# SOME REMARKS ON CARTAN-BRAUER-HUA THEOREM[1)]

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Concerning Cartan-Brauer-Hua theorem, T. Nagahara and H. Tominaga proved the following [3, Lemma 3.5]:

*Let $U$ be a ring with 1, and $B$ a two-sided simple subring of $U$ containing 1. If $A$ is a division subring of $U$ containing 1 such that $B$ is invariant relative to all inner automorphisms determined by non-zero elements of $A$, then either $A\subseteq B$ or $A\subseteq V_U(B)$.*

In what follows, by making use of the same method as in the proof of this fact, we shall present a slight generalization of [3, Lemma 3.5] (Theorem 1) and an extension of [2, Theorem 7.13.1 (2)] (Theorem 2). And finally, we shall prove that Theorem 2 is still valid for inner automorphisms provided $A$ is a simple ring (Theorem 3). Throughout the present note, a ring will mean a ring with the identity element 1, and a subring one with this identity element.

Our first theorem containing [3, Lemma 3.5] can be stated as follows:

**Theorem 1.** *Let $U$ be a ring, and $A$ and $B$ a subring of $U$ satisfying minimum condition for right ideals and a two-sided simple subring of $U$ respectively. If for each $a\in A$ and $b\in B$ there exists an element $b_1\in B$ such that $ab=b_1a$, then either $A\subseteq B$ or $A\subseteq V_U(B)$.*

*Proof.* To be easily seen from the proof of [3, Lemma 3.5], it suffices to prove that $A=(A\frown B)\smile V_A(B)$. Let $a$ be an arbitrary element of A. If $a$ and 1 are linearly left independent over $B$, then for each $b\in B$, $ab=b_1a$ and $(a+1)b=b_2(a+1)$ yield $(b_1-b_2)a+(b-b_2)=0$, whence it follows $b_1=b_2=b$. Consequently, we obtain $a\in V_A(B)$. If, on the other hand, $a$ and 1 are linearly dependent, then there holds $d_1a=d_2$ for some non-zero $d_1\in B$. In case $d_2\neq0$, since $B$ is two-sided simple, we obtain $da=1$ for some $d\in B$. And so, recalling that $A$ satisfies minimum condition for right ideals, one will readily see that $a$ is a regular element of $A$. And then, $aB=Ba=B$ will yield at once $a\in B$. In case $d_2=0$ too, since $d_1(a+1)=d_1\neq0$, we obtain $a+1\in B$. Thus, in either case, $a$ is contained in $B$. We have proved tnerefore $A=(A\frown B)\smile V_A(B)$.

Combining our method with the one employed in the proof of [2,

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Theorem 7.13.1 (2)], we can obtain the following:

**Theorem 2.**[2] *Let $U$ be a ring and $B$ a two-sided simple subring of $U$. If $B$ is not of characteristic 2 and $A$ is a subring of $U$ such that $B$ is invariant relative to all inner derivations determined by elements of $A$, then either $A \subseteq B$ or $A \subseteq V_U(B)$.*

*Proof.* Let $a$ be an arbitrary element of $A$. For any element $b \in B$, we set $[b, a]=ba-ab=b_1$, $[[b, a]a]=b_2$, $[b, a^2]=b_3$ where $b_1, b_2$ and $b_3$ are in $B$. Then, one will easily see that $2b_1a=2(ba^2-aba)=b_2+b_3 \in B$. And, if $a$ and 1 are linearly left independent over $B$, we obtain $b_1=0$. This means obviously that $a \in V_A(B)$. On the other hand, if $a$ and 1 are linearly dependent: $b^*a-b^{**}=0$ with non-zero $b^* \in B$, then noting that $B$ is two-sided simple, it will be easy to see that $a \in A \frown B$. We have proved therefore $A=(A \frown B) \smile V_A(B)$. Now, the rest of the proof is the same with the latter half of the proof of [3, Lemma 3.5].

Finally, we shall present the following:

**Theorem 3.** *Let $U$ be a ring and $B$ a two-sided simple subring of $U$. If $B$ is not of characteristic 2, and $A$ a simple subring of $U$ such that $B$ is invariant relative to all inner automorphisms determined by regular elements of $A$, then either $A \subseteq B$ or $A \subseteq V_U(B)$.*

*Proof.* Let $K$ be the prime field of $A$ (which is evidently contained in the center of $B$), and let $a$ be an arbitrary $\alpha$-biregular element[3] of $A$ $(0 \neq \alpha \in K)$. If $a$ and 1 are linearly left independent over $B$, then for an arbitrary $b \in B$, $ab=b^*a$ and $(a-\alpha)b=b^{**}(a-\alpha)$ yield at once $(b^*-b^{**})a+\alpha(b^{**}-b)=0$, whence it follows $b^*=b^{**}=b$. Hence we obtain $a \in V_A(B)$. On the other hand, if $a$ and 1 are linearly dependent, then it will be easy to see that $a \in B$. Since each element of $A$ is a sum of biregular elements by [1], the fact proved above will show that $B$ is invariant relative to all inner derivations determined by elements of $A$. Hence, our assertion is a direct consequence of Theorem 2.

## References

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[2] N. Jacobson: Structure of rings, Providence (1956).

[3] T. Nagahara and H. Tominaga: On Galois and locally Galois extensions of simple rings, Math. J. Okayama Univ., 10 (1961) to appear.

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3) Cf. [1].