PARTIALLY ORDERED ABELIAN SEMIGROUPS. IV

ON THE EXTENTION OF THE CERTAIN NORMAL PARTIAL ORDER DEFINED ON ABELIAN SEMIGROUPS

By

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In Part I[1] of this series, I noted that for any two elements x and y non-comparable in the strong partial order P defined on an abelian semigroup S there exists an extension Q of P such $x>y$ in Q if and only if P is normal. In this Part IV, I shall discuss the extension of the partial order under the weak condition than strongness.

Definition 1. A set S is said to be *a partially ordered abelian semigroup* (p.o. semigroup), when S is (I) an abelian semigroup (not necessarily contains the unit element), (II) a partially ordered set, and satisfies (III) the homogeneity: $a\geqq b$ implies $ac\geqq bc$ for any c of S.

A partial order which satisfies the condition (III) is called a *partial order defined on an abelian semigroup.*

Moreover, if a partial order defined on an abelian semigroup S is a linear order, then S is said to be a *linearly ordered abelian semigroup* (l.o. semigroup).

We write $a//b$ in P for a and b are non-comparable in P.

Definition 2. Let P be a partial order defined on an abelian semigroup S. We consider the following conditions for the partial order P:

(E): $ac>bc$ in P implies $a>b$ in P. (order cancellation law)

(G): Let x and y be any two elements non-comparable in P. Then there exists an extension of P in which $x>y$.

(H): If $a//b$ in P, then $ua\neq ub$ for any u in S.

(K): If $a//b$ in P, then $ua//ub$ in P for any u in S.

(L): Let $a//b$ and $u//v$ in P respectively. If $au\neq bv$, then $au//bv$ in P.

1) Partically ordered abelian semigroup. I. On the extension of the strong partial order defined on abelian semigroups. Journ. Fac. Sci., Hokkaido University, Series I, vol. XI (1951), pp. 181–189.

Strongness: $ac \geqq bc$ in P implies $a \geqq b$ in P.

Normality: $a^n \geqq b^n$ in P for some positive integer n implies $a \geqq b$ in P.

Theorem 1. *Let P be a partial order defined on an abelian semigroup S. Then P satisfies the condition* (K) *if and only if P satisfies the conditions* (E) *and* (H).

Proof. Clearly the condition (K) implies the condition (H). If P satisfies the condition (K) and $ac > bc$ in P, then a and b are comparable in P. And hence we have $a > b$ in P.

Conversely, let P satisfy the conditions (H) and (E) and let $a // b$ in P. Then we have $ua \neq ub$ for any u in S by the condition (H). If ac and bc are comparable in P for some c in S, say that $ac > bc$ in P, then we have $a > b$ in P by the condition (E), this is impossible.

Theorem 2. *Let P be a normal partial order defined on abelian semigroup S which satisfies the condition* (K). *If $a // b$ in P, then $u^i a^j // u^i b^j$ in P for any u in S and any integers $i\,(\geqq 0)$ and $j\,(>0)$, where if $i=0$, $u^i a^j // u^i b^j$ means that $a^j // b^j$.*

Proof. By the normality, $a // b$ in P implies $a^j // b^j$ in P for any positive integer j. And hence we have $u^i a^j // u^i b^j$ in P by the condition (K).

Theorem 3. *Let P be a normal partial order defined on an abelian semigroup S which satisfies the condition* (K) *and x and y be any two elements non-comparable in P. Then there exists a normal extension Q of P such that $x > y$ in Q.*

Proof. Let P be a normal partial order defined on S and the elements x and y are not comparable in P. Let us define a relation Q as follows:

We put $a > b$ in Q if and only if $a \neq b$ and there exist two non-negative integers n and m, such that not both zero and

(§) $$a^n y^m \geqq b^n x^m \quad \text{in } P,$$

where if $m=0$ or $n=0$ (§) means that $a^n \geqq b^n$ or $y^m \geqq x^m$ in Prespectively.

First we note that n is never zero, for otherwise we should have $y^m \geqq x^m$ in P, whence by the normality we have $y \geqq x$ in P against the hypothesis.

(i) We being with verifying that $a > b$ and $b > a$ in Q are contradictory. Suppose that $a > b$ and $b > a$ in Q, namely $a^n y^m \geqq b^n x^m$ and $b^i y^j \geqq a^i x^j$ in P for some non-negative integers n, m, i, j. By multiplying i times the first, n times the second inequality, we obtain $(ab)^{ni} y^{mi+nj} \geqq (ab)^{ni} x^{mi+nj}$ in P, which contradicts the condition (K). If $m=j=0$, then we have $a > b$ and $b > a$ in P, which is impossible.

(ii) We show the transitivity of Q. Assume that $a>b$ and $b>c$ in Q, i.e., for some non-negative integers n, m, i, j, $a^n y^m \geqq b^n x^m$ and $b^i y^j \geqq c^i x^j$ in P. By multiplying as in (i) we get $a^{ni} y^{mi+nj} \geqq c^{ni} x^{mi+nj}$ in P. Here ni is not zero, and $a=c$ is impossible by the condition (K), so that $a>c$ in Q. If $m=j=0$, then we have $a>b$, $b>c$ in P, and hence $a>c$ in $P(Q)$.

(iii) We prove next the homogeneity of Q. Suppose that $a>b$ in Q. If $ac \neq bc$, from $(ac)^n y^m \geqq (bc)^n x^m$ in P we have $ac>bc$ in Q. Therefore $a>b$ in Q implies $ac \geqq bc$ in Q for any c of S.

(iv) Q is an extension of P, for if $a>b$ in P, then $ay^0 > bx^0$ in P, therefore $a>b$ in Q.

(v) It is clear that $x>y$ in Q. In fact, $xy \geqq yx$ in P.

(vi) We may prove the normality of Q. Indeed, supposing $a^n > b^n$ in Q fer some positive integer n, i.e., $(a^n)^i y^j \geqq (b^n)^i x^j$ in P, we see at once that $a>b$ in Q.

(vii) If $a // b$ in Q, then $a // b$ in P, and hence $ua \neq ub$ for any u in S. Therefore, Q satisfies the condition (H).

Theorem 4. *Let P be a partial order defined on an abelian semigroup S which satisfies the condition* (G) *and let $a // b$, $u // v$ in P. If* $au \neq bv$ and $av = bu$, *then $au // bv$ in P.*

Proof. Suppose that au and bv are comparable in P, say that $au>bv$ in P. There exists an extension Q of P such that $v>u$ in Q. Then we have $bv \geqq bu = av \geqq au$ in Q, that is, we have $bv \geqq au$ in Q. This contradicts the assumption.

Theorem 5. *Let P be a partial order defined on an abelian semigroup S which satisfies the condition* (G) *and let $a // b$, $u // v$ in P. If $au \neq bv$ and $av \neq bu$, then $au // bv$ or $av // bu$ in P.*

Proof. Suppose that au and bv are comparable in P, say that $au>bv$ in P. If $bu>av$ in P, then we consider an extension Q of P such that $v>u$ in Q. Then we have $bv \geqq bu > av \geqq au$ in Q, that is, $bv>au$ in Q, this is absurd. If $av>bu$ in P, then we consider an extension Q of P such that $b>a$ in Q. Then we have $bv \geqq av > bu \geqq au$ in Q, that is, $bv>au$ in Q, which leads the contradiction also. Therefore, $bu // av$ in P.

Theorem 6. *Let P be a normal partial order defined on an abelian semigroup S which satisfies the condition* (K). *If $a>b$ and $x // y$ in P, then $a^n y^m > b^n x^m$ or $a^n y^m // b^n x^m$ in P ($a^n x^m > b^n y^m$ or $a^n x^m // b^n y^m$ in P) for any integers m ($\geqq 0$) and n (>0).*

Proof. If $a^n y^m = b^n x^m$ for some positive integeas m and n, then we have $a^n x^m \geqq b^n x^m = a^n y^m \geqq b^n y^m$ in P, that is, $b^n x^m \geqq b^n y^m$ in P which contradicts the condition (K).

By the existence of the extension Q of P such that $y > x$ in Q, we have $a^n y^m \geqq b^n x^m$ in Q. Hence, if $a^n y^m$ and $b^n x^m$ are comparable in P, then we have $a^n y^m > b^n x^m$ in P.

Theorem 7. *Let P be a normal partial order defined on an abelian semigroup S which satisfies the conditions* (K) *and* (L) *and let $x // y$ in P. For two distinct elements a and b, the following two properties are equivalent to each other:*

(1) $a > b$ *in* P *or* $a^n y^m = b^n x^m$

(2) $a^n y^m \geqq b^n x^m$ *in* P

for some integers m ($\geqq 0$) and n (>0), where if $m=0$, $a^n y^m$ and $b^n x^m$ means that a^n and b^n respectively.

Proof. (1) implies (2): If $a > b$ in P, then we can write $ay^o \geqq bx^o$ in P.

(2) implies (1): If $a // b$ in P, then by the normality we have $m > 0$ and $a^n // b^n$, $y^m // x^m$ in P. Therefore, $a^n y^m = b^n x^m$ by the condition (L).

If $m=0$, then $a^n \geqq b^n$, and hence $a > b$ in P.

If $m > 0$ and $b > a$ in P, then $b^n > a^n$, and hence we have $b^n x^m \geqq b^n x^m$, $b^n y^m \geqq a^n y^m$ in P. Therefore, we have $b^n y^m \geqq a^n y^m \geqq b^n x^m \geqq a^n x^m$ in P, that is, $b^n y^m \geqq b^n x^m$ in P which contradicts the condition (K). Therefore, we have $a > b$ in P.

Moreover, in this case, $a > b$ in P if and only if $a^n y^m > b^n x^m$ in P for some integers m ($\geqq 0$) and n (>0).

Theorem 8. *Let P be a normal partial order defined on an abelian semigroup S which satisfies the conditions* (K) *and* (L) *and x and y be any two elements non-comparable in P. Then there exists a normal extension Q, which satisfies the condition* (K), *of P such that $x > y$ in Q.*

Proof. By Theorem 3, there exists the normal extension Q of P which satisfies the condition (H) such that $x > y$ in Q.

The order-relation Q is as follows:

$a > b$ in Q if and only if $a > b$ in P, or $a // b$ in P and $a^n y^m = b^n x^m$ for some positive integers m and n.

(viii) Suppose that $ac > bc$ in Q. If $ac > bc$ in P, then we have $a > b$ in $P(Q)$. If $ac // bc$ in P, then $a // b$ in P and $(ac)^n y^m = (bc)^n x^m$, i.e., $c^n(a^n y^m) = c^n(b^n x^m)$ for some positive integers m and n. By the condition (K) of P, $a^n y^m$ and $b^n x^m$ are comparable in P. And hence we have $a^n y^m = b^n x^m$

by the condition (L) of P. Therefore, we have $a>b$ in Q. Thus Q satisfies the conditions (H) and (E), that is, the condition (K).

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