A NOTE ON STRICTLY GALOIS EXTENSION OF PRIMARY RINGS

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Let $R\ni 1$ be a primary ring with minimum condition (for one-sided ideals). One of the present authors proved in [1][1] that if R is strictly Galois with respect to $\mathfrak{G}$ then R possesses a $\mathfrak{G}$-normal basis element. The purpose of this note is to present a slight generalization of this fact.

In what follows, R be always a primary ring with minimum condition which is strictly Galois with respect to (an F-group) $\mathfrak{G}$ of order n, $N\ni 1$ a subring of R with minimum condition such that $N\mathfrak{G}=N$ and R possesses a linearly independent right N-basis consisting of t elements.[2] Further, we set $t=nq+r$, where $0\leq r<n$. Under this situation, our theorem can be stated as follows:

Theorem. *There exist q elements $x_1, \cdots, x_q\in R$ and a $\mathfrak{G}N_r$-submodule M of R such that*

(1) *M is $\mathfrak{G}N_r$-homomorphic to $\mathfrak{G}N_r$ and possesses a linearly independent right N-basis consisting of r elements,*

(2) $R=\sum_{i=1}^{q}\oplus\sum_{\sigma\in\mathfrak{G}}\oplus(x_i\sigma)N\oplus M.$

Proof. As is shown in [1], $\mathrm{Hom}_{S_l}(R, R)=\mathfrak{G}R_r=\sum_{\sigma\in\mathfrak{G}}\oplus\sigma R_r$, where $S=J(\mathfrak{G}, R)$. Since $[R:S]_l=n$, and so, since R is S-left regular, R is $\mathrm{Hom}_{S_l}(R, R)$-right regular too. In fact, $R^{(n)}$ is $\mathfrak{G}R_r$-isomorphic to $\mathfrak{G}R_r$, where $R^{(n)}$ means the direct sum of n-copies of R as $\mathfrak{G}R_r$-module. Accordingly, $R^{(n)}$ is $\mathfrak{G}N_r$-isomorphic to $\mathfrak{G}R_r$ of course. Now let $R=u_1N\oplus\cdots\oplus u_tN$. Then, we have $\mathfrak{G}R_r=R_r\mathfrak{G}=\sum_{i=1}^{t}u_{ir}N_r\mathfrak{G}=\sum_{i=1}^{t}\oplus u_{ir}\mathfrak{G}N_r$. Hence, $\mathfrak{G}R_r$ is $\mathfrak{G}N_r$-isomorphic to $(\mathfrak{G}N_r)^{(t)}$, and so we have eventually that $R^{(n)}$ is $\mathfrak{G}N_r$-isomorphic to $(\mathfrak{G}N_r)^{(t)}$. Here let $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ be all the non-isomorphic directly indecomposable direct summands of the $\mathfrak{G}N_r$-module R (or $\mathfrak{G}N_r$ itself). And, in the Remak decompositions of $\mathfrak{G}N_r$-modules R and $\mathfrak{G}N_r$, the re-

1) As to notations and terminologies used in this note, we follow [1]. And we will use freely the results cited in [1].

2) N does not necessarily contain the subring $S=J(\mathfrak{G}, R)$.

spective numbers of directly indecomposible components which are isomorphic to $\mathfrak{p}_i$ will be denoted by n_i and m_i. Then, our isomorphism mentioned above yields at once $n_i n = m_i t = m_i(nq+r)$, whence we have $m_i r = nk_i$ with some non-negative integer $k_i < m_i$. Consequently, it follows that $n_i = m_i q + k_i (i=1, \cdots, s)$. This proves clearly the existence of a $\mathfrak{G}N_r$-isomorphism φ of R onto $(\mathfrak{G}N_r)^{(q)} \oplus T$, where $T = \sum_{i=1}^{s} \oplus \mathfrak{p}_i^{(k_i)}$. Recalling here $m_i > k_i$, we see that T is $\mathfrak{G}N_r$-homomorphic to $\mathfrak{G}N_r$. Now, let $y_i = (0, \cdots, 0, \overset{(i)}{1}, \cdots, 0) \in (\mathfrak{G}N_r)^{(q)}$. Then, one will easily verify that $x_i = \varphi^{-1}\{y_i\}$ $(i=1, \cdots, q)$ and $M = \varphi^{-1}\{T\}$ are desired ones.

Our theorem may be considered as a generalization of [1, Theorem 1]. Moreover, in case R is a division ring we obtain the following which secures the existence of the so-called semi-normal basis.

Corollary. *Let R be a devision ring which is strictly Galois with respect to $\mathfrak{G}$ of order n, and N a division subring of R with $N\mathfrak{G} = N$ and $[R:N]_r = t$. If $t = nq + r$ $(0 \leq r < n)$ then there exist some $x_0, x_1, \cdots, x_q \in R$ such that $R = \sum_{i=1}^{q} \oplus \sum_{\sigma \in \mathfrak{G}} \oplus (x_i\sigma)N \oplus \sum_{\tau} \oplus (x_0\tau)N$, where τ runs over a suitable subset of $\mathfrak{G}$ consisting of r elements.*

References

[1] H. Tominaga: A note on Galois theory of primary rings, Math. J. Okayama Univ., 8 (1958), 117–124.

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