A GENERALIZATION OF MAZUR-ORLICZ THEOREM ON FUNCTION SPACES

By

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1. Introduction. Let $\Omega(\boldsymbol{B}, \mu)$ be a locally finite[1] measure space. By many investigators various function spaces consisting of locally almost finite $\boldsymbol{B}$-measurable functions[2] on Ω have been considered as a generalization of the so-called $\boldsymbol{L}_p$-spaces on Ω $(1 \leqq p \leqq +\infty)$. One of them is $\boldsymbol{L}_{M(u,\omega)}$-space (Musielak-Orlicz [3], [4]).

Let $M(u, \omega)$ be a function on $[0, +\infty] \times \Omega$ with the following properties (it will be called (M)-*function*);

(M)

1) $0 \leqq M(u, \omega) \leqq +\infty$ for all $(u, \omega) \in [0, +\infty] \times \Omega$,
2) $\lim_{u \to 0} M(u, \omega) = 0$ for all $\omega \in \Omega$,
3) $M(u, \omega)$ is a non-decreasing and left continuous[3] function of u for all $\omega \in \Omega$,
4) $\lim_{u \to \infty} M(u, \omega) > 0$ for all $\omega \in \Omega$,
5) $M(u, \omega)$ is locally $\boldsymbol{B}$-measurable[4] as a function of ω for all $u \in [0, +\infty]$.

Using this function $M(u, \omega)$ we can define a functional $\rho_M(x)$ on locally almost finite $\boldsymbol{B}$-measurable functions $x(\omega)$ $(\omega \in \Omega)$ by the formula

$$(1) \qquad \rho_M(x) = \int_\Omega M[|x(\omega)|, \omega] d\mu \text{ [5]}$$

If $\boldsymbol{L}_{M(u,\omega)}$ denotes the set of all $x(\omega)$ such that $\rho_M(\alpha x) < +\infty$ for a positive number $\alpha = \alpha(x)$ depending on x, $\boldsymbol{L}_{M(u,\omega)}$ is a vector space.

As special cases, $\boldsymbol{L}_{M(u,\omega)}$ coincides with four typical spaces respectively:

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1) Ω is covered by the family of measurable sets of finite measure.

2) Correctly speaking, we shall consider only the functions which are almost finite real valued and $\boldsymbol{B}$-measurable in every measurable set of finite measure. And two functions $x(\omega)$ and $y(\omega)$ are identified if $x(\omega) = y(\omega)$ except on a set of measure zero in every measurable set of finite measure.

3) Since $M(u, \omega)$ can be replaced by $M(u-0, \omega)$, the left side continuity is not essential for the definition of the space $\boldsymbol{L}_{M(u,\omega)}$.

4) It is unnecessary for $M(u, \omega)$ to be almost finite valued.

5) (M)-2) and 3) imply the measurability of a function $M[|x(\omega)|, \omega]$. The integration on Ω means the supremum of integrations on every finite measured set.

1) L_p-space $(0<p\leqq+\infty)$, when $M(u,\omega)=u^p$,[6]
2) $L_{N(u)}$-space (Orlicz [7]), when $M(u,\omega)=N(u)$ and $N(u)$ is a convex function of u,
3) $L_{M(u)}$-space (Mazur-Orlicz [2]), when $M(u,\omega)=M(u)$,
4) $L_{N(u,\omega)}$-space[7] (Nakano [5]), when $M(u,\omega)=N(u,\omega)$, and $N(u,\omega)$ is a convex function of u for all $\omega\in\Omega$.

In view of generalization of a constructive method, the relation between above four spaces is shown with the following schema,

$$(2)\qquad \begin{array}{ccccc} & & L_p\,(1\leqq p\leqq+\infty) & & \\ & & \downarrow & & \\ & & L_{N(u)} & \longrightarrow & L_{N(u,\omega)} \\ & & \downarrow & & \downarrow \\ L_p\,(0<p<1) & \longrightarrow & L_{M(u)} & \longrightarrow & L_{M(u,\omega)} \end{array}$$

In the spaces $L_{N(u)}$ and $L_{N(u,\omega)}$, if we put

$$(3)\qquad \|x\|_N=\inf\{\varepsilon>0\,;\,\rho_N(x/\varepsilon)\leqq1\}\,,$$

we have a complete norm (B-norm) on $L_{N(u)}$ and $L_{N(u,\omega)}$ respectively ([1], [5]). In the spaces $L_{M(u)}$ and $L_{M(u,\omega)}$, putting

$$(4)\qquad \|x\|_M=\inf\{\varepsilon>0\,;\,\rho_M(x/\varepsilon)\leqq\varepsilon\}\,,$$

we have a complete quasi-norm (F-norm) on $L_{M(u)}$ and $L_{M(u,\omega)}$ respectively ([2], [3]). We can see easily $\lim_{n\to\infty}\|x_n\|_N=0$ ($\lim_{n\to0}\|x_n\|_M=0$) if and only if $\lim_{n\to\infty}\rho_N(\alpha x_n)=0$ ($\lim_{n\to\infty}\rho_M(\alpha x_n)=0$) for all $\alpha\geqq0$.

Mazur-Orlicz has shown in [2] the following result[8]:

Given $L_{M(u)}$-space, the necessary and sufficient condition for to exist a convex (M)-function $N(u)$ such as $L_{M(u)}=L_{N(u)}$ is that the linear topology induced by the quasi-norm $\|x\|_M$ is locally convex.

The purpose of this paper is to generalize this result to the problem of the relation between $L_{M(u,\omega)}$ and $L_{N(u,\omega)}$. In §2 we shall define the abstract $L_{M(u,\omega)}$-space, and in §3 the problem will be studied in an abstract form. If $\Omega(\boldsymbol{B},\mu)$ is non-atomic, we obtain a similar result to the above Mazur-Orlicz theorem (Theorem 2). Although in general it does not hold in an atomic case, under some assumption it can be proved also (Theorem 3).

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6) If $p=+\infty$, then we put $u^{+\infty}=0$ $(0\leqq u\leqq1)$ and $=+\infty$ $(u>1)$.

7) H. Nakano calls $L_{N(u,\omega)}$ a *modulared function space* in [5] (appendix).

8) It has been proved under an additional condition: $M(2u)\leqq KM(u)$ for all $u\geqq u_0>0$ (non-atomic case) or $M(2u)\leqq KM(u)$ for all $0\leqq u\leqq u_0$ (atomic case).

2. Modulared vector lattice. First of all, we shall define a *modulared vector latice* $R(\rho)$ as the abstraction of $L_{M(u,\omega)}$-spaces. Let R be a conditionally complete[9] vector lattice. A functional on R with values $0 \leqq \rho(x) \leqq +\infty$ will be called a *modular*[10] ([4], [5], [6]) when the following conditions are satisfied;

(ρ)
1) $\rho(\alpha x)=0$ for all $\alpha \geqq 0$ if and only if $x=0$,
2) $\inf_{\alpha>0} \rho(\alpha x)=0$ for all $x \in R$,
3) $|x| \leqq |y|$ implies $\rho(x) \leqq \rho(y)$,
4) $x \frown y=0$ implies $\rho(x+y)=\rho(x)+\rho(y)$,
5) $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$[11] implies $\sup_{\lambda \in \Lambda} \rho(x_\lambda)=\rho(x)$,
6) for any orthogonal system $x_\lambda \geqq 0$ $(\lambda \in \Lambda)$ such as $\sum_{\lambda \in \Lambda} \rho(x_\lambda) < +\infty$ we can find $x \in R$ and $x=\sum_{\lambda \in \Lambda} x_\lambda$[12] (orthogonal completeness).

Moreover, if ρ satisfies the following condition (C), ρ will be called a *convex modular*;

(C) $\rho(\alpha x)$ is a convex function of α for all $x \in R$.

We shall call R where a (convex) modular is defined a (*convex*) *modulared vector lattice.* A convex modulared vector lattice will be said briefly the *Nakano space*[13]. We can see easily that $L_{M(u,\omega)}(\rho_M)$ is a modulared vector lattice and $L_{N(u,\omega)}(\rho_N)$ is the Nakano space.

The (ρ)-condition implies some properties;

(5) $\rho(x \smile y)+\rho(x \frown y)=\rho(x)+\rho(y)$ for $x, y \geqq 0$,

(6) $\rho(\alpha x+\beta y) \leqq \rho(x)+\rho(y)$ for $x, y \in R$, $\alpha, \beta \geqq 0$, $\alpha+\beta=1$.

It has been shown in [3] and [4] that the property (6) defines a ordered[14] quasi-norm $\|x\|_\rho$ on R by the formula

(7) $\|x\|_\rho=\inf\{\varepsilon>0 ; \rho(x/\varepsilon) \leqq \varepsilon\}$ $(x \in R)$.

We can see easily $\lim_{n\to\infty} \|x_n\|_\rho=0$ if and only if $\lim_{n\to\infty} \rho(\alpha x_n)=0$ for all $\alpha \geqq 0$.

9) Every upper-bounded system of elements has a supremum in R.

10) For the first time the name 'modular' was used by H. Nakano, when (ρ)-1)~5) and (C) were satisfied. The convex modular defined in this paper coincides with the *monotone-complete modular* in Nakano's terminology ([5]). The orthogonal completeness ((ρ)-6)) implies the monotone completeness (cf. Remark of Lemma 1). The condition (ρ) is stronger than that in [4] and of the *quasi-modular* in [8].

11) For any $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda_3 \in \Lambda$ such as $x_{\lambda_1} \cup x_{\lambda_2} \leqq x_{\lambda_3}$ and $\bigcup_{\lambda \in \Lambda} x_\lambda = x$.

12) $\sum_{\lambda \in \Lambda} x_\lambda = \bigcup_{\Lambda' \subset \Lambda} \sum_{\lambda \in \Lambda'} x_\lambda$, where Λ' is a finite subset of Λ.

13) In [5] it is called a *monotone-complete modulared semi-ordered linear space.*

14) $|x| \leqq |y|$ implies $\|x\|_\rho \leqq \|y\|_\rho$.

In this section we shall prove that $||x||_\rho$ is a *complete* quasi-norm on R.

Lemma 1. *The necessary and sufficient condition for a directed system of positive elements* $0\leqq x_\lambda\uparrow_{\lambda\in\Lambda}$ *to be order-bounded is that the following two conditions are satisfied;*

(i) $\sup_{\lambda\in\Lambda}\rho(\alpha x_\lambda)<+\infty$ *for some* $\alpha>0$,

(ii) *for any* $p\in R$ $(p\neq 0)$ *we can find two positive numbers* $\beta_2>\beta_1>0$ *such that* $\sup_{\lambda\in\Lambda}(\beta_1[p]x_\lambda)<\rho(\beta_2 p)$[15].

Proof. Supposing $0\leqq x_\lambda\uparrow_{\lambda\in\Lambda} x$, then (ρ)–2) and 3) imply (i). Since $\sup_{\alpha>0}\rho(\alpha p)>0$ $(p\neq 0)$ and $\inf_{\alpha>0}\rho(\alpha[p]x)=0$, we have easily (ii).

Sufficiency: First, (ii) implies the fact that for a given $p>0$ we can find $0<[q]\leqq[p]$[16] such that $[q]x_\lambda$ $(\lambda\in\Lambda)$ is order-bounded. Because; in the contrary case, we can obtain the decomposition of $[p]$, $[p]=[q_1]\oplus\cdots\oplus[q_n]$[17], and $\lambda_i\in\Lambda$ $(1\leqq i\leqq n)$ such that $\beta_2[q_i]p<\beta_1[q_i]x_{\lambda_i}$ $(1\leqq i\leqq n)$, hence $\beta_2 p=\sum_{i=1}^{n}\beta_2[q_i]p\leqq\sum_{i=1}^{n}\beta_1[q_i]x_{\lambda_i}=\bigcup_{i=1}^{n}\beta_1[q_i]x_{\lambda_i}\leqq\beta_1[p]x_{\lambda_0}$ for some $\lambda_0\in\Lambda$. This implies the contradiction: $\rho(\beta_2 p)\leqq\rho(\beta_1[p]x_{\lambda_0})<\rho(\beta_2 p)$.

Therefore, if we put $[p_\gamma]$ $(\gamma\in\Gamma)$ a maximal orthogonal system of projectors such as $[p_\gamma]x_\lambda$ $(\lambda\in\Lambda)$ is order-bounded, then we have $\sum_{\gamma\in\Gamma}[p_\gamma]=\mathbf{I}$.[18] Putting $[p_\gamma]x_\lambda\uparrow_{\lambda\in\Lambda} y_\gamma$, since $\rho(\alpha y_\gamma)=\sup_{\lambda\in\Lambda}\rho(\alpha[p_\gamma]x_\lambda)$, we see $\sum_{\gamma\in\Gamma}\rho(\alpha y_\gamma)=\sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}\rho(\alpha[p_\gamma]x_\lambda)=\sup_{\lambda\in\Lambda}\rho(\alpha x_\lambda)<+\infty$ ((i)). Hence the orthogonal completeness $((\rho)$–6)) implies the existence $x\in R$ such as $x=\sum_{\gamma\in\Gamma}\alpha y_\gamma=\bigcup_{\lambda\in\Lambda}\alpha x_\lambda$, that is $x_\lambda\uparrow_{\lambda\in\Lambda} x/\alpha$.

Remark. When $\sup_{\alpha>0}\rho(\alpha p)=+\infty$ $(p\neq 0)$ is satisfied, (ii) follows from (i).

Theorem 1. $||x||_\rho$ $(x\in R)$ *is a complete quasi-norm on* $R(\rho)$.

Proof. Let x_ν $(\nu=1,2,\cdots)$ be a Cauchy sequence, and we assume $||x_{\nu+1}-x_\nu||_\rho\leqq 1/2^\nu$ $(\nu=1,2,\cdots)$. Putting $|x_2-x_1|+\cdots+|x_n-x_{n-1}|=z_n$ $(n\geqq 2)$ and $\sum_{\nu=n}^{m}|x_{\nu+1}-x_\nu|=y_{n,m}$ $(n\geqq 1, m\geqq n)$, we see $y_{1,m}=z_n+y_{n,m}$, $||y_{n,m}||_\rho\leqq 1/2^{n-1}$, that

15) $[p]$ is a projection operator and defined as follows $[p]x=\bigcup_{\nu=1}^{\infty}(x\frown\nu\,|\,p\,|)$ for all $x\geqq 0$, it is called a *projector* ([5]).

16) $[q]x\leqq[p]x$ for all $x\geqq 0$.

17) $[p]x=\sum_{i=1}^{l}[q_i]x$ for all $x\in R$ and $[q_i][q_j]=0$ $(i\neq j)$.

18) $\sum_{\gamma\in\Gamma}[p_\gamma]x=x$ for all $x\geqq 0$.

is, $\rho(2^{n-1}y_{n,m})\leqq 1/2^{n-1}$, and $y_{n,m}\uparrow_{m\geqq n}$. The non-decreasing sequence $y_{1,m}\uparrow_{m\geqq 1}$ satisfies (i) and (ii) in the provious Lemma 1. First, $\sup_{m\geqq 1}\rho(y_{1,m})\leqq 1$ follows from $\| y_{1,m}\|_\rho\leqq 1$ $(m\geqq 1)$. Next for any $p\neq 0$ we can find a positive number $\beta_2>0$ and an integer $n\geqq 1$ such as $1/2^{n-2}<\rho(\beta_2 p)$, and further $\beta_1>0$ such as $2\beta_1<2^{n-1}$ and $\rho(2\beta_1 z_n)<1/2^{n-1}$. Hence $\rho(\beta_1[p]y_{1,m})\leqq\rho(2\beta_1 z_n)+\rho(2\beta_1 y_{n,m})\leqq 1/2^{n-1}+\rho(2^{n-1}y_{n,m})\leqq 1/2^{n-1}+1/2^{n-1}=1/2^{n-2}<\rho(\beta_2 p)$. By Lemma 1 we can put $\sum_{\nu=n}^{\infty}|x_{\nu+1}-x_\nu|=\bigcup_{m\geqq n}y_{n,m}=y_n$ $(n\geqq 1)$. This implies also that the sequence $x_\nu(\nu=1,2,\cdots)$ coverges to an element x_0 in order[19], that is, $\text{o-}\lim_{\nu\to\infty}x_\nu=x_0$. We see $|x_0-x_n|=|\text{o-}\lim_{m\to\infty}\sum_{\nu=n}^{m}(x_{\nu+1}-x_\nu)|\leqq\bigcup_{m\geqq n}y_{n,m}=y_n$, hence $\|x_0-x_n\|_\rho\leqq\|y_n\|_\rho=$[20]$\sup_{m\geqq n}\|y_{n,m}\|_\rho\leqq 1/2^{n-1}$, that is, $\lim_{n\to\infty}\|x_0-x_n\|_\rho=0$. Q.E.D.

3. Local convexity of the linear topology in modulared vector lattices.

A. Non-atomic case. Let $R(\rho)$ be a non-atomic[21] modulared vector lattice, we have the following main theorem.

Theorem 2. *In a non-atomic modulared vector lattice $R(\rho)$ the following four conditions are equivalent each other;*

a) *the metric linear topology induced by $\|x\|_\rho$ is normable,*
b) *the metric linear topology induced by $\|x\|_\rho$ is locally convex,*
c) *there exists a convex modular $m(x)$ on $R(\rho)$ (R is the Nakano space),*
d) *there exists a complete ordered norm $|||x|||$ on $R(\rho)$ (R is a Banach lattice).*

Proof. (b)→(c). First, we shall prove the following fact:

For any $\varepsilon>0$ we can find a positve number $\delta=\delta(\varepsilon)>0$ such that

$$(*)\qquad \rho(x/\varepsilon)>\varepsilon \quad\text{implies}\quad \sum_{i=1}^{l}\rho(n_i x_i/\delta)/n_i>\delta,$$

where $\{x_i; 1\leqq i\leqq l\}$ is an arbitrary orthogonal decomposition of x, $x=\sum_{i=1}^{l}\oplus x_i$[22], and n_i $(1\leqq i\leqq l)$ are arbitrary positive integers.

19) $x_0=\bigcap_{n=1}^{\infty}\bigcup_{\nu\geqq n}x_\nu=\bigcup_{n=1}^{\infty}\bigcap_{\nu\geqq n}x_\nu$ and it is denoted by $\text{o-}\lim_{\nu\to\infty}x_\nu=x_0$.

20) (ρ)-5) implies $\sup_{\lambda\in\Lambda}\|x_\lambda\|_\rho=\|x\|_\rho$ for all $0\leqq x_\lambda\uparrow_{\lambda\in\Lambda}x$.

21) For every $a\in R$, $a>0$ we can find $b, c>0$ such as $a=b+c$ and $b\frown c=0$.

22) $x=\sum_{i=1}^{l}x_i$ and $|x_i|\frown|x_j|=0$ $(i\neq j)$.

Because, from the local convexity of $\|x\|_\rho$, for any $\varepsilon>0$ we can find a positive number $\delta=\delta(\varepsilon)>0$ such that $\|x_i\|_\rho\leqq\delta$ $(1\leqq i\leqq l)$ imply $\|\sum_{i=1}^{l}x_i/l\|_\rho\leqq\varepsilon$, that is,

$$(8)\qquad \rho(x_i/\delta)\leqq\delta\ (1\leqq i\leqq l)\quad\text{imply}\quad \rho\Big(\sum_{i=1}^{l}x_i/\varepsilon l\Big)\leqq\varepsilon.$$

Hence, if $\sum_{i=1}^{l}\rho(n_ix_i/\delta)/n_i\leqq\delta$, $x=\sum_{i=1}^{l}\oplus x_i$ and n_i $(1\leqq i\leqq l)$ are positive integers, then in view of the assumption that R is non-atomic, we can find an orthogonal decomposition of x_i such that

$$(9)\qquad \begin{cases} x_i=\sum_{\nu=1}^{n_i}\oplus x_{i,\nu} & (1\leqq i\leqq l)\\ \rho(n_ix_{i,\nu}/\delta)=\rho(n_ix_i/\delta)/n_i & (1\leqq\nu\leqq n_i,\ 1\leqq i\leqq l)^{23)}. \end{cases}$$

If we put $y_{\nu_1,\nu_2,\dots,\nu_l}=\sum_{i=1}^{l}\oplus n_ix_{i,\nu_i}$ $(1\leqq\nu_i\leqq n_i)$, then the total number of elements $y_{\nu_1,\nu_2,\dots,\nu_l}$ is $n_1n_2\cdots n_l$ and the sum of them equals to $n_1n_2\cdots n_lx$, because the multiplicity of $n_ix_{i,j}$ in the summation is $n_1n_2\cdots n_l/n_i$, we have

$$(10)\qquad \sum_{\substack{1\leqq\nu_i\leqq n_i\\ 1\leqq i\leqq l}}y_{\nu_1,\nu_2,\dots,\nu_l}=\sum_{i=1}^{l}\sum_{j=1}^{n_i}n_1n_2\cdots n_lx_{i,j}$$
$$=\sum_{l=1}^{l}n_1n_2\cdots n_lx_i=n_1n_2\cdots n_lx.$$

On the other hand

$$(11)\qquad \rho(y_{\nu_1,\nu_2,\dots,\nu_l}/\delta)=\sum_{i=1}^{l}\rho(n_ix_{i,\nu_i}/\delta)=\sum_{i=1}^{l}\rho(n_ix_i/\delta)/n_i\leqq\delta.$$

Therefore from (8), (10) and (11) we see

$$\rho\Big(\sum_{\substack{1\leqq\nu_i\leqq n_i\\ 1\leqq i\leqq l}}y_{\nu_1,\nu_2,\dots,\nu_l}/\varepsilon n_1n_2\cdots n_l\Big)=\rho(x/\varepsilon)\leqq\varepsilon.$$

Thus (*) has been proved.

And the following fact is a direct consequence of (*),

$$(12)\qquad \sup_{\alpha\geqq0}\rho(\alpha x)<+\infty\ \text{ if and only if }\ x=0.$$

Since $\sup_{\alpha\geqq0}\rho(\alpha x)=\gamma<+\infty$ implies $\inf_{n\geqq1}\rho(nx/\delta)/n=0$ for all $\delta>0$, by (*) $\rho(x/\varepsilon)\leqq\varepsilon$ for all $\varepsilon>0$, hence $\|x\|_\rho=0$, that is, $x=0$.

Putting, for $\delta_1=\delta(1)>0$,

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23) This is a method used oftenly in non-atomic cases. Confer [4] or [5].

(13) $$\bar{\rho}(x)=\inf_{\substack{x=\Sigma\oplus x_i\\ \eta_i\geqq 1}}\sum_i \rho(2\eta_i x_i/\delta_1)/\delta_1\eta_i \quad (x\in R),$$

the functional $\bar{\rho}(x)$ $(x\in R)$ has the following properties;

$(\bar{\rho})$

1) $\bar{\rho}(x)\leqq 1/\delta_1\cdot\rho(2x/\delta_1)$ for all $x\in R$,
2) $\bar{\rho}(x)\leqq 1$ implies $\rho(x)\leqq 1$,
3) $|x|\leqq|y|$ implies $\bar{\rho}(x)\leqq\bar{\rho}(y)$,
4) $x\frown y=0$ implies $\bar{\rho}(x+y)=\bar{\rho}(x)+\bar{\rho}(y)$,
5) $\bar{\rho}(tx)/t$ $(t>0)$ is a non-decreasing function of $t>0$ for all $x\in R$,
6) $[p_\lambda]\uparrow_{\lambda\in\Lambda}[x]$[24] and $\bar{\rho}(x)<+\infty$ imply $\sup_{\lambda\in\Lambda}\bar{\rho}([p_\lambda]x)=\bar{\rho}(x)$.

$(\bar{\rho})$–1) is obvious from the definition of $\bar{\rho}$. $(\bar{\rho})$–2) is a simple consequence of $(*)$; if $\rho(x)>1$, $x=\sum_{i=1}^{l}\oplus x_i$ and $\eta_i\geqq 1$ $(1\leqq i\leqq l)$, then we see $\sum_{i=1}^{l}\rho(2\eta_i x_i/\delta_1)/\eta_i \geqq\sum_{i=1}^{l}\rho(2n_i x_i/\delta_1)/2\delta_1 n_i$ where n_i $(1\leqq i\leqq l)$ are positive integers such as $n_i\leqq\eta_i<n_i+1$ $(1\leqq i\leqq l)$. $(\bar{\rho})$–3) and 4) are easily implied from (ρ)–3) and 4) respectively. Next we shall check $(\bar{\rho})$–5); for $t_2>t_1>0$ we have

$$\bar{\rho}(t_1x)/t_1=\inf_{\substack{x=\Sigma\oplus x_i\\ \eta_i\geqq 1}}\sum_i\rho(2t_1\eta_i x_i/\delta_1)/t_1\eta_i\delta_i=\inf_{\substack{x=\Sigma\oplus x_i\\ \xi_i\geqq t_1}}\sum_i\rho(2\xi_i x_i/\delta_1)/\xi_i\delta_i$$
$$\leqq\inf_{\substack{x=\Sigma\oplus x_i\\ \xi'_i\geqq t_2}}\sum_i\rho(2\xi'_i x_i/\delta_1)/\xi'_i\delta_1=\bar{\rho}(t_2x)/t_2\,.$$

$(\bar{\rho})$–6) is shown as follows; $\bar{\rho}(x)<+\infty$ if and only if $\rho(2x/\delta_1)<+\infty$. Hence

$$0\leqq\bar{\rho}(x)-\bar{\rho}([p_\lambda]x)=\bar{\rho}(x-[p_\lambda]x)\leqq\rho(2(x-[p_\lambda]x)/\delta_1)/\delta_1<+\infty,$$

and $\inf_{\lambda\in\Lambda}\rho(2(x-[p_\lambda]x)/\delta_1)=0$ is effected by (ρ)–5).

Next we put $\bar{\bar{\rho}}(x)$ $(x\in R)$ as follows;

(14) $$\bar{\bar{\rho}}(x)=\begin{cases}\sup\bar{\rho}([p_\lambda]x), & \text{if there exists } [p_\lambda]\uparrow_{\lambda\in\Lambda}[x] \text{ and}\\ & \bar{\rho}([p_\lambda]x)<+\infty \text{ for all } \lambda\in\Lambda,\\ +\infty, & \text{elsewhere.}\end{cases}$$

We see obviously $\bar{\bar{\rho}}(x)\leqq\bar{\rho}(x)$ $(x\in R)$ and $\bar{\bar{\rho}}(x)=\bar{\rho}(x)$, if $\bar{\rho}(x)<+\infty$. The functional $\bar{\bar{\rho}}(x)$ $(x\in R)$ has the same properties as $(\bar{\rho})$ and moreover has the stronger property than 6) of $(\bar{\rho})$:

(15) If $[p_\lambda]\uparrow_{\lambda\in\Lambda}[x]$, then $\sup\bar{\bar{\rho}}([p_\lambda]x)=\bar{\bar{\rho}}(x)$.

Now we can construct a convex modular $m(x)$ $(x\in R)$:

(16) $$m(x)=\int_0^1\bar{\bar{\rho}}(tx)/t\,dt \quad (x\in R)$$

24) $[p_\lambda]y\uparrow_{\lambda\in\Lambda}[x]y$ for all $y\geqq 0$.

Evidently we see

(17) $$\bar{\bar{\rho}}(x/2) \leqq m(x) \leqq \bar{\rho}(x) \quad (x \in R).$$

It is obvious also that this functional $m(x)$ on R satisfies (C) from the fact that $\bar{\bar{\rho}}(tx)/t$ is a non-decreasing function of $t>0$.

We shall check the modular condition (ρ) about $m(x)$ $(x \in R)$. (ρ)–1); $\sup_{\alpha \geqq 0} m(\alpha x)=0$ implies $\sup_{\alpha \geqq 0} \bar{\bar{\rho}}(\alpha x)=0$, hence from the definition of $\bar{\bar{\rho}}$ and $(\bar{\rho})$–2) we can see $\sup_{\alpha \geqq 0} \rho(\alpha x) \leqq 1$, therefore $x=0$ follows from (12). (ρ)–2) is evident from (17) and $(\bar{\rho})$–1): $0 \leqq m(x) \leqq \rho(2x/\delta_1)/\delta_1$. (ρ)–3) and 4) are almost evident. (ρ)–5); from (15) we see $\sup_{\lambda \in \Lambda} m([p_\lambda]x)=m(x)$ for $[p_\lambda] \uparrow_{\lambda \in \Lambda} [x]$, and since $m(\alpha x)=\int_0^\alpha \bar{\bar{\rho}}(tx)/t\, dt$, $m(\alpha x)$ is a left-continuous function of $\alpha \geqq 0$, therefore $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies $\sup_{\lambda \in \Lambda} m(x_\lambda)=m(x)$. (ρ)–6); for the orthogonal system $x_\lambda \geqq 0$ $(\lambda \in \Lambda)$ such as $\sum_{\lambda \in \Lambda} m(x_\lambda) < +\infty$ we see $\sum_{\lambda \in \Lambda} \bar{\bar{\rho}}(x_\lambda/2) \leqq \sum_{\lambda \in \Lambda} m(x_\lambda) < +\infty$, hence $\sum_{\lambda \in \Lambda'} \bar{\bar{\rho}}(x_\lambda/2) \leqq 1$ for some $\Lambda' \subset \Lambda$ such as $\Lambda - \Lambda'$ is a finite set. $(\bar{\rho})$–2) and 4) imply $\sum_{\lambda \in \Lambda'} \rho(x_\lambda/2) \leqq 1$, whence $\sum_{\lambda \in \Lambda'} x_\lambda$ exists by the orthogonal completeness of ρ.

c)→d). Let $m(x)$ $(x \in R)$ be a convex modular on $R(\rho)$, then we have an ordered norm $|||x|||$ on R by the formula ([5]):

(18) $$|||x|||=\inf\{\varepsilon>0\,;\, m(x/\varepsilon) \leqq 1\}.$$

We can see easily $\lim_{n \to \infty} |||x_n|||=0$ if and only if $\lim_{n \to \infty} m(\alpha x_n)=0$ for all $\alpha \geqq 0$. Hence $\lim_{n \to \infty} |||x_n|||=0 \rightleftarrows \lim_{n \to \infty} \|x_n\|_m=0$. Therefore the completeness of $|||x|||$ $(x \in R)$ follows from Theorem 1.

d)→a). Let $|||x|||$ $(x \in R)$ be a complete ordered norm on $R(\rho)$. It is sufficient to show the fact $\|x\|_\rho$ is equivalent to $|||x|||$. In general we can prove the following lemma.

Lemma 2. *If R is a σ-complete*[25] *vector lattice and $\|x\|_1$, $\|x\|_2$ $(x \in R)$ are two complete ordered quasi-norm on R, then they are equivalent each other.*

Proof. For any $\varepsilon>0$ we can find positive numbers δ_ε and γ_ε such that

(19) $$\|x\|_1 \leqq \delta_\varepsilon \text{ implies } \|\gamma_\varepsilon x\|_2 \leqq \varepsilon.$$

Because; in the contrary case, there exist $\varepsilon_0>0$ and $0 \leqq x_\nu \in R$ $(\nu=1, 2, \cdots)$ such as $\|x_\nu\|_1 \leqq 1/2^\nu$ and $\|x_\nu/\nu\|_2 \geqq \varepsilon_0$ $(\nu=1, 2, \cdots)$. Since

25) Every upper-bounded sequence of elements has a supremum in R.

$\sum_{\nu=1}^{\infty}\|x_\nu\|_1\leqq 1$, from the completeness of $\|x\|_1$ we can find $x_0\in R$ and $\lim_{n\to 0}\|x_0-\sum_{i=1}^{n}x_\nu\|_1=0$. And

$$0\leqq\|x_0\frown x_\nu-x_\nu\|_1=\left\|x_0\frown x_\nu-\left(\sum_{i=1}^{n}x_i\right)\frown x_\nu\right\|_1^{26)}\leqq\left\|x_0-\sum_{i=1}^{n}x_i\right\|_1\to 0\ (n\to\infty),$$

hence $x_0\frown x_\nu=x_\nu$, that is, $x_0\geqq x_\nu$ ($\nu=1, 2, \cdots$). Therefore we have a contradiction: $\|x_0/\nu\|_2\geqq\|x_\nu/\nu\|_2\geqq\varepsilon_0$ and $\lim_{\nu\to\infty}\|x_0/\nu\|_2\geqq\varepsilon_0>0$.

Thus, given $\lim_{n\to\infty}\|y_n\|_1=0$, for any $\varepsilon>0$ we have $\|y_n/\gamma_\varepsilon\|_1\leqq\delta_\varepsilon$ for almost all n, hence $\|\gamma_\varepsilon y_n/\gamma_\varepsilon\|_2=\|y_n\|_2\leqq\varepsilon$ for almost all n, that is, $\lim_{n\to\infty}\|y_n\|_2=0$. Q.E.D.

Remark. Under the assumption $\sup_{\alpha>0}\rho(\alpha x)=+\infty$ $(x\neq 0)$, the condition b) in the above Theorem 2 may be replaced with the following.

b′) for some $\varepsilon_0>0$ $\{x;\|x\|_\rho\leqq\varepsilon_0\}$ contains a convex neighbourhead of 0.

The application to function spaces. The detailed proof will be omited. Let $m(x)$ be a convex modular $L_{M(u,\omega)}$. By Radon-Nikodym's theorem we can find a convex (M)-function $N(u,\omega)$ and $m(x)$ dan be represented as follows

$$m(x)=\int_\Omega N[|x(\omega)|,\omega]d\mu \qquad (x(\omega)\in L_{M(u,\omega)}).$$

The orthogonal completeness of m implies $L_{N(u,\omega)}=L_{M(u,\omega)}$. Thus by Theorem 2 Mazur-Orlicz's result in §1 can be generalized;

Given $L_{M(u,\omega)}$-space on non-atomic measure space $\Omega(\boldsymbol{B},\mu)$, the necessary and sufficient condition to exist a convex (M)*-function $N(u,\omega)$ such as $L_{M(u,\omega)}=L_{N(u,\omega)}$ is that the linear topology induced by $\|x\|_M$ on $L_{M(u,\omega)}$ is locally convex.*

B. Atomic case. In an atomic modulared vector lattice $R(\rho)$ the above Theorem 2 does not hold in general. The so-called S-space is a counter example. Putting $\Omega=\{\omega_1,\omega_2,\cdots\}$, $\mu(\omega_n)=1$ and $M(u,\omega_n)=u/2^n(1+u)$ $(n=1,2,\cdots)$, then $L_{M(u,\omega)}$ is S-space on Ω. It is easily proved that $\|x\|_M$ on $S(\Omega)$ is locally convex, but not normable.

Now we shall consider the following assumption:

$$(**)\qquad R=\sum_{\nu=1}^{\infty}\oplus R_\nu{}^{27)} \text{ and } R(\rho)\cong R_\nu(\rho)^{28)}\ (\nu\geqq 1),$$

26) Since $|x\smile z-y\smile z|+|x\frown z-y\frown z|=|x-y|$ ([5]), we have $\|x\frown z-y\frown z\|_\rho\leqq\|x-y\|_\rho$.

27) Every R_ν is a normal subspace of R, that is, R_ν is a linear subspaceand if $R_\nu\ni x$ and $|x|\leqq|y|$ then $y\in R_\nu$ and $0\leqq x_\lambda\in R_\nu$ $(\lambda\in\Lambda)$, $x_\lambda\uparrow_{\lambda\in\Lambda}x$ imply $x\in R_\nu$ ([5]). $R=\sum_{\nu=1}^{\infty}\oplus R_\nu$ means that $|x|\frown|y|=0$ for all $x\in R_\nu$, $y\in R_\mu$ $(\nu\neq\mu)$ and every $x\geqq 0$ can be represented in $x=\sum_{\nu=1}^{\infty}x_\nu$ for some $0\leqq x_\nu\in R_\nu$ $(\nu\geqq 1)$.

28) There exists an isomorphism I_ν from R onto R_ν such as $\rho(x)=\rho(I_\nu x)$ for all $x\in R$.

where R_ν ($\nu=1, 2, \cdots$) are normal subspaces[27] and orthogonal each other. For instance, if $\Omega(2^\Omega, \mu)$ is an atomic measure space and for any $\omega_0 \in \Omega$ $\Omega_{\omega_0}=\{\omega ; M(u, \omega_0)\mu(\omega_0)=M(u, \omega)\mu(\omega)$ for all $u \geqq 0\}$ is an infinite set, then $L_{M(u,\omega)}$ satisfies $(**)$. As a special case, $L_{M(u)}$ on an atomic measure space $\Omega(2^\Omega, \mu)$, where $\mu(\omega)=1$ ($\omega \in \Omega$) and Ω is infinite, satisfies $(**)$.

Theorem 3. *In the modulared vector lattice $R(\rho)$ satisfying the assumption $(**)$, four conditions in Theorem 2 are equivalent each other. Moreover the condition* b) *can be replaced by the weaker condition :*

b′) *for some* $\varepsilon_0>0$ $\{x ; \|x\|_\rho \leqq \varepsilon_0\}$ *contains a convex neighbourhood of* 0.

Proof. It is sufficient to show b′)→c). It follows from b′) that there is $\delta_0>0$ such that

$$\rho(x_i/\delta_0) \leqq \delta_0 \quad (1 \leqq i \leqq l) \text{ imply } \rho\Big(\sum_{i=1}^{l} x_i/\varepsilon_0 l\Big) \leqq \varepsilon_0 . \tag{20}$$

Using the assumption $(**)$ we shall show

$$\rho(x/\delta_0) \leqq \delta_0 \text{ implies } \sum_{i=1}^{l} n_i \rho(x_i/\varepsilon_0 n_i) \leqq \varepsilon_0 , \tag{$\overset{**}{*}$}$$

where $\{x_i ; 1 \leqq i \leqq l\}$ is an arbitrary orthogonal decomposition of x, $x=\sum_{i=1}^{l} \oplus x_i$, and n_i ($1 \leqq i \leqq l$) are arbitrary positive integers.

Because ; from $(**)$ we can find $x_{i,\nu} \in R_\nu$ ($1 \leqq \nu \leqq n_i$, $1 \leqq i \leqq l$) such that

$$\begin{cases} x_{i,\nu} \frown x_{j,\mu}=0 \quad ((i, \nu) \neq (j, \mu)) \\ \rho(\alpha x_{i,\nu})=\rho(\alpha x_i) \text{ for all } \alpha \geqq 0. \quad (1 \leqq \nu \leqq n_i,\ 1 \leqq i \leqq l). \end{cases} \tag{21}$$

If we put $y_{\nu_1,\nu_2,\ldots,\nu_l}=\sum_{i=1}^{l} \oplus x_{i,\nu_i}$, then the total number of elements $y_{\nu_1,\nu_2,\ldots,\nu_l}$ is $n_1 n_2 \cdots n_l$ and we have

$$\sum_{\substack{1 \leqq \nu_i \leqq n_i \\ 1 \leqq i \leqq l}} y_{\nu_1,\nu_2,\ldots,\nu_l}/n_1 n_2 \cdots n_l=\sum_{i=1}^{l} \oplus \sum_{\nu=1}^{n_i} \oplus x_{i,\nu}/n_i . \tag{22}$$

On the other hand

$$\begin{aligned} \rho(y_{\nu_1,\nu_2,\ldots,\nu_l}/\delta_0) &= \sum_{i=1}^{l} \rho(x_{i,\nu_i}/\delta_0) \\ &= \sum_{i=1}^{l} \rho(x_i/\delta_0)=\rho(x/\delta_0) \leqq \delta_0 , \end{aligned} \tag{23}$$

therefore (20) and (22) imply

$$\begin{aligned} \sum_{i=1}^{l} n_i \rho(x_i/\varepsilon_0 n_i) &= \sum_{i=1}^{l} \sum_{\nu=1}^{n_i} \rho(x_{i,\nu}/\varepsilon_0 n_i) \\ &= \rho\Big(\sum_{\substack{1 \leqq \nu_i \leqq n_i \\ 1 \leqq i \leqq l}} y_{\nu_1,\nu_2,\ldots,\nu_l}/\varepsilon_0 n_1 n_2 \cdots n_l\Big) \leqq \varepsilon_0 . \end{aligned}$$

Next, we put

(24) $$\tilde{\rho}(x)=\sup_{\substack{x=\Sigma\oplus x_i \\ 0<\xi_i\leq 1}}\sum\rho(\xi_i x_i/\varepsilon_0)/\xi_i \quad (x\in R)\,.$$

The functional $\tilde{\rho}(x)$ $(x\in R)$ has the following properties;

$(\tilde{\rho})$

1) $\rho(x/\varepsilon_0)\leqq\tilde{\rho}(x)$ for all $x\in R$,
2) $\rho(x/\delta_0)\leqq\delta_0$ implies $\tilde{\rho}(x)\leqq 2\varepsilon_0$,
3) $|x|\leqq|y|$ implies $\tilde{\rho}(x)\leqq\tilde{\rho}(y)$,
4) $x\frown y=0$ implies $\tilde{\rho}(x+y)=\tilde{\rho}(x)+\tilde{\rho}(y)$,
5) $\tilde{\rho}(tx)/t$ $(t>0)$ is a non-decreasing function of $t>0$ for all $x\in R$.
6) $[p_\lambda]\uparrow_{\lambda\in\Lambda}[x]$ implies $\sup\tilde{\rho}([p_\lambda]x)=\tilde{\rho}(x)$.

$(\tilde{\rho})$–2) is a direct consequence of $(^*_*)$, and other properties are obvious from the definition of $\tilde{\rho}$.

Now we can construct a convex modular $m(x)$ on R:

(25) $$m(x)=\int_0^1\tilde{\rho}(tx)/t\,dt \quad (x\in R)\,.$$

Evidently we see

(26) $$\rho(x/2\varepsilon_0)\leqq\tilde{\rho}(x/2)\leqq m(x)\leqq\tilde{\rho}(x) \quad (x\in R)\,.$$

It is easy to check the convex modular condition: (ρ) and (C). (C) follows from $(\tilde{\rho})$–5). (ρ)–1) and 6) are implied by (26). (ρ)–3), 4) and 5) are almost obvious. (ρ)–2): It is sufficient to see $m(\alpha x)<+\infty$ for some $\alpha=\alpha(x)>0$. For x we can find $\alpha>0$ such as $\rho(\alpha x/\delta_0)\leqq\delta_0$, hence from $(\tilde{\rho})$–2) and (26) we have $m(\alpha x)\leqq\tilde{\rho}(\alpha x)\leqq 2\varepsilon_0<+\infty$. Q.E.D.

Finally we remark that in an atomic modulared vector lattice $R(\rho)$ it can be proved that $\|x\|_\rho$ is normable if and only if there is a convex modular on $R(\rho)$ (cf. [8]). It will be studied in another paper.

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