ON F-NORMS OF QUASI-MODULAR SPACES

By

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§1. Introduction. Let R be a *universally continuous semi-ordered linear space* (*i.e. a conditionally complete vector lattice* in Birkhoff's sense [1]) and ρ be a functional which satisfies the following four conditions:

(ρ.1) $0 \leqq \rho(x) = \rho(-x) \leqq +\infty$ for all $x \in R$;

(ρ.2) $\rho(x+y) = \rho(x) + \rho(y)$ for any $x, y \in R$ with $x \perp y$[1];

(ρ.3) If $\sum_{\lambda \in \Lambda} \rho(x_\lambda) < +\infty$ for a mutually orthogonal system $\{x_\lambda\}_{\lambda \in \Lambda}$[2], there exists $x_0 \in R$ such that $x_0 = \sum_{\lambda \in \Lambda} x$ and $\rho(x_0) = \sum_{\lambda \in \Lambda} \rho(x_\lambda)$;

(ρ.4) $\overline{\lim_{\xi \to 0}} \rho(\xi x) < +\infty$ for all $x \in R$.

Then, ρ is called a *quasi-modular* and R is called a *quasi-modular space.*

In the previous paper [2], we have defined a quasi-modular space and proved that if R is a non-atomic quasi-modular space which is semi-regular, then we can define a modular[3] m on R for which every universally continuous linear functional[4] is continuous with respect to the norm defined by the modular[5] m [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\boldsymbol{\rho}$ on a linear space L which satisfies the following conditions:

(A.1) $\boldsymbol{\rho}(x) \geqq 0$ and $\boldsymbol{\rho}(x) = 0$ if and only if $x = 0$;

(A.2) $\boldsymbol{\rho}(-x) = \boldsymbol{\rho}(x)$;

(A.3) $\boldsymbol{\rho}(\alpha x + \beta y) \leqq \boldsymbol{\rho}(x) + \boldsymbol{\rho}(y)$ for every $\alpha, \beta \geqq 0$ with $\alpha + \beta = 1$;

(A.4) $\alpha_n \to 0$ implies $\boldsymbol{\rho}(\alpha_n x) \to 0$ for every $x \in R$;

(A.5) for any $x \in L$ there exists $\alpha > 0$ such that $\boldsymbol{\rho}(\alpha x) < +\infty$.

They showed that L is a quasi-normed space with a quasi-norm $\|\cdot\|_0$ defined by the formula;

1) $x \perp y$ means $|x| \frown |y| = 0$.

2) A system of elements $\{x_\lambda\}_{\lambda \in \Lambda}$ is called *mutually orthogonal,* if $x_\lambda \perp x_\gamma$ for $\lambda \neq \gamma$.

3) For the definition of a modular, see [3].

4) A linear functional f is called *universally continuous,* if $\inf_{\lambda \in \Lambda} f(a_\lambda) = 0$ for any $a_\lambda \downarrow_{\lambda \in \Lambda} 0$. R is called *semi-regular,* if for any $x \neq 0$, $x \in R$, there exists a universally continuous linear functional f such that $f(x) \neq 0$.

5) This modular ρ is a generalization of a modular m in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geqq 0$ for each $x \in R$.

(1.1) $$\|x\|_0=\inf\left\{\xi\,;\,\rho\left(\frac{1}{\xi}x\right)\leqq\xi\right\}$$ [6]

and $\|x_n\|_0\to 0$ is equivalent to $\rho(\alpha x_n)\to 0$ for all $\alpha\geqq 0$.

In the present paper, we shall deal with a general quasi-modular space R (i. e. without the assumption that R is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on R and to investigate the condition under which R is an F-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular ρ on R does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: $(\rho.1)\sim(\rho.4)$ with those of ρ [6], we can not apply the formula (1.1) directly to ρ to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular ρ^* which satisfies (A.2)$\sim$(A.5) on an arbitrary quasi-modular space R in §2 (Theorems 2.1 and 2.2). Since R may include a normal manifold $R_0=\{x: x\in R,\ \rho^*(\xi x)=0$ for all $\xi\geqq 0\}$ and we can not define a quasi-norm on R_0 in general, we have to exclude R_0 in order to proceed with the argument further. We shall prove in §3 that a quasi-norm $\|\cdot\|_0$ on $R_0^\perp$ defined by ρ^* according to the formula (1.1) is semi-continuous, and in order that $R_0^\perp$ is an F-space with $\|\cdot\|_0$ (i. e. $\|\cdot\|_0$ is complete), it is necessary and sufficient that ρ satisfies

$(\rho.4')$ $$\sup_{x\in R}\{\overline{\lim_{\alpha\to 0}}\ \rho(\alpha x)\}<+\infty$$

(Theorem 3.2).

In §4, we shall show that we can define another quasi-norm $\|\cdot\|_1$ on $R_0^\perp$ which is equivalent to $\|\cdot\|_0$ such that $\|x\|_0\leqq\|x\|_1\leq 2\|x\|_0$ holds for every $x\in R_0^\perp$ (Formulas (4.1) and (4.3)). $\|\cdot\|_1$ has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4 ; §83]. At last in §5 we shall add shortly the supplementary results concerning the relations between $\|\cdot\|_0$-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases [3].

Throughout this paper R denotes a *universially continuous semi-ordered linear space* and ρ a *quasi-modular* defined on R. For any $p\in R$, $[p]$ is a *projector*: $[p]x=\bigcup_{n=1}^{\infty}(n|p|\cap x)$ for all $x\geqq 0$ and $1-[p]$ is a *projection operator* onto the normal manifold $N=\{p\}^\perp$, that is, $x=[p]x+(1-[p])x$.

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].

§2. **The conversion of a quasi-modular.** From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. *For any quasi-modular* ρ, *we have*

(2.1) $\rho(0)=0$;

(2.2) $\rho([p]x)\leqq\rho(x)$ *for all* $p, x\in R$;

(2.3) $\rho([p]x)=\sup_{\lambda\in\Lambda}\rho([p_\lambda]x)$ *for any* $[p_\lambda]\uparrow_{\lambda\in\Lambda}[p]$.

In the argument below, we have to use the additional property of ρ:

(ρ.5) $\rho(x)\leqq\rho(y)$ if $|x|\leqq|y|$, $x, y\in R$,

which is not valid for an abitrary ρ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular ρ satisfies (ρ.5).

Tneorem 2.1. *Let* R *be a quasi-modular space with quasi-modular* ρ. *Then there exists a quasi-modular* ρ' *for which* (ρ.5) *is valid.*

Proof. We put for every $x\in R$,

$$\rho'(x)=\sup_{0\leqq|y|\leqq|x|}\rho(y). \tag{2.4}$$

It is clear that ρ' satisfies the conditions (ρ.1), (ρ.2) and (ρ.5).

Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be an orthogonal system such that $\sum_{\lambda\in\Lambda}\rho'(x_\lambda)<+\infty$, then

$$\sum_{\lambda\in\Lambda}\rho(x_\lambda)<+\infty,$$

because

$$\rho(x)\leqq\rho'(x) \qquad \text{for all } x\in R.$$

We have

$$x_0=\sum_{\lambda\in\Lambda}x_\lambda\in R$$

and

$$\rho(x_0)=\sum_{\lambda\in\Lambda}\rho(x_\lambda) \qquad \text{in virtue of } (\rho.3).$$

For such x_0,

$$\rho'(x_0)=\sup_{0\leqq|y|\leqq|x_0|}\rho(y)=\sup_{0\leqq|y|\leqq|x_0|}\sum_{\lambda\in\Lambda}\rho([x_\lambda]y)$$
$$=\sum_{\lambda\in\Lambda}\sup_{0\leqq|y|\leqq|x_0|}\rho([x_\lambda]y)=\sum_{\lambda\in\Lambda}\rho'(x_\lambda)$$

holds, i. e. ρ' fulfils (ρ.3).

If ρ' does not fulfil (ρ.4), we have for some $x_0\in R$,

$$\rho'\left(\frac{1}{n}x_0\right)=+\infty \qquad \text{for all } n\geqq1.$$

By (ρ.2) and (ρ.4), x_0 can not be written as $x_0=\sum_{\nu=1}^{\kappa}\xi_\nu e_\nu$, where e_ν is an atomic element for each ν with $1\leqq\nu\leqq\kappa$, namely, we can decompose x_0 into an infinite number of orthogonal elements. First we decompose into

$$x_0=x_1+x_1', \quad x_1\perp x_1',$$

where $\rho'\left(\frac{1}{\nu}x_1\right)=+\infty$ $(\nu=1, 2, \cdots)$ and $\rho'(x_1')>1$. For the definition of ρ', there exists $0\leqq y_1\leqq |x_1'|$ such that $\rho(y_1)\geqq 1$. Next we can also decompose x_1 into

$$x_1=x_2+x_2', \quad x_2\perp x_2',$$

where

$$\rho'\left(\frac{1}{\nu}x_2\right)=+\infty \quad (\nu=1, 2, \cdots)$$

and

$$\rho'\left(\frac{1}{2}x_2'\right)>2.$$

There exists also $0\leqq y_2\leqq |x_2'|$ such that $\rho\left(\frac{1}{2}y_2\right)\geqq 2$. In the same way, we can find by induction an orthogonal sequence $\{y_\nu\}_{\nu=1,2,\dots}$ such that

$$\rho\left(\frac{1}{\nu}y_\nu\right)\geqq\nu$$

and

$$0\leqq|y_\nu|\leqq|x|$$

for all $\nu\geqq 1$.

Since $\{y_\nu\}_{\nu=1,2,\dots}$ is order-bounded, we have in virtue of (2.3)

$$y_0=\sum_{\nu=1}^{\infty} y_\nu \in R$$

and

$$\rho\left(\frac{1}{\nu}y_0\right)\geqq\rho\left(\frac{1}{\nu}y_\nu\right)\geqq\nu,$$

which contradicts (ρ.4). Therefore ρ' has to satisfy (ρ.4). Q.E.D.

Hence, in the sequel, we denote by ρ' a quasi-modular defined by the formula (2.4).

If ρ satisfies (ρ.5), ρ does also (A.3) in §1:

$$\rho(\alpha x+\beta y)\leqq\rho(x)+\rho(y)$$

for $\alpha, \beta\geqq 0$ with $\alpha+\beta=1$.

Because, putting $[p]=[(|x|-|y|)^+]$, we obtain

$$\begin{aligned}
&\rho(\alpha x+\beta y)\leqq\rho(\alpha|x|+\beta|y|)\\
&\leqq\rho(\alpha[p]|x|+\alpha(1-[p])|y|+\beta[p]|x|+(1-[p])\beta|y|)\\
&=\rho([p]|x|+(1-[p])|y|)\\
&=\rho([p]x)+\rho((1-[p])y)\\
&\leqq\rho(x)+\rho(y)\,.
\end{aligned}$$

Remark 1. As is shown above, the existence of ρ' as a quasi-modular depends essentially on the condition (ρ.4). Thus, in the above theorems, we cannot replace (ρ.4) by the weaker condition:

(ρ.4″) for any $x\in R$, there exists $\alpha\geqq 0$ such that $\rho(\alpha x)<+\infty$.

In fact, the next example shows that there exists a functional ρ_0 on a universally continuous semi-ordered linear space satisfying (ρ.1), (ρ.2), (ρ.3) and (ρ.4″), but does not (ρ.4). For this ρ_0, we obtain

$$\rho_0'(x)=\sup_{|y|\leqq|x|}\rho_0(y)=+\infty$$

for all $x\neq 0$.

Example. $L_1[0,1]$ is the set of measurable functions $x(t)$ which are defined in $[0,1]$ with

$$\int_0^1|x(t)|\,dt<+\infty\,.$$

Putting

$$\rho_0(x)=\rho_0(x(t))=\int_0^1|x(t)|\,dt+\sum_{i=1}^{\infty}i\text{ mes}\left\{t:x(t)=\frac{1}{i}\right\},$$

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: (A.4), namely,

$$(\rho.6)\qquad \lim_{\xi\to 0}\rho(\xi x)=0 \qquad \text{for all } x\in R.$$

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

Theorem 2.2. *Let ρ be a quasi-modular on R. We can find a functional ρ^* which satisfies (ρ.1)$\sim$(ρ.6) except (ρ.3).*

Proof. In virtue of Theorem 2.1, there exists a quasi-modular ρ' which satisfies (ρ.5). Now we put

$$(2.5)\qquad d(x)=\lim_{\xi\to 0}\rho'(\xi x)\,.$$

It is clear that $0\leqq d(x)=d(|x|)<+\infty$ for all $x\in R$ and

$$d(x+y)=d(x)+d(y) \qquad \text{if } x \perp y.$$

Hence, putting

$$\rho^*(x)=\rho'(x)-d(x) \qquad (x \in R). \tag{2.6}$$

we can see easily that (ρ.1), (ρ.2), (ρ.4) and (ρ.6) hold true for ρ^*, since

$$d(x) \leqq \rho'(x)$$

and

$$d(\alpha x)=d(x)$$

for all $x \in R$ and $\alpha>0$.
We need to prove that (ρ.5) is true for ρ^*. First we have to note

$$\inf_{\lambda \in \Lambda} d([p_\lambda]x)=0 \tag{2.7}$$

for any $[p_\lambda] \downarrow_{\lambda \in \Lambda} 0$. In fact, if we suppose the contrary, we have

$$\inf_{\lambda \in \Lambda} d([p_\lambda]x_0) \geqq \alpha>0$$

for some $[p_\lambda] \downarrow_{\lambda \in \Lambda} 0$ and $x_0 \in R$.

Hence,

$$\rho'\left(\frac{1}{\nu}[p_\lambda]x_0\right) \geqq d([p_\lambda]x_0) \geqq \alpha$$

for all $\nu \geqq 1$ and $\lambda \in \Lambda$. Thus we can find a subsequence $\{\lambda_n\}_{n \geqq 1}$ of $\{\lambda\}_{\lambda \in \Lambda}$ such that

$$[p_{\lambda_n}] \geqq [p_{\lambda_{n+1}}]$$

and

$$\rho'\left(\frac{1}{n}([p_{\lambda_n}]-[p_{\lambda_{n+1}}])x_0\right) \geqq \frac{\alpha}{2}$$

for all $n \geqq 1$ in virtue of (ρ.2) and (2.3). This implies

$$\rho'\left(\frac{1}{n}x_0\right) \geqq \sum_{m \geqq n} \rho'\left(\frac{1}{m}([p_{\lambda_m}]-[p_{\lambda_{m+1}}])x_0\right)=+\infty,$$

which is inconsistent with (ρ.4). Secondly we shall prove

$$d(x)=d(y), \qquad \text{if } [x]=[y]. \tag{2.8}$$

We put $[p_n]=[(|x|-n|y|)^+]$ for $x, y \in R$ with $[x]=[y]$ and $n \geqq 1$. Then, $[p_n] \downarrow_{n=1}^{\infty} 0$ and $\inf_{n=1,2,\cdots} d([p_n]x)=0$ by (2.7). Since $(1-[p_n])n|y| \geqq (1-[p_n])|x|$ and

$$d(\alpha x)=d(x)$$

for $\alpha>0$ and $x \in R$, we obtain

$$d(x)=d([p_n]x)+d((1-[p_n])x)$$
$$\leqq d([p_n]x)+d(n(1-[p_n])y)$$
$$\leqq d([p_n]x)+d(y).$$

As n is arbitrary, this implies

$$d(x)\leqq \inf_{n=1,2,\cdots} d([p_n]x)+d(y),$$

and also $d(x)\leqq d(y)$. Therefore we conclude that (2.8) holds.

If $|x|\geqq|y|$, then

$$\rho^*(x)=\rho^*([y]x)+\rho^*(([x]-[y])x)$$
$$=\rho'([y]x)-d([y]x)+\rho^*(([x]-[y])x)$$
$$\geqq\rho'(y)-d(y)+\rho^*(([x]-[y])x)$$
$$\geqq\rho^*(y).$$

Thus ρ^* satisfies (ρ.5). Q.E.D.

Theorem 2.3. *ρ^* (which is constructed from ρ according to the formulas (2.4), (2.5) and (2.6)) satisfies (ρ.3) (that is, ρ^* is a quasi-modular), if and only if ρ satisfies*

(ρ.4′) $$\sup_{x\in R}\{\overline{\lim_{\xi\to 0}}\,\rho(\xi x)\}=K<+\infty .$$

Proof. Let ρ satisfy (ρ.4). We need to prove

(2.9) $$\sup_{x\in R} d(x)=\sup_{x\in R}\{\lim_{\xi\to 0}\rho'(\xi x)\}=K'<+\infty ,$$

where

$$\rho'(x)=\sup_{0\leqq|y|\leqq|x|}\rho(y).$$

Since ρ' is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $n_0(x)=\rho(x)$ and $n_\nu(x)=\rho'\left(\frac{1}{\nu}x\right)$ for $\nu\geqq 1$ and $x\in R$. Hence we can find positive numbers ε, γ, a natural number ν_0 and a finite dimensional normal manifold N_0 such that $x\in N_0^\perp$ with

$$\rho(x)\leqq\varepsilon \quad \text{implies} \quad \rho'\left(\frac{1}{\nu_0}x\right)\leqq\gamma .$$

In N_0, we have obviously

$$\sup_{x\in N_0}\{\lim \rho'(\xi x)\}=\gamma_0<+\infty .$$

If $\varepsilon\leqq 2K$, for any $x_0\in N_0^\perp$, we can find $\alpha_0>0$ such that $\rho(\alpha x_0)\leqq 2K$ for all $0\leqq\alpha\leqq\alpha_0$ by (ρ.4′), and hence there exists always an orthogonal decomposition such that

$$\alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z$$

where $\frac{\varepsilon}{2} < \rho(x_i) \leqq \varepsilon$ $(i=1, 2, \cdots, n)$, y_j is an atomic element with $\rho(y_j) > \varepsilon$ for every $j=1, 2, \cdots, m$ and $\rho(z) \leqq \frac{\varepsilon}{2}$. From above, we get $n \leqq \frac{4K}{\varepsilon}$ and $m \leqq \frac{2K}{\varepsilon}$. This yields

$$\begin{aligned}\rho'\left(\frac{1}{\nu_0}\alpha_0 x_0\right) &\leqq \sum_{i=1}^{n} \rho'\left(\frac{1}{\nu_0}x_i\right) + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0} \\ &\leqq n\gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0} \\ &\leqq \frac{4K}{\varepsilon}\gamma + \frac{2K}{\varepsilon}\left\{\sup_{0 \leqq \alpha \leqq \alpha_0} \rho(\alpha x)\right\} + \gamma .\end{aligned}$$

Hence, we obtain

$$\lim_{\xi \to 0} \rho'(\xi x_0) \leqq \rho'\left(\frac{\alpha_0}{\nu_0} x_0\right) \leqq \left(\frac{4K+\varepsilon}{\varepsilon}\right)\gamma + \left(\frac{4K^2}{\varepsilon}\right)$$

in case of $\varepsilon \leqq 2K$. If $2K \leqq \varepsilon$, we have immediately for $x \in N_0^{\perp}$

$$\lim_{\xi \to 0} \rho'(\xi x) \leqq \gamma .$$

Therefore, we obtain

$$\sup_{x \in R} \{\lim_{\xi \to 0} \rho'(\xi x)\} \leqq \gamma'$$

where

$$\gamma' = \frac{4K+\varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0 .$$

Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be an orthogonal system with $\sum_{\lambda \in \Lambda} \rho^*(x_\lambda) < +\infty$. Then for arbitrary $\lambda_1, \cdots, \lambda_k \in \Lambda$, we have

$$\sum_{\nu=1}^{k} d(x_{\lambda_\nu}) = d(\sum_{\nu=1}^{k} x_{\lambda_\nu}) = \lim_{\xi \to 0} \rho'(\xi \sum_{\nu=1}^{k} x_{\lambda_\nu}) \leqq \gamma' ,$$

which implies $\sum_{\lambda \in \Lambda} d(x_\lambda) \leqq \gamma'$. It follows that

$$\sum_{\lambda \in \Lambda} \rho'(x_\lambda) = \sum_{\lambda \in \Lambda} \rho^*(x_\lambda) + \sum_{\lambda \in \Lambda} d(x_\lambda) < +\infty ,$$

which implies $x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in R$ and $\sum_{\lambda \in \Lambda} \rho^*(x_\lambda) = \rho^*(x_0)$ by $(\rho.4)$ and (2.7). Therefore ρ^* satisfies $(\rho.3)$.

On the other hand, suppose that ρ^* satisfies $(\rho.3)$ and $\sup_{x \in R} d(x) = +\infty$. Then we can find an orthogonal sequence $\{x_\nu\}_{\nu \geqq 1}$ such that

$$\sum_{\nu=1}^{\mu} d(x_\nu) = d(\sum_{\nu=1}^{\mu} x_\nu) \geqq \mu$$

for all $\mu \geqq 1$ in virtue of (2.8) and the orthogonal additivity of d. Since $\lim_{\xi\to 0} \rho^*(\xi x)=0$, there exists $\{\alpha_\nu\}_{\nu\geqq 1}$ with $0<\alpha_\nu$ $(\nu\geqq 1)$ and $\sum_{\nu=1}^{\infty}\rho^*(\alpha_\nu x_\nu)<+\infty$. It follows that $x_0=\sum_{\nu=1}^{\infty}\alpha_\nu x_\nu \in R$ and $d(x_0)=\sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu)$ from (ρ.3). For such x_0, we have for every $\xi \geqq 0$,

$$\rho'(\xi x_0)=\sum_{\nu=1}^{\infty}\rho'(\xi\alpha_\nu x_\nu)\geqq \sum_{\nu=1}^{\infty} d(x_\nu)=+\infty ,$$

which is inconsistent with (ρ.4). Therefore we have

$$\sup_{x\in R}(\lim_{\xi\to 0}\rho(\xi x))\leqq \sup_{x\in R} d(x)<+\infty . \qquad \text{Q.E.D.}$$

§3. **Quasi-norms.** We denote by R_0 the set:

$$R_0=\{x : x\in R,\ \rho^*(nx)=0 \text{ for all } n\geqq 1\},$$

where ρ^* is defined by the formula (2.6). Evidently R_0 is a semi-normal manifold[7] of R. We shall prove that R_0 is a normal manifold of R. In fact, let $x=\bigcup_{\lambda\in\Lambda} x_\lambda$ with $R_0\ni x_\lambda \geqq 0$ for all $\lambda\in\Lambda$. Putting

$$[p_{n,\lambda}]=[(2nx_\lambda - nx)^+],$$

we have

$$[p_{n,\lambda}]\uparrow_{\lambda\in\Lambda}[x] \quad \text{and} \quad 2n[p_{n,\lambda}]x_\lambda \geqq [p_{n,\lambda}]nx ,$$

which implies $\rho^*(n[p_{n,\lambda}]x)=0$ and $\sup_{\lambda\in\Lambda}\rho^*(n[p_{n,\lambda}]x)=\rho^*(nx)=0$. Hence, we obtain $x\in R_0$, that is, R_0 is a normal manifold of R.

Therefore, R is orthogonally decomposed into

$$R=R_0 \oplus R_0^{\perp} .^{8)}$$

In virtue of the definition of ρ^*, we infer that for any $p\in R_0$, $[p]R_0$ is *universally complete*, i.e. for any orthogonal system $\{x_\lambda\}_{\lambda\in\Lambda}(x_\lambda\in[p]R_0)$, there exists $x_0=\sum_{\lambda\in\Lambda} x_\lambda \in [p]R$. Hence we can also verify without difficulty that R_0 has no universally continuous linear functional except 0, if R_0 is non-atomic. When R_0 is discrete, it is isomorphic to $S(\Lambda)$[9]-space. With respect to such a universally complete space R_0, we can not always construct a linear metric topology on R_0, even if R_0 is discrete.

In the following, therefore, we must exclude R_0 from our consideration. Now we can state the theorems which we aim at.

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7) A linear manifold S is said to be *semi-normal*, if $a\in S$, $|b|\leqq|a|$, $b\in R$ implies $b\in S$. Since R is univerfally continuous, a semi-normal manifold S is normal if and only if $\bigcup_{\lambda\in\Lambda} x_\lambda \in R$, $0\leqq x_\lambda\in S\,(\lambda\in\Lambda)$ implies $\bigcup_{\lambda\in\Lambda} x_\lambda\in S$.

8) This means that $x\in R$ is written by $x=y+z$, $y\in R_0$ and $z\in R_0^{\perp}$.

9) $S(\Lambda)$ is the set of all real functions defined on Λ.

Theorem 3.1. *Let R be a quasi-modular space. Then $R_0^\perp$ becomes a quasi-normed space with a quasi-norm $\|\cdot\|_0$ which is semi-continuous, i.e.*

$$\sup_{\lambda\in\Lambda}\|x_\lambda\|_0=\|x\|_0 \qquad \textit{for any } 0\leqq x_\lambda\uparrow_{\lambda\in\Lambda}x\,.$$

Proof. In virtue of Theorems 2.1 and 2.2, ρ^* satisfies $(\rho.1)\sim(\rho.6)$ except $(\rho.3)$. Now we put

$$\|x\|_0=\inf\left\{\xi\,;\,\rho^*\left(\frac{1}{\xi}x\right)\leqq\xi\right\}\,. \tag{3.1}$$

Then,

i) $0\leqq\|x\|_0=\|-x\|_0<\infty$ and $\|x\|_0=0$ is equivalent to $x=0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R_0^\perp$.

ii) $\|x+y\|_0\leqq\|x\|_0+\|y\|_0$ for any $x, y\in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{\alpha_n\to 0}\|\alpha_n x\|_0=0$ and $\lim_{\|x_n\|_0\to 0}\|\alpha x_n\|_0=0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $\|\cdot\|_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha\to\alpha_0}\|\alpha x\|_0=\|\alpha_0 x\|_0$ for all $x\in R_0^\perp$ and $\alpha_0\geqq 0$. If $x\in R_0^\perp$ and $[p_\lambda]\uparrow_{\lambda\in\Lambda}[p]$, for any positive number ξ with $\|[p]x\|_0>\xi$ we have $\rho^*\left(\frac{1}{\xi}[p]x\right)>\xi$, which implies $\sup_{\lambda\in\Lambda}\rho^*\left(\frac{1}{\xi}[p_\lambda]x\right)>\xi$ and hence $\sup_{\lambda\in\Lambda}\|p_\lambda]x\|_0\geqq\xi$. Thus we obtain

$$\sup_{\lambda\in\Lambda}\|[p_\lambda]x\|_0=\|[p]x\|_0\,,\ \text{if}\ [p_\lambda]\uparrow_{\lambda\in\Lambda}[p]\,.$$

Let $0\leqq x_\lambda\uparrow_{\lambda\in\Lambda}x$. Putting

$$[p_{n,\lambda}]=\left[\left(x_\lambda-\left(1-\frac{1}{n}\right)x\right)^+\right]$$

we have

$$[p_{n,\lambda}]\uparrow_{\lambda\in\Lambda}[x]\ \text{and}\ [p_{n,\lambda}]x_\lambda\geqq[p_{n,\lambda}]\left(1-\frac{1}{n}\right)x \qquad (n\geqq 1)\,.$$

As is shown above, since

$$\sup_{\lambda\in\Lambda}\|[p_{n,\lambda}]x_\lambda\|_0\geqq\sup_{\lambda\in\Lambda}\left\|[p_{n,\lambda}]\left(1-\frac{1}{n}\right)x\right\|_0=\left\|\left(1-\frac{1}{n}\right)x\right\|_0,$$

we have

$$\sup_{\lambda\in\Lambda}\|x_\lambda\|_0\geqq\left\|\left(1-\frac{1}{n}\right)x\right\|_0$$

and also $\sup_{\lambda\in\Lambda}\|x_\lambda\|_0\geqq\|x\|_0$. As the converse inequality is obvious by iv), $\|\cdot\|_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim_{n\to\infty}\|x_n\|_0=0$ if and only if $\lim_{n\to\infty}\rho(\xi x_n)=0$ for all $\xi\geqq 0$.

In order to prove the completeness of quasi-norm $\|\cdot\|_0$, the next Lemma is necessary.

Lemma 2. *Let* $p_{n,\nu}$, $x_\nu \geqq 0$ and $a \geqq 0$ $(n, \nu=1, 2, \cdots)$ *be the elements of* $R_0^\perp$ *such that*

$$(3.2) \qquad [p_{n,\nu}]\uparrow_{\nu=1}^{\infty}[p_n] \text{ with } \bigcap_{n=1}^{\infty}[p_n]a=[p_0]a \neq 0 ;$$

$$(3.3) \qquad [p_{n,\nu}]x_\nu \geqq n[p_{n,\nu}]a \text{ for all } n, \nu \geqq 1 .$$

Then $\{x_\nu\}_{\nu\geqq 1}$ *is not a Cauchy sequence of* $R_0^\perp$ *with respect to* $\|\cdot\|_0$.

Proof. We shall show that there exist a sequence of projectors $[q_m]\downarrow_{m=1}^{\infty}(m\geqq 1)$ and sequences of natural numbers ν_m, n_m such that

$$(3.4) \qquad \|[q_m]a\|_0 > \frac{\delta}{2} \quad \text{and} \quad [q_m]x_{\nu_m} \geqq n_m[q_m]a \qquad (m=1, 2, \cdots)$$

and

$$(3.5) \qquad n_m[q_m]a \geqq [q_m]x_{\nu_{m-1}}, \qquad n_{m+1} > n_m \ (m=2, 3, \cdots),$$

where $\delta = \|[p_0]a\|_0$.

In fact, we put $n_1=1$. Since $[p_{1,\nu}][p_0]\uparrow_{\nu=1}^{\infty}[p_0]$ and $\|\cdot\|_0$ is semi-continuous, we can find a natural number ν_1 such that

$$\|[p_{1,\nu_1}][p_0]a\|_0 > \frac{\|[p_0]a\|_0}{2} = \frac{\delta}{2} .$$

We put $[q_1]=[p_{1,\nu_1}][p_0]$. Now, let us assume that $[q_m]$, ν_m, n_m $(m=1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na-x_{\nu_k})^+]\uparrow_{n=1}^{\infty}[a]$ and $\|[q_k]a\|_0 > \frac{\delta}{2}$, there exists n_{k+1} with

$$\|(n_{k+1}a-x_{\nu_k})^+[q_k]a\|_0 > \frac{\delta}{2} .$$

For such n_{k+1}, there exists also a natural number ν_{k+1} such that

$$\|[p_{n_{k+1},\nu_{k+1}}][(n_{k+1}a-x_{\nu_k})^+][q_k]a\|_0 > \frac{\delta}{2} .$$

in virtue of (3.2) and semi-continuity of $\|\cdot\|_0$. Hence we can put

$$[q_{k+1}]=[p_{n_{k+1},\nu_{k+1}}][(n_{k+1}a-x_{\nu_k})^+][q_k] ,$$

because

$$[q_{k+1}] \leqq [q_k] , \ \|[q_{k+1}]a\| > \frac{\delta}{2} , \ [q_{k+1}]x_{\nu_{k+1}} \geqq n_{k+1}[q_{k+1}]a$$

by (3.3) and $[q_{k+1}]n_{k+1}a \geqq [q_{k+1}]x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geqq 3$

$$\| x_{\nu_{k+1}} - x_{\nu_{k-1}} \|_0 \geqq \| [q_{k+1}](x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0$$
$$\geqq \| n_{k+1}[q_{k+1}]a - n_k[q_{k+1}]a) \|_0 \geqq \| [q_{k+1}]a_0 \|_0 \geqq \frac{\delta}{2},$$

since $[q_{k+1}] \leqq [q_k] \leqq [(n_k a - x_{\nu-1})^+]$ implies $[q_{k+1}]n_k a \geqq [q_{k+1}]x_{\nu_{k-1}}$ by (3.4). It follows from the above that $\{x_\nu\}_{\nu\geqq 1}$ is not a Cauchy sequence.

Theorem 3.2. *Let R be a quasi-modular space with quasi-modnlar ρ. Then $R_0^\perp$ is an F-space with $\|\cdot\|_0$ if and only if ρ satisfies (ρ.4′).*

Proof. If ρ satisfies (ρ.4′), ρ^* is a quasi-modular which fulfils also (ρ.5) and (ρ.6) in virtue of Theorem 2.3. Since $\| x \|_0 \left(= \inf \left\{ \xi \,;\, \rho^*\left(\frac{x}{\xi}\right) \leqq \xi \right\} \right)$ is a quasi-norm on $R_0^\perp$, we need only to verify completeness of $\|\cdot\|_0$. At first let $\{x_\nu\}_{\nu\geqq 1} \subset R_0^\perp$ be a Cauchy sequence with $0 \leqq x_\nu \uparrow_{\nu=1,2,\dots}$. Since ρ^* satisfies (ρ.3), there exists $0 \leqq x_0 \in R_0^\perp$ such that $x_0 = \bigcup_{\nu=1}^{\infty} x_\nu$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,\nu}] = [(x_\nu - n x_0)^+]$ and $\bigcup_{\nu=1}^{\infty} [p_{n,\nu}] = [p_n]$, we obtain

$$[p_{n,\nu}]x_\nu \geqq n[p_{n,\nu}]x_0 \qquad \text{for all } n, \nu \geqq 1 \tag{3.6}$$

and $[p_n] \downarrow_{n=1}^{\infty} 0$. Since $\{x_\nu\}_{\nu\geqq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^{\infty} [p_n] = 0$, that is, $\bigcup_{n=1}^{\infty} ([x_0] - [p_n]) = [x_0]$. And

$$(1 - [p_{n,\nu}]) \geqq (1 - [p_n]) \qquad (n, \nu \geqq 1)$$

implies

$$n(1 - [p_n])x_0 \geqq (1 - [p_n])x_\nu \geqq 0 .$$

Hence we have

$$y_n = \bigcup_{\nu=1}^{\infty} (1 - [p_n])x_\nu \in R_0^\perp ,$$

because $R_0^\perp$ is universally continuous. As $\{x_\nu\}_{\nu\geqq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $\|\cdot\|_0$

$$\gamma = \sup_{\nu\geqq 1} \| x_\nu \|_0 < +\infty ,$$

which implies

$$\| y_n \|_0 = \sup_{\nu\geqq 1} \| (1 - [p_n])x_\nu \|_0 \leqq \gamma$$

for every $n \geqq 1$ by semi-continuity of $\|\cdot\|_0$. We put $z_1 = y_1$ and $z_n = y_n - y_{n-1}$ ($n \geqq 2$). It follows from the definition of y_n that $\{z_\nu\}_{\nu\geqq 1}$ is an orthogonal sequence with $\| \sum_{\nu=1}^{n} z_\nu \|_0 = \| y_n \|_0 \leqq \gamma$. This implies

$$\sum_{\nu=1}^{n} \rho^*\left(\frac{z_\nu}{1+\gamma}\right) = \rho^*\left(\frac{y_n}{1+\gamma}\right) \leqq \gamma$$

for all $n \geqq 1$ by the fomula (3.1). Then (ρ.3) assures the existence of $z = \sum_{\nu=1}^{\infty} z_\nu = \bigcup_{\nu=1}^{\infty} y_\nu$. This yields $z = \bigcup_{\nu=1}^{\infty} x_\nu$. Truly, it follows from

$$z = \bigcup_{n=1}^{\infty} y_n = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-[p_n])x_\nu = \bigcup_{\nu=1}^{\infty} \bigcup_{n=1}^{\infty} (1-[p_n])x_\nu = \bigcup_{\nu=1}^{\infty} [x_0]x_\nu = \bigcup_{\nu=1}^{\infty} x_\nu .$$

By semi-continuity of $\|\cdot\|_0$, we have

$$\| z - x_\nu \|_0 \leqq \sup_{\mu \geqq \nu} \| x_\mu - x_\nu \|_0$$

and furthermore $\lim_{\nu \to \infty} \| z - x_\nu \|_0 = 0$.

Secondly let $\{x_\nu\}_{\nu \geqq 1}$ be an arbitrary Cauchy sequence of $R_0^\perp$. Then we can find a subsequence $\{y_\nu\}_{\nu \geqq 1}$ of $\{x_\nu\}_{\nu \geqq 1}$ such that

$$\| y_{\nu+1} - y_\nu \|_0 \leqq \frac{1}{2^\nu} \qquad \text{for all } \nu \geqq 1 .$$

This implies

$$\| \sum_{\nu=m}^{n} | y_{\nu+1} - y_\nu | \|_0 \leqq \sum_{\nu=m}^{n} \| y_{\nu+1} - y_\nu \|_0 \leqq \frac{1}{2^{m-1}} \qquad \text{for all } n > m \geqq 1 .$$

Putting $z_n = \sum_{\nu=1}^{n} | y_{\nu+1} - y_\nu |$, we have a Cauchy sequence $\{z_n\}_{n \geqq 1}$ with $0 \leqq z_n \uparrow_{n=1}^{\infty}$. Then by the fact proved just above,

$$z_0 = \bigcup_{n=1}^{\infty} z_n = \sum_{\nu=1}^{\infty} | y_{\nu+1} - y_\nu | \in R_0^\perp \quad \text{and} \quad \lim_{n \to \infty} \| z_0 - z_n \|_0 = 0 .$$

Since $\sum_{\nu=1}^{\infty} | y_{\nu+1} - y_\nu |$ is convergent, $y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_\nu)$ is also convergent and

$$\| y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_\nu) - y_n \|_0 = \| \sum_{\nu=n}^{\infty} (y_{\nu+1} - y_\nu) \|_0 \leqq \| z_0 - z_n \|_0 \to 0 .$$

Since $\{y_\nu\}_{\nu \geqq 1}$ is a subsequence of the Cauchy sequence $\{x_\nu\}_{\nu \geqq 1}$, it follows that

$$\lim_{\mu \to \infty} \| y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_\nu) - x_\mu \|_0 = 0 .$$

Therefore $\|\cdot\|_0$ is complete in $R_0^\perp$, that is, $R_0^\perp$ is an F-space with $\|\cdot\|_0$.

Conversely if $R_0^\perp$ is an F-space, then for any orthogonal sequence $\{x_\nu\}_{\nu \geqq 1} \in R_0^\perp$, we have $\sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R_0^\perp$ for some real numbers $\alpha_\nu > 0$ (for all $\nu \geqq 1$). Hence we can see that $\sup_{x \in R} d(x) < +\infty$ by the same way applied in Theorem 2.1. It follows that ρ must satisfy (ρ.4$'$). Q.E.D.

Since R_0 contains a normal manifold which is universally complete, if $R_0 \neq 0$, we can conclude directly from Theorems 3.1 and 3.2

Corollary. *Let R be a quasi-modular space which includes no universally complete normal manifold. Then R becomes a quasi-normed space with a quasi-norm $\|\cdot\|_0$ defined by* (3.1) *and R becomes an F-space with $\|\cdot\|_0$ if and only if ρ fulfils* (ρ.4′).

§4. Another Quasi-norm. Let L be a *modular space* in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

$$\| x \|_1 = \inf_{\xi>0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10)} \tag{4.1}$$

and show that $\|\cdot\|_1$ is also a quasi-norm on L and

$$\| x \|_0 \leqq \| x \|_1 \leqq 2 \| x \|_0 \qquad \text{for all } x \in L \tag{4.2}$$

hold, where $\|\cdot\|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leqq \| x \|_1 = \| -x \|_1 < +\infty$ ($x \in L$) and that $\| x \|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^{\infty} 0$ implies $\lim_{n\to\infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n\to\infty} \| x_n \|_1 = 0$ implies $\lim_{n\to\infty} \rho(\xi x_n) = 0$ for all $\xi \geqq 0$, we obtain that $\lim_{n\to 0} \| \alpha_n x \|_1 = 0$ for all $\alpha_n \downarrow_{n=1}^{\infty} 0$ and that $\lim_{n\to\infty} \| x_n \|_1 = 0$ implies $\lim_{n\to\infty} \| \alpha x_n \|_1 = 0$ for all $\alpha > 0$. If $\| x \|_1 < \alpha$ and $\| y \|_1 < \beta$, there exist $\xi, \eta > 0$ such that

$$\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta .$$

This yields

$$\| x+y \| \leqq \frac{\xi+\eta}{\xi\eta} + \rho\left(\frac{\xi\eta}{\xi+\eta}(x+y) \right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi+\eta}(\xi x) + \frac{\xi}{\xi+\eta}(\eta y) \right)$$

$$\leqq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta ,$$

in virtue of (A.3). Therefore $\| x+y \|_1 \leqq \| x \|_1 + \| y \|_1$ holds for any $x, y \in L$ and $\|\cdot\|_1$ is a quasi-norm on L. If $\xi\rho(\xi x) \leqq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leqq \frac{1}{\xi}$ and hence

$$\frac{1}{\xi} \leqq \frac{1}{\xi} + \rho(\xi x) \leqq \frac{2}{\xi} .$$

10) For the convex modular m, we can define two kinds of norms such as

$$\| x \| = \inf_{\xi\to 0} \frac{1+m(\xi x)}{\xi} \quad \text{and} \quad |\!|\!| x |\!|\!| = \inf_{m(\xi x)\leqq 1} \frac{1}{|\xi|}$$

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi\rho(\xi x)$ in $|\!|\!|\cdot|\!|\!|$ and $\|\cdot\|$ respectively.

This yields (4.2), since we have $\|x\|_0 \leqq \frac{1}{\xi}$ and $\rho(\eta x) > \frac{1}{\eta}$ for every η with $\|x\|_0 > \frac{1}{\eta}$. Therefore we can obtain from above

Theorem 4.1. *If L is a modular space with a modular satisfying* (A.1)~(A.5) *in* §1, *then the formula* (4.1) *yields a quasi-norm* $\|\cdot\|_1$ *on* L *which is equivalent to* $\|\cdot\|_0$ *defined by Musielak and Orlicz in* [6] *as is shown in* (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

Theorem 4.2. *If R is a quasi-modular space with a quasi-modular ρ, then*

$$\|x\|_1 = \inf_{\xi>0}\left\{\frac{1}{\xi} + \rho^*(\xi x)\right\} \qquad (x \in R) \tag{4.3}$$

is a semi-continuous quasi-norm on $R_0^{\perp}$ *and* $\|\cdot\|_1$ *is complete if and only if* ρ *satisfies* (ρ.4′), *where* ρ^* *and* R_0 *are the same as in* §2 *and* §3. *And further we have*

$$\|x\|_0 \leqq \|x\|_1 \leqq 2\|x\|_0 \qquad \textit{for all } x \in R_0^{\perp}\,. \tag{4.4}$$

§5. **A quasi-norm-convergence.** Here we suppose that a quasi-modular ρ^* on R satisfies (ρ.1)~(ρ.6) except (ρ.3) and $\rho^*(\xi x)$ is not identically zero as a function of $\xi \geqq 0$ for each $0 \neq x \in R$ (i.e. $R_0 = \{0\}$). A sequence of elements $\{x_\nu\}_{\nu\geqq1}$ is called *order-convergent* to a and denoted by $\text{o-}\lim_{\nu\to\infty} x_\nu = a$, if there exists a sequence of elements $\{a_\nu\}_{\nu\geqq1}$ such that $|x_\nu - a| \leqq a_\nu$ ($\nu \geqq 1$) and $a_\nu \downarrow_{\nu=1}^{\infty} 0$. And a sequence of elements $\{x_\nu\}_{\nu\geqq1}$ is called *star-convergent* to a and denoted by $\text{s-}\lim_{\nu\to\infty} x_\nu = a$, if for any subsequence $\{y_\nu\}_{\nu\geqq1}$ of $\{x_\nu\}_{\nu\geqq1}$, there exists a subsequence $\{z_\nu\}_{\nu\geqq1}$ of $\{y_\nu\}_{\nu\geqq1}$ with $\text{o-}\lim_{\nu\to0} z_\nu = a$. A quasi-norm $\|\cdot\|$ on R is termed to be *continuous*, if $\inf_{\nu\geqq1}\|a_\nu\| = 0$ for any $a_\nu \downarrow_{\nu=1}^{\infty} 0$. In the sequel, we write by $\|\cdot\|_0$ (or $\|\cdot\|_1$) the quasi-norm defined on R by ρ^* in §3 (resp. in §4).

Now we prove

Theorem 5.1. *In order that* $\|\cdot\|_0$ (*or* $\|\cdot\|_1$) *is continuous, it is necessary and sufficient that the following condition is satisfied*:

(5.1) *for any $x \in R$ there exists an orthogonal decomposition $x = y + z$ such that $[z]R$ is finite dimensional and $\rho(y) < +\infty$.*

Proof. Necessity. If (5.1) is not true for some $x \in R$, we can find a sequence of projector $\{[p_n]\}_{n\geqq 1}$ such that $\rho([p_n]x)=+\infty$ and $[p_n]\downarrow_{n=1}^{\infty}0$. Hence by (3.1) it follows that $\|[p_n]x\|_0>1$ for all $n\geqq 1$, which contradicts the continuity of $\|\cdot\|_0$.

Sufficiency. Let $a_\nu\downarrow_{\nu=1}^{\infty}0$ and put $[p_n^\varepsilon]=[(a_n-\varepsilon a_1)^+]$ for any $\varepsilon>0$ and $n\geqq 1$. It is easily seen that $[p_n^\varepsilon]\downarrow_{n=1}^{\infty}0$ for any $\varepsilon>0$ and

$$a_n=[a_1]a_n=[p_n^\varepsilon]a_n+(1-[p_n^\varepsilon])a_n\leqq[p_n^\varepsilon]a_1+\varepsilon a_1 .$$

This implies

$$\rho^*(\xi a_n)\leqq\rho^*(\xi[p_n^\varepsilon]a_1)+\rho^*(\xi\varepsilon(1-[p_n^\varepsilon])a_1)$$

for all $n\geqq 1$ and $\xi\geqq 0$. In virtue of (5.1) and $[p_n^\varepsilon]\downarrow_{n=1}^{\infty}0$, we can find n_0 (depending on ξ and ε) such that $\rho^*(\xi[p_n^\varepsilon]a_1)<+\infty$, and hence $\inf_{n\geqq 1}\rho^*(\xi[p_n^\varepsilon]a_1)=0$ by (2.3) in Lemma 1 and (ρ.2). Thus we obtain

$$\inf_{n\geqq 1}\rho^*(\xi a_n)\leqq\rho^*(\xi\varepsilon a_1) .$$

Since ε is arbitrary, $\lim_{n\to\infty}\rho^*(\xi a_n)=0$ follows. Hence we infer that $\inf_{n\geqq 1}\|a_n\|_0=0$ and $\|\cdot\|_0$ is continuous in view of Remark 2 in §3. Q.E.D.

In view of the proof of the above theorem we get obviously

Corollary. *$\|\cdot\|_0$ is continuous, if*

(5.2) $$\rho^*(a_\nu)\to 0 \quad \textit{implies} \quad \rho^*(\alpha a_\nu)\to 0 \qquad \textit{for every } \alpha\geqq 0 .$$

From the definition, it is clear that $\text{s-}\lim_{\nu\to\infty}x_\nu=0$ implies $\lim_{\nu\to\infty}\|x_\nu\|_0=0$, if $\|\cdot\|_0$ is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

Theorem 5.2. $\lim_{\nu\to\infty}\|x_\nu\|_0=0$ (*or* $\lim_{\nu\to\infty}\|x_\nu\|_1=0$) *implies* $\text{s-}\lim_{\nu\to\infty}x_\nu=0$, *if* $\|\cdot\|_0$ *is complete* (i.e. ρ^* *satisfies* (ρ.3)).

If we replace $\lim_{\nu\to\infty}\|x_\nu\|=0$ by $\lim_{\nu\to\infty}\rho(x_\nu)=0$, Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

(5.3) $$\rho^*(x)=0 \quad \text{implies} \quad x=0 .$$

Truly we obtain

Theorem 5.3. *If ρ^* satisfies* (5.3) *and $\|\cdot\|_0$ is complete, $\rho(a_\nu)\to 0$ implies* $\text{s-}\lim_{\nu\to\infty}a_\nu=0$.

Proof. We may suppose without loss of generality that ρ^* is semi-continuous,[11] i.e. $\rho^*(x)=\sup_{\lambda\in\Lambda}\rho^*(x_\lambda)$ for any $0\leqq x\uparrow_{\lambda\in\Lambda}x$. If

11) If ρ^* is not semi-continuous, putting $\rho_*(x)=\inf_{y_\lambda\uparrow_{\lambda\in\Lambda}x}\{\sup_{\lambda\in\Lambda}\rho^*(y_\lambda)\}$, we obtain a quasi-modular ρ_* which is semi-continuous and $\rho^*(x_\nu)\to 0$ is equivalent to $\rho_*(x_\nu)\to 0$.

$$\rho(a_\nu) \leqq \frac{1}{2^\nu} \quad (\nu \geqq 1),$$

we can prove by the similar way as in the proof of Lemma 2 that there exists $\bigcup_{\nu=1}^{\infty} |a_\nu| \in R$ in virtue of (ρ.3).

Now, since

$$\rho\left(\bigcup_{\mu \geqq \nu}^{\infty} |a_\mu|\right) \leqq \sum_{\mu \geqq \nu}^{\infty} \rho(a_\mu) \leqq \frac{1}{2^{\nu-1}}$$

holds for each $\nu \geqq 1$, $\rho\left\{\bigcap_{\nu=1}^{\infty}\left(\bigcup_{\mu \geqq \nu}^{\infty} |a_\mu|\right)\right\}=0$ and hence (5.3) implies

$$\bigcap_{\nu=1}^{\infty}\left(\bigcup_{\mu \geqq \nu}^{\infty} |a_\mu|\right)=0 .$$

Thus we see that $\{a_\mu\}_{\mu \geqq 1}$ is order-convergent to 0.

For any $\{b_\nu\}_{\nu \geqq 1}$ with $\rho(b_\nu) \to 0$, we can find a subsequence $\{b'_\nu\}_{\nu \geqq 1}$ of $\{b_\nu\}_{\nu \geqq 1}$ with $\rho(b'_\nu) \leqq \frac{1}{2^\nu}$ ($\nu=1, 2, \cdots$). Therefore we have $\text{s-}\lim_{\nu \to \infty} b_\nu = 0$. Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analougous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

Theorem 5.4. *If ρ^* satisfies* (5.3) *and $\|\cdot\|_0$ is complete and continuous, then* (5.2) *holds.*

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