sum_{\nu=1}^{\infty} \bar{f}_\nu(\eta_\nu) \leqq 1 .$$

On the other hand we obtain

$$\sum_{\nu=1}^{\infty} \xi_\nu^{(\mu)} \eta_\nu \geqq \sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} \xi_\nu^{(\mu)} \eta_\nu = \sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} \xi_\nu^{(\mu)} f'_\nu(\xi_\nu^{(\mu)}) \geqq \sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} f_\nu(\xi_\nu^{(\mu)}) > \varepsilon ,$$

which is impossible, because $x_\mu (\mu = 1, 2, \cdots)$ is weakly convergent to 0.

(3.2) *If a modulared sequence space $l(f_\nu)$ satisfies the following conditions:*

1) *The weak convergence and the strong convergence coincide;*
2) $0 < \inf_{\nu \geqq 1} f_\nu(1) \leqq \sup_{\nu \geqq 1} f_\nu(1) < +\infty$;
3) f'_ν, $\bar{f}'_\nu$ *are strictly increasing,*

then, for any number $\varepsilon > 0$ *we have*

$$\chi_{f_\nu} < 1 + \varepsilon$$

for almost all ν .

Suppose that

$$\chi_{f_\nu} > 1 + \varepsilon \qquad (\nu = 1, 2, \cdots).$$

for some $\varepsilon > 0$. Then, it is obvious that

$$\chi_{\bar{f}_\nu} < \frac{1+\varepsilon}{\varepsilon} \qquad (\nu = 1, 2, \cdots)$$

for the modular conjugate space $l(\bar{f}_\nu)$. (See (1.3)).

Now, take such elements $e_\mu = (\eta_\nu^{(\mu)}) (\mu = 1, 2, \cdots)$ that

$$\eta_\nu^{(\mu)} = \begin{cases} 1 & \text{for} \quad \nu = \mu , \\ 0 & \text{for} \quad \nu \neq \mu , \end{cases}$$

then, for any $x = (\xi_\nu) \in l(\bar{f}_\nu)$, we have

$$(x, e_\mu) = \xi_\mu \qquad (\mu = 1, 2, \cdots) .$$

Since the sequence is bounded, putting $\xi_0 = \sup_{\nu \geqq 1} |\xi_\nu|$, we obtain

$$\bar{f}_\mu \left(\frac{\xi_\mu}{\xi_0} \right) \geqq \left(\frac{\xi_\mu}{\xi_0} \right)^{\chi \bar{f} \nu} \cdot \bar{f}_\mu(1) \geqq \left(\frac{\xi_\mu}{\xi_0} \right)^{\frac{1+\varepsilon}{\varepsilon}} \bar{f}_\mu(1) ,$$

and

$$\lim_{\mu \to \infty} \bar{f}_\mu \left(\frac{\xi_\mu}{\xi_0} \right) = 0 .$$

Hence it follows that

$$\lim_{\mu \to \infty} \xi_\mu = 0 ,$$

so that the sequence of elements $e_\mu (\mu = 1, 2, \cdots)$ is convergent weakly to 0.

On the other hand, we see easily that

$$m(e_\mu) = f_\mu(1) \geqq \inf_{\nu \geqq 1} f_\mu(1) > 0$$

for every $\mu = 1, 2, \cdots$, which contradicts the assumption.