RATIONAL APPROXIMATIONS TO ALGEBRAIC FUNCTIONS

By

Saburô UCHIYAMA

1. Introduction. There is a classical theorem due to J. Liouville on the approximability of algebraic numbers by rational numbers. Liouville's result states that if α is an algebraic number of degree $n \geqq 2$ then

$$\left|\alpha - \frac{p}{q}\right| \geqq \frac{A}{q^n}$$

for all rational integers p, $q\,(q>0)$, where A is a positive constant depending only on α. This theorem has been improved successively by A. Thue, C. L. Siegel, F. J. Dyson, and K. F. Roth. It is proved by Roth [5]*) that if α is an algebraic number of degree $n \geqq 2$ then for each $\kappa > 2$ the inequality

$$(1) \qquad \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\kappa}$$

has only finitely many solutions in integers p, $q\,(q>0)$. This is best possible in the sense that for every irrational number α, whether algebraic or not, there are infinitely many integers p, $q\,(q>0)$ satisfying (1) with $\kappa=2$.

It is well known that the theorem of Liouville for algebraic numbers has an analogue in algebraic function fields and, as was shown by K. Mahler [2], the analogue of Liouville's theorem for algebraic functions cannot be improved, in general, if the field of constants is of positive characteristic. On the other hand, the present author [6] has pointed out that it is possible to obtain an analogue of the theorem of Roth in algebraic function fields with the constant field of characteristic 0. The result is known to be the best possible of its kind.

The purpose of the present paper is to give a full account of general theorems on the approximation to algebraic functions by rational functions, with an arbitrary field of constants. A particular case of some of

*) Numbers in brackets refer to the references at the end of this paper.

our results presented here has been treated by Mahler [2] and by the writer [6] as a supplement to Mahler's paper [2].

2. The valuations. Let K be an arbitrary field of characteristic χ, χ being 0 or a prime number. Let t be an indeterminate and let $K[t]$ denote the ring of all polynomials in t with coefficients in K and $K(t)$ the field of all rational functions in t with coefficients in K.

If $\xi=\xi(t)$ is an element of $K(t)$, there exist polynomials $p=p(t)$, $q=q(t)\neq 0$ in $K[t]$ such that $\xi=p/q$. We define

$$\deg \xi \equiv \deg p - \deg q .$$

We shall be concerned in the following with (non-trivial) valuations on $K(t)$ that are trivial on K. Thus there are two kinds of such valuations, namely:

The valuation $|\ |$. For $\alpha=\alpha(t)$ in $K(t)$ we define $|\alpha|$ by putting

$$|\alpha|=\begin{cases}0 & \text{if } \alpha=0,\\ c^{\deg \alpha} & \text{if } \alpha\neq 0,\end{cases}$$

where $c>1$ is a constant fixed throughout this paper.

A valuation $|\ |_\tau$. Let τ be a fixed primary irreducible polynomial in $K[t]$. For $\alpha=\alpha(t)$ in $K[t]$ we define $|\alpha|_\tau$ by putting

$$|\alpha|_\tau=\begin{cases}0 & \text{if } \alpha=0,\\ c^{-\nu \deg \tau} & \text{if } \alpha\neq 0,\end{cases}$$

where $\nu=\mathrm{ord}_\tau \alpha$, i.e. $\tau^{-\nu}\alpha$ contains the factor τ in neither numerator nor denominator.

These valuations are so-called normal valuations on $K(t)$ and there holds the product formula:

$$|\alpha|\prod_\tau |\alpha|_\tau=|\alpha|_0 ,$$

where the product is taken over all primary irreducible polynomials τ in $K[t]$, and where $|\ |_0$ is the trivial valuation on $K(t)$, i.e. for $\alpha=\alpha(t)$ in $K(t)$

$$|\alpha|_0=\begin{cases}0 & \text{if } \alpha=0,\\ 1 & \text{if } \alpha\neq 0.\end{cases}$$

In particular, if $\alpha=\alpha(t)\neq 0$ is a polynomial of $K[t]$, then we have

$$|\alpha|\cdot|\alpha|_\tau\geqq 1 \tag{2}$$

for any valuation $|\ |_\tau$ on $K(t)$, or more generally,

$$|a|\prod_{j=1}^{s}|a|_{\tau_j}\geqq 1$$

for any valuations $|\ |_{\tau_j}$ $(1\leqq j\leqq s)$, mutually inequivalent on $K(t)$ and finite in number.

Now, let $K\langle t^{-1}\rangle$ denote the completion of $K(t)$ under the valuation $|\ |$ and $K\langle\tau\rangle$ denote the completion of $K(\tau)$ under the valuation $|\ |_\tau$. Thus $K\langle t^{-1}\rangle$ is the field of all formal power series of the type

$$\sum_{j=0}^{\infty} a_j t^{l-j} \qquad (a_j\in K),$$

where l is a certain non-negative integer, and $K\langle\tau\rangle$ is the field of all elements of the form

$$\sum_{j=0}^{\infty} a_j \tau^{j-l} \qquad (a_j\in K[t],\ \deg a_j<\deg\tau),$$

l being a non-negative integer.

3. Main results. The following theorem is an analogue of Liouville's theorem on rational approximations to real algebraic numbers:

Theorem 1. *Let K be an arbitrary field.*

(i) *Let $\alpha=\alpha(t)$ be an element of $K\langle t^{-1}\rangle$ algebraic of degree $n\geqq 2$ over $K(t)$. Then there is a constant $A_1>0$ such that*

$$\left|\alpha-\frac{p}{q}\right|\geqq\frac{A_1}{|q|^n} \tag{3}$$

for all pairs of polynomials $p=p(t)$, $q=q(t)\neq 0$ in $K[t]$. If K is of characteristic $\chi>0$, the inequality (3) *cannot be improved in general.*

(ii) *Let $\alpha=\alpha(t)$ be an element of $K\langle\tau\rangle$ algebraic of degree $n\geqq 2$ over $K(t)$. Then there is a constant $A_2>0$ such that*

$$|p-q\alpha|_\tau\geqq\frac{A_2}{|p,q|^n} \tag{4}$$

for all pairs of polynomials $p=p(t)$, $q=q(t)$ in $K[t]$ with $|p,q|>0$, where

$$|p,q|=\max(|p|,|q|).$$

If K is of characteristic $\chi>0$, the inequality (4) *cannot be improved in general.*

The part (i) of this theorem is proved by Mahler [2].

If the constant field K is of characteristic 0, then Theorem 1 can be improved to the form:

Theorem 2. *Let K be a field of characteristic* 0.

(i) *Let $\alpha=\alpha(t)\neq 0$ be any element of $K\langle t^{-1}\rangle$ algebraic over $K(t)$. Then for each $\kappa>2$, the inequality*

$$(5) \qquad \left|\alpha-\frac{p}{q}\right|<\frac{1}{|q|^{\kappa}}$$

is satisfied by only a finite number of pairs of polynomials $p=p(t)$, $q=q(t)\neq 0$ in $K[t]$ with $(p, q)=1$.

(ii) *Let $\alpha=\alpha(t)\neq 0$ be any element of $K\langle\tau\rangle$ algebraic over $K(t)$. Then for each $\kappa>2$, the inequality*

$$(6) \qquad |p-q\alpha|_{\tau}<\frac{1}{|p, q|^{\kappa}}$$

is satisfied by only a finite number of pairs of polynomials $p=p(t)$, $q=q(t)$ in $K[t]$ with $(p, q)=1$.

We observe that Theorem 2 is the best possible of its kind, as so is Roth's theorem on rational approximations to algebraic numbers. In fact we shall prove:

Theorem 3. *Let K be an arbitrary field of characteristic* 0.

(i) *Let $\alpha=\alpha(t)$ be any element of $K\langle t^{-1}\rangle$, not a rational function. Then there exist infinitely many pairs of polynomials $p=p(t)$, $q=q(t)\neq 0$ in $K[t]$ with $(p, q)=1$ satisfying the inequality*

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{|q|^{2}}.$$

(ii) *Let $\alpha=\alpha(t)$ be any element of $K\langle\tau\rangle$, not a rational function. Then there exist infinitely many pairs of polynomials $p=p(t)$, $q=q(t)$ in $K[t]$ with $(p, q)=1$ satisfying the inequality*

$$|p-q\alpha|_{\tau}<\frac{1}{|p, q|^{2}}.$$

In §4 we prove Theorem 1, (ii). We shall give a proof for Theorem 2 in §§5~8, and a proof for Theorem 3 in §9. While our proof of Theorem 2 follows, in the main, lines analogous to Roth's [5], there are essential differences in details. In §10 we note some further results allied to Theorem 2. Several applications of these theorems will be given in §11.

4. Proof of Theorem 1, (ii). If $\alpha=\alpha(t)$ is an element of $K\langle\tau\rangle$ algebraic of degree $n\geqq 2$ over $K(t)$, it satisfies an irreducible equation $f(x)=0$, where

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n,$$

the coefficients $a_0\neq 0$, $a_1,\cdots,a_n$ being polynomials in $K[t]$. Following Mahler, we consider the polynomial

$$g(x)=\sum_{j=0}^{n-1}(a_0\alpha^j+a_1\alpha^{j-1}+\cdots+a_j)x^{n-1-j}.$$

Then $f(x)/(x-\alpha)=(f(x)-f(\alpha))/(x-\alpha)=g(x)$ identically in x, and so

$$x-\alpha=\frac{f(x)}{g(x)}.$$

Put

$$c_1=\max(1,\ |\alpha|_\tau).$$

Let $p=p(t)$, $q=q(t)\neq 0$ be any elements of $K[t]$. If

$$\left|\frac{p}{q}\right|_\tau>c_1\geqq|\alpha|_\tau,$$

then we have, on account of (2),

$$|p-q\alpha|_\tau=|p|_\tau\geqq\frac{1}{|p|}\geqq\frac{1}{|p,q|^n},$$

since $|p,q|\geqq\max(1,\ |p|)$. If

$$\left|\frac{p}{q}\right|_\tau\leqq c_1,$$

then

$$\left|q^{n-1}g\left(\frac{p}{q}\right)\right|_\tau\leqq c_1^{n-1},$$

Now, the expression

$$q^nf\left(\frac{p}{q}\right)=a_0p^n+a_1p^{n-1}q+\cdots+a_nq^n$$

lies in $K[t]$ and does not vanish since $f(x)$ is an irreducible polynomial of degree $n\geqq 2$ with coefficients in $K[t]$. Hence, by (2),

$$\left| q^n f\left(\frac{p}{q}\right) \right|_\tau \geqq \frac{1}{\left| q^n f\left(\frac{p}{q}\right) \right|} \geqq \frac{1}{c_2 | p, q |^n},$$

where

$$c_2 = \max(|a_0|, |a_1|, \cdots, |a_n|).$$

Therefore

$$| p - q\alpha |_\tau = \frac{\left| q^n f\left(\frac{p}{q}\right) \right|_\tau}{\left| q^{n-1} g\left(\frac{p}{q}\right) \right|_\tau} \geqq \frac{1}{c_1^{n-1} c_2 | p, q |^n}.$$

Thus it suffices to put

$$A_2 = \min\left(1, \frac{1}{c_1^{n-1} c_2}\right).$$

This proves the first part of Theorem 1, (ii).

To prove the second part of Theorem 1, (ii), let $\chi > 0$ be the characteristic of K and consider the element

$$\alpha = \tau + \tau^\chi + \tau^{\chi^2} + \cdots$$

of $K\langle\tau\rangle$. We have

$$\alpha = \tau + (\tau + \tau^\chi + \cdots)^\chi = \tau + \alpha^\chi,$$

and so α is a root of the algebraic equation

$$x^\chi - x + \tau = 0.$$

Since τ is an irreducible polynomial in $K[t]$, it follows that α is of exact degree χ over $K(t)$. Put

$$p_j = \tau + \tau^\chi + \cdots + \tau^{\chi^{j-1}}, q_j = 1 \quad (j = 1, 2, \cdots).$$

Then

$$| p_j, q_j | = c^{\chi^{j-1} \deg \tau}$$

and

$$| p_j - q_j \alpha |_\tau = | \tau^{\chi^j} + \cdots | = c^{-\chi^j \deg \tau} = | p_j, q_j |^{-\chi},$$

completing the proof of our assertion.

5. Some lemmas. In what follows we shall suppose throughout that the ground field K is of characteristic 0.

Consider polynomials of the type

$$P(x_1, \cdots, x_m) = \sum_{\substack{0 \leqq j_\mu \leqq r_\mu \\ (1 \leqq \mu \leqq m)}} C(j_1, \cdots, j_m) x_1^{j_1} \cdots x_m^{j_m}$$

in m indeterminates x_μ $(1 \leqq \mu \leqq m)$ with coefficients $C(j_1, \cdots, j_m)$ in $K[t]$. We define

$$H(P) = \max |C(j_1, \cdots, j_m)|$$

and write

$$P_{i_1 \cdots i_m} = \left(\prod_{\mu=1}^{m} \frac{1}{i_\mu !} \frac{\partial^{i_\mu}}{\partial x_x^{i_\mu}} \right) P$$

for any non-negative integers i_μ $(1 \leqq \mu \leqq m)$. We shall say that P has the index I at $(\alpha_1, \cdots, \alpha_m)$ with respect to $(s_1, \cdots, s_m)$, where $\alpha_1, \cdots, \alpha_m$ are any elements algebraic over $K(t)$ and $s_1, \cdots, s_m$ are positive integers, if I is the least value of

$$\sum_{\mu=1}^{m} \frac{i_\mu}{s_\mu}$$

for which

$$P_{i_1 \cdots i_m}(\alpha_1, \cdots, \alpha_m) \neq 0 .$$

Clearly such $i_1, \cdots, i_m$ exist except when P vanishes identically.

Now let $r_1, \cdots, r_m$ be positive integers, $B \geqq 1$. We consider the set

$$M_m = M_m(B ; r_1, \cdots, r_m)$$

of polynomials $P(x_1, \cdots, x_m)$ satisfying the conditions:

(a) P has coefficients in $K[t]$ and is not identically zero;

(b) P is of degree at most r_μ in x_μ $(1 \leqq \mu \leqq m)$;

(c) $H(P) \leqq B$.

Let $p_1 = p_1(t), \cdots, p_m = p_m(t)$, $q_1 = q_1(t), \cdots, q_m = q_m(t)$ be any polynomials of $K[t]$ such that $q_\mu \neq 0$, $(p_\mu, q_\mu) = 1$ $(1 \leqq \mu \leqq m)$. Let $I(P)$ denote the index of P at $(p_1/q_1, \cdots, p_m/q_m)$ with respect to $(r_1, \cdots, r_m)$. We define

$$I_m(B ; h_1, \cdots, h_m ; r_1, \cdots, r_m) = \sup I(P) ,$$

the supremum being taken over all P in M_m and all $(p_1/q_1, \cdots, p_m/p_m)$ with $|q_1| = h_1, \cdots, |q_m| = h_m$ $(|p_1, q_1| = h_1, \cdots, |p_m, q_m| = h_m)$, where $h_\mu \geqq 1$ $(1 \leqq \mu \leqq m)$.

Lemma 1. We have

$$I_1(B\,;h_1\,;r_1)\leqq\frac{\log B}{r_1\log h_1}.$$

Let $P(x_1)$ be a polynomial in M_1 and let $p_1, q_1\neq 0$ be any elements of $K[t]$ with $|q_1|=h_1$ ($|p_1, q_1|=h_1$). If I is the index of P at (p_1/q_1) with respect to (r_1), then we have

$$\mathrm{P}(x_1)=(q_1x_1-p_1)^{Ir_1}Q(x_1),$$

where Q is a polynomial in x_1 with coefficients in $K[t]$ since $(p_1, q_1)=1$. It follows that

$$H(P)\geqq|p_1, q_1|^{Ir_1}\geqq h_1^{Ir_1},$$

whence the required result.

After the manner of Roth's method [5], we can prove, using generalized Wronskians defined over $K(t)$, the following inductive lemma:

Lemma 2. Let $2\leqq\mu\leqq m$ and let $r_1, \cdots, r_\mu$ be positive integers such that

$$r_{j-1}/r_j>\delta^{-1}\qquad(2\leqq j\leqq\mu),$$

where $0<\delta<1$. Then

$$I_\mu(B\,;h_1, \cdots, h_\mu\,;r_1, \cdots, r_\mu)\leqq 2\max(\Phi+\Phi^{\frac{1}{2}}+\delta^{\frac{1}{2}}),$$

where the maximum is taken over integers l satisfying

$$1\leqq l\leqq r_\mu+1,$$

and where

$$\Phi=I_1(B^l\,;h_\mu\,;lr_\mu)+I_{\mu-1}(B^l\,;h_1, \cdots, h_{\mu-1}\,;lr_1, \cdots, lr_{\mu-1}).$$

Lemma 3. Let m be a positive integer and let δ satisfy

$$0<\delta<1.$$

Let $r_1, \cdots, r_m$ be positive integers satisfying

$$r_{j-1}/r_j>\delta^{-1}\qquad(2\leqq j\leqq m).$$

Let $h_1, \cdots, h_m$ be positive numbers satisfying

$$r_j\log h_j\geqq r_1\log h_1\qquad(2\leqq j\leqq m).$$

Then

$$I_m(h_1^{\delta r_1}\,;\,h_1, \cdots, h_m\,;r_1, \cdots, r_m)<\eta,$$

where

$$\eta=\eta(m,\delta)=7^m\delta^{2^{-m}}.$$

For $m=1$ the result follows at once from Lemma 1. Suppose that $\mu\geqq 2$ is an integer and that the present lemma holds for $m=\mu-1$. We have, by Lemma 1 again,

$$I_1(h_1^{\delta lr_1}; h_\mu : lr_\mu)<\delta$$

and, using the induction hypothesis,

$$I_{\mu-1}(h_1^{\delta lr_1}; h_1 \cdots, h_{\mu-1}; lr_1, \cdots, lr_{\mu-1})<\eta(\mu-1,\delta).$$

Hence

$$\Phi<\delta+\eta(\mu-1,\delta)<2\eta(\mu-1,\delta).$$

It now follows from Lemma 2 that

$$\begin{aligned}&I_\mu(h_1^{\delta r_1}; h_1, \cdots, h_\mu; r_1, \cdots, r_\mu)\\ &\leqq 2(2\eta(\mu-1,\delta)+2^{\frac{1}{2}}\eta(\mu-1,\delta)^{\frac{1}{2}}+\delta^{\frac{1}{2}})\\ &\leqq 2\left(\frac{2}{7}+\frac{2^{\frac{1}{2}}}{7^{\frac{3}{2}}}+\frac{1}{7^2}\right)\cdot\eta(\mu,\delta)\\ &<\eta(\mu,\delta).\end{aligned}$$

This completes the induction.

Lemma 4. For any positive integers $r_1, \cdots, r_m$ and a real number $\lambda>0$ the number of sets of integers $i_1, \cdots, i_m$ such that

$$\sum_{\mu=1}^{m}\frac{i_\mu}{r_\mu}\leqq\frac{1}{2}(m-\lambda),\quad 0\leqq i_\mu\leqq r_\mu\quad(1\leqq\mu\leqq m)$$

is at most

$$(2m)^{\frac{1}{2}}\lambda^{-1}(1+r_1)\cdots(1+r_m).$$

This is a slightly sharpened form for the corresponding lemma of Roth [5, Lemma 8], a very simple proof of which is given by J. W. S. Cassels [1].

Lemma 5. (i) Let $\alpha=\alpha(t)$ be an element of $K\langle t^{-1}\rangle$ satisfying the equation

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n=0 \qquad (a_0\neq 0), \tag{7}$$

where $a_0, a_1, \cdots, a_n$ are polynomials of $K[t]$. Then

$$|\alpha|\leqq H(f).$$

(ii) Let $\alpha=\alpha(t)$ be an element of $K\langle\tau\rangle$ satisfying the equation (7). Then

$$|\alpha|_\tau \leqq H(f).$$

More generally, we have, if $\alpha_j=\alpha_j(t)\in K\langle\tau_j\rangle$, $f(\alpha_j)=0$ $(1\leqq j\leqq s)$,

$$\prod_{j=1}^{s} \max(1, |\alpha_j|_{\tau_j}) \leqq H(f).$$

where τ_j $(1\leqq j\leqq s)$ are distinct primary irreducible polynomials in $K[t]$.

We may suppose that $\alpha\neq 0$ since otherwise there is nothing to prove. From the relation

$$a_0\alpha=-(a_1+a_2\alpha^{-1}+\cdots+a_n\alpha^{-n+1})$$

we find that

$$|a_0||\alpha|\leqq\max(|a_1|,\cdots,|a_n|)\leqq H(f),$$

if $|\alpha|>1$. Hence, for $|\alpha|>1$,

$$|\alpha|\leqq\frac{H(f)}{|a_0|}\leqq H(f).$$

This inequality is obviously true also for $|\alpha|\leqq 1$.

Similarly we find that, if $|\alpha|_\tau>1$,

$$|a_0|_\tau |\alpha|_\tau\leqq\max(|a_1|_\tau,\cdots,|a_n|_\tau)\leqq 1,$$

whence

$$|\alpha|_\tau\leqq\frac{1}{|a_0|_\tau}\leqq|a_0|\leqq H(f),$$

and this inequality also holds if $|\alpha|_\tau\leqq 1$.

6. Construction of approximation polynomials. Let $\alpha=\alpha(t)\neq 0$ be an integral algebraic function of degree n over $K(t)$, i.e. one which satisfies an algebraic equation

$$f(x)=0,$$

where

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n \qquad (a_0=1)$$

is an irreducible polynomial with coefficients in $K[t]$.

Put

$$c_3=H(f).$$

Let $p_1=p_1(t),\cdots,p_m=p_m(t)$, $q_1=q_1(t),\cdots,q_m=q_m(t)$ be any elements of $K[t]$ such that $q_\mu\neq 0$, $(p_\mu,q_\mu)=1$ $(1\leqq\mu\leqq m)$ and $|q_1|=h_1,\cdots,|q_m|=h_m$ $(|p_1,q_1|=h_1,\cdots,|p_m,q_m|=h_m)$, where $h_\mu\geqq 1$ $(1\leqq\mu\leqq m)$. Suppose that the numbers $m,\delta,h_1,\cdots,h_m,r_1,\cdots,r_m$ satisfy the following conditions:

$$0<\delta<1\,, \tag{8}$$

$$2\eta(m,\delta)+(1+2\delta)n(2m)^{\frac{1}{2}}<m\,, \tag{9}$$

$$r_{j-1}/r_j>\delta^{-1} \qquad (2\leqq j\leqq m)\,, \tag{10}$$

$$\log h_1>\delta^{-2}(\log c+m\log c_3)\,, \tag{11}$$

$$r_j\log h_j\geqq r_1\log h_1 \qquad (2\leqq j\leqq m)\,. \tag{12}$$

We set

$$\lambda=(1+2\delta)n(2m)^{\frac{1}{2}}\,,$$

$$\gamma=\frac{1}{2}(m-\lambda)\,,$$

$$B_1=h_1^{\delta r_1}\,.$$

Lemma 6. If the conditions (8), (9), (10), (11) and (12) are satisfied, then there exists a polynomial

$$Q(x_1,\cdots,x_m)$$

in $M_m^*=M_m(B_1\,;r_1,\cdots,r_m)$ such that

(a) the index of Q at $(\alpha,\alpha,\cdots,\alpha)$ with respect to $(r_1,\cdots,r_m)$ is at least $\gamma-\eta$;

(b) $Q(p_1/q_1,\cdots,p_m/q_m)\neq 0$;

(c) for any non-negative integers $i_1,\cdots,i_m$ we have

$$|Q_{i_1\cdots i_m}(\alpha,\cdots,\alpha)|\leqq B_1^{1+\delta} \quad \text{if } \alpha\in K\langle t^{-1}\rangle\,,$$

$$|Q_{i_1\cdots i_m}(\alpha,\cdots,\alpha)|_\tau\leqq B_1^{\delta} \quad \text{if } \alpha\in K\langle \tau\rangle\,.$$

To prove this lemma, consider a general polynomial

$$P(x_1,\cdots,x_m)=\sum_{\substack{0\leqq j_\mu\leqq r_\mu\\(1\leqq\mu\leqq m)}}C(j_1,\cdots,j_m)x_1^{j_1}\cdots x_m^{j_m}$$

in M_m^*. Then each of the coefficients $C(j_1,\cdots,j_m)$, as a polynomial in t, possesses exactly

$$1+\left[\frac{\log B_1}{\log c}\right]$$

distinct terms. Hence the total number N of coefficients, whose values being in K, in the polynomials $C(j_1,\cdots,j_m)$ $(0\leqq j_\mu\leqq r_\mu,\ 1\leqq\mu\leqq m)$ is equal to

$$(1+r_1)\cdots(1+r_m)\left(1+\left[\frac{\log B_1}{\log c}\right]\right).$$

Next, the number of derivatives

$$P_{i_1\cdots i_m}(x_1, \cdots, x_m)\,,$$

where

(13) $$\sum_{\mu=1}^{m}\frac{i_\mu}{r_\mu}\leqq\gamma\,,\quad 0\leqq i_\mu\leqq r_\mu \qquad (1\leqq\mu\leqq m)\,,$$

does not exceed, by Lemma 4, the bound

$$(2m)^{\frac{1}{2}}\lambda^{-1}(1+r_1)\cdots(1+r_m)\,.$$

For each set of integers $i_1, \cdots, i_m$ satisfying (13) we form the polynomial $P_{i_1\cdots i_m}(x\cdots x)$ in the single indeterminate x and then devide this polynomial by $f(x)$, obtaining the remainder

$$R(i_1, \cdots, i_m\,; x)=\sum_{j=0}^{n-1}C_j x^j\,.$$

The coefficients C_j are linear combinations of the $C(j_1, \cdots, j_m)$ with coefficients in $K[t]$. It is easy to see that the C_j are, as polynomials in $K[t]$, of degree at most

$$\frac{\log c_3^{mr_1}B_1}{\log c}<(1+\delta)\frac{\log B_1}{\log c}\,.$$

It follows that the total number of such coefficients of the C_j in $R(i_1, \cdots, i_m\,; x)$ for all sets of integers $i_1, \cdots, i_m$ satisfying (13) does not exceed

$$(2m)^{\frac{1}{2}}\lambda^{-1}(1+r_1)\cdots(1+r_m)n(1+2\delta)\frac{\log B_1}{\log c}\,,$$

which is less than N by the definition of λ, since

$$\frac{\log B_1}{\log c}<1+\left[\frac{\log B_1}{\log c}\right].$$

Thus we conclude that there exists a polynomial P in M_m^* such that

$$P_{i_1\cdots i_m}(\alpha, \cdots, \alpha)=0$$

for all sets of integers $i_1, \cdots, i_m$ satisfying (13): in other words, the index of P at $(\alpha, \cdots, \alpha)$ with respect to $(r_1, \cdots, r_m)$ is at least γ. The polynomial P being a member of M_m^*, there exists, by Lemma 3, a derivative

$$Q(x_1, \cdots, x_m)=P_{j_1\cdots j_m}(x_1, \cdots, x_m)$$

with

$$\sum_{\mu=1}^{m}\frac{j_\mu}{r_\mu}<\eta$$

such that

$$Q(p_1/q_1, \cdots, p_m/q_m)\neq 0\,.$$

The index of Q at $(\alpha, \cdots, \alpha)$ with respect to $(r_1, \cdots, r_m)$ is at least $\gamma-\eta$. Thus the polynomial Q satisfies the conditions (a) and (b) of Lemma 6. To verify that Q satisfies the condition (c) as well is immediate. Proof of Lemma 6 is now complete.

7. Proof of Theorem 2, (i). First we prove the following

Lemma 7. Let $\alpha=\alpha(t)$ be an arbitrary element of $K\langle t^{-1}\rangle$ and let $p_i=p_i(t)$, $q_i=q_i(t)\neq 0$ $(i=1, 2)$ be any polynomials in $K[t]$ such that $p_1/q_1 \neq p_2/q_2$, $|q_1|=|q_2|$. Then for each $\kappa>2$,

$$\left|\alpha-\frac{p_1}{q_1}\right|<|q_1|^{-\kappa} \quad \text{implies} \quad \left|\alpha-\frac{p_2}{q_2}\right|\geqq|q_2|^{-\kappa}.$$

If not, we would have

$$|q_1|^{-2}\leqq\left|\frac{p_1}{q_1}-\frac{p_2}{q_2}\right|=\left|\left(\alpha-\frac{p_1}{q_1}\right)-\left(\alpha-\frac{p_2}{q_2}\right)\right|<|q_1|^{-\kappa},$$

which is impossible since $\kappa>2$.

Now, let $\alpha=\alpha(t)\neq 0$ be an element of $K\langle t^{-1}\rangle$ algebraic of degree n over $K(t)$. Suppose that Theorem 2, (i) is false, so that for some $\kappa>2$, the inequality (5) has infinitely many solutions $p=p(t), q=q(t)\neq 0$ in $K[t]$ with $(p, q)=1$. Denote by E the set of all such solutions (p, q) of (5). It follows from Lemma 7 that $|q|$ is not bounded when (p, q) runs through the elements of E, and so we may suppose that α is an integral algebraic function. For, if not, there is a (non-zero) polynomial $a=a(t)$ in $K[t]$ such that $a\alpha$ is an integral algebraic function, and for arbitrary $\varepsilon>0$ and for all (p, q) in E with sufficiently large $|q|$

$$0<\left|a\alpha-\frac{ap}{q}\right|<|a|\cdot|q|^{-\kappa}<|q|^{-\kappa+\varepsilon},$$

where ε can be chosen so small that $\kappa-\varepsilon>2$.

We take an integer m so large that $m>n(2m)^{\frac{1}{2}}$ and

$$\frac{2m}{m-n(2m)^{\frac{1}{2}}}<\kappa,$$

which is possible since $\kappa>2$. Let δ be a sufficiently small positive number satisfying the conditions (8) and (9), and the inequality

$$\frac{2m(1+\delta)+2\delta(1+\delta)}{m-(1+2\delta)n(2m)^{\frac{1}{2}}-2\eta}<\kappa,$$

which is equivalent to

(14) $$\frac{m(1+\delta)+\delta(1+\delta)}{\gamma-\eta}<\kappa .$$

We now choose a solution (p_1, q_1) from E with $|q_1|=h_1$ so large as to satisfy (11). We then choose further solutions (p_j, q_j) $(2\leqq j\leqq m)$ from E such that $|q_j|=h_j$ $(2\leqq j\leqq m)$, where

$$\frac{\log h_j}{\log h_{j-1}}>\frac{2}{\delta} \qquad (2\leqq j\leqq m) .$$

Let r_1 be any integer such that

$$r_1>\frac{\log h_m}{\delta \log h_1}$$

and define r_j $(2\leqq j\leqq m)$ by

$$\frac{r_1 \log h_1}{\log h_j}\leqq r_j<\frac{r_1 \log h_1}{\log h_j}+1 .$$

Then the condition (12) is satisfied. Also, for $2\leqq j\leqq m$,

$$\frac{r_j \log h_j}{r_1 \log h_1}<1+\frac{\log h_j}{r_1 \log h_1}\leqq 1+\frac{\log h_m}{r_1 \log h_1}<1+\delta ,$$

whence

$$\frac{r_{j-1}}{r_j}>\frac{\log h_j}{\log h_{j-1}}(1+\delta)^{-1}>\delta^{-1}$$

and the condition (10) is satisfied. Hence there exists a polynomial $Q(x_1, \cdots, x_m)$ in M_m^* with the properties listed in Lemma 6.

On one hand, we have

$$|Q(q_1/q_1, \cdots, p_m/q_m)|\geqq h_1^{-r_1} \cdots h_m^{-r_m}>h_1^{-mr_1(1+\delta)} .$$

On the other hand, we find that

$$Q(p_1/q_1, \cdots, p_m/q_m)=\sum_{i_1=0}^{r_1} \cdots \sum_{i_m=0}^{r_m} Q_{i_1 \cdots i_m}(\alpha, \cdots, \alpha)\cdot$$
$$(p_1/q_1-\alpha)^{i_1} \cdots (p_m/q_m-\alpha)^{i_m} ,$$

whence

$$|Q(p_1/q_1, \cdots, p_m/q_m)|\leqq B_1^{1+\delta} \max (h_1^{i_1} \cdots h_m^{i_m})^{-\kappa} ,$$

where the maximum is taken over all integers $i_1, \cdots, i_m$ satisfying the inequalities

$$\sum_{\mu=1}^{m} \frac{i_\mu}{r_\mu}\geqq\gamma-\eta , \quad 0\leqq i_\mu\leqq r_\mu \qquad (1\leqq\mu\leqq m) .$$

Thus

$$\max(h_1^{i_1}\cdots h_m^{i_m})^{-\kappa}=\max\{h_1^{i_1/r_1}\cdots(h_m^{r_m/r_1})^{i_m/r_m}\}^{-r_1\kappa}$$
$$\leqq\max(h_1^{i_1/r_1}\cdots h_1^{i_m/r_m})^{-r_1\kappa}$$
$$\leqq h_1^{-r_1(\gamma-\eta)\kappa},$$

and so

$$|Q(p_1/q_1,\cdots,p_m/q_m)|\leqq h_1^{\delta(1+\delta)r_1-r_1(\gamma-\eta)\kappa}.$$

Combining these estimates for $Q(p_1/q_1,\cdots,p_m/q_m)$, we obtain

$$h_1^{-r_1m(1+\delta)}\leqq h_1^{\delta(1+\delta)r_1-r_1(\gamma-\eta)\kappa},$$

or

$$\kappa\leqq\frac{m(1+\delta)+\delta(1+\delta)}{\gamma-\eta},$$

which contradicts (14). This completes the proof of Theorem 2, (i).

8. Proof of Theorem 2, (ii). We require the following

Lemma 8. Let $\alpha=\alpha(t)$ be an arbitrary element of $K\langle\tau\rangle$ and let $p_i=p_i(t)$, $q_i=q_i(t)$ $(i=1,2)$ be any polynomials in $K[t]$ such that $p_1q_2-p_2q_1\neq 0$, $|p_1,q_1|=|p_2,q_2|$. Then for each $\kappa>2$,

$$|p_1-q_1\alpha|_\tau<|p_1,q_1|^{-\kappa}\quad\text{implies}\quad|p_2-q_2\alpha|_\tau\geqq|p_2,q_2|^{-\kappa}.$$

If not, we would have

$$|p_1,q_1|^{-2}\leqq|p_1q_2-p_2q_1|_\tau=|(p_1-q_1\alpha)q_2-(p_2-q_2\alpha)q_1|_\tau<|p_1,q_1|^{-\kappa},$$

which is impossible since $\kappa>2$.

Now, let $\alpha=\alpha(t)\neq 0$ be any element of $K\langle\tau\rangle$ algebraic of degree n over $K(t)$. Suppose that Theorem 2, (ii) is false, so that for some $\kappa>2$, the inequality (6) has infinitely many solutions $p=p(t)$, $q=q(t)$ in $K[t]$ with $(p,q)=1$. Denote by M the set of all such solutions (p,q) of (6). It follows from Lemma 8 that $|p,q|$ is not bounded when (p,q) runs through the elements of M, and so we may suppose again that α is an integral algebraic function. For, if not, there is a (non-zero) polynomial $a=a(t)$ in $K[t]$ such that $a\alpha$ is an integral algebraic function, and for arbitrary $\varepsilon>0$ and for all (p,q) in M with sufficiently large $|p,q|$

$$0<|ap-q(a\alpha)|_\tau<|a|_\tau|p,q|^{-\kappa}\leqq|a|^\kappa|ap,q|^{-\kappa}<|ap,q|^{-\kappa+\varepsilon},$$

where ε can be chosen so small that $\kappa-\varepsilon>2$.

The rest of the proof of Theorem 2, (ii) is quite similar to that of (i). We take m and δ to satisfy the conditions (8) and (9) and the inequality (14). We then choose solutions $(p_1,q_1),\cdots,(p_m,q_m)$ from M with $|p_j,q_j|=h_j$ $(1\leqq j\leqq m)$ and define $r_1,\cdots,r_m$ as in §7. The conditions for

Lemma 6 are all satisfied, and so there exists a polynomial $Q(x_1, \cdots, x_m)$ with the properties stated there. We have on one hand

$$|q_1^{r_1} \cdots q_m^{r_m} Q(p_1/q_1, \cdots, p_m/q_m)|_\tau \geqq B_1^{-1} h_1^{-r_1} \cdots h_m^{-r_m}$$
$$> h_1^{-\delta r_1 - r_1 m(1+\delta)},$$

and on the other hand

$$|q_1^{r_1} \cdots q_m^{r_m} Q(p_1/q_1, \cdots, p_m/q_m)|_\tau \leqq B_1^{\delta} \max (h_1^{i_1} \cdots h_m^{i_m})^{-\kappa}$$
$$\leqq h_1^{\delta^2 r_1 - r_1 (\gamma - \eta)\kappa},$$

as in §7. Thus we find that

$$h_1^{-\delta r_1 - r_1 m(1+\delta)} \leqq h_1^{\delta^2 r_1 - r_1(\gamma-\eta)\kappa},$$

which again contradicts (14), completing the proof of Theorem 2, (ii).

9. Proof of Theorem 3. Let K be an arbitrary field of characteristic 0.

First we prove the part (ii). Let

$$\alpha = \sum_{i=0}^{\infty} c_i \tau^{i-l} \qquad (c_j \in K[t], \deg c_i < \deg \tau)$$

be any element of $K\langle\tau\rangle$, not belonging to $K(t)$. We may suppose without loss of generality that $l=0$. We wish to show that, given non-negative integers d_1, d_2, there exist polynomials $p=p(t)$, $q=q(t)\neq 0$ in $K[t]$ with $|p, q|>0$ such that

$$p = \sum_{j=0}^{d_1} a_j \tau^j \qquad (a_j \in K[t], \deg a_j < \deg \tau),$$

$$q = \sum_{k=0}^{d_2} b_k \tau^k \qquad (b_k \in K[t], \deg b_k < \deg \tau),$$

and $\alpha - p/q$, as an element of $K\langle\tau\rangle$, does not contain the first d_1+d_2+1 terms in it. This follows from the fact that every linear homogeneous equations with coefficients in K with unknowns more than the equations in number has always a non-trivial solution in K. For instance, if τ is a linear polynomial in $K[t]$, then the coefficients c_i, a_j, b_k lie in K and we must solve the equations

$$a_j = b_0 c_j + b_1 c_{j-1} + \cdots + b_j c_0 \qquad (0 \leqq j \leqq d_1), \tag{15}$$

$$0 = b_0 c_{k+d_1} + b_1 c_{k+d_1-1} + \cdots + b_{d_2} c_{k+d_1-d_2} \qquad (1 \leqq k \leqq d_2), \tag{16}$$

where we put $c_i = 0$ for $i<0$. The system (16), consisting of d_2 linear homogeneous equations in d_2+1 unknowns, has a non-trivial solution

$b_0, b_1, \cdots, b_{d_2}$ in K. We then determine $a_0, a_1, \cdots, a_{d_1}$ by the relations (15). The general case where τ is not necessarily linear can be treated by a similar but somewhat more complicated arguments. This proves Theorem 3, (ii).

To prove the part (i), let

$$\alpha = \sum_{i=0}^{\infty} c_i t^{l-i} \qquad (c_i \in K)$$

be any element of $K\langle t^{-1}\rangle$, not belonging to $K(t)$. Again, there is no loss in generality in supposing that $l=0$. For a prescribed non-negative integer d, put

$$p = \sum_{j=0}^{d} a_j t^j \qquad (a_j \in K),$$

$$q = \sum_{k=0}^{d} b_k t^k \qquad (b_k \in K).$$

We see that

$$\frac{p}{q} = \frac{\sum_0^d a_j t^{-j}}{\sum_0^d b_k t^{-k}}.$$

Hence, we can determine, just as in the above, the coefficients $a_0, a_1, \cdots, a_d, b_0, b_1, \cdots, b_d$ of p, q in such a way that $q \neq 0$, and $\alpha - p/q$, as an element of $K\langle t^{-1}\rangle$, does not contain the first $2d+1$ terms in it.

Theorem 3 is thus completely proved.

10. Further results. Let K be an arbitrary field of characteristic 0. In this section we wish to note some partial refinements of Theorem 2.

The following theorem is an analogue of a result of D. Ridout [4].

Theorem 4. *Let $a=a(t)\neq 0$ be any element of $K\langle t^{-1}\rangle$ algebraic over $K(t)$. Let $P_1=P_1(t), \cdots, P_m=P_m(t)$, $Q_1=Q_1(t), \cdots, Q_n=Q_n(t)$ be a finite set of distinct irreducible polynomials in $K[t]$. Let μ, ν, C be real numbers satisfying*

$$0 \leqq \mu \leqq 1, \quad 0 \leqq \nu \leqq 1, \quad C > 0.$$

Let $p=p(t)$, $q=q(t)$ be restricted to polynomials in $K[t]$ of the form

$$p = p^* P_1^{a_1}, \cdots, P_m^{a_m}, \quad q = q^* Q_1^{b_1}, \cdots, Q_n^{b_n},$$

where $a_1, \cdots, a_m$, $b_1, \cdots, b_n$ are non-negative integers and $p^*=p^*(t)$ are polynomials in $K[t]$ such that

$$0<|p^*|<C|p|^{\mu}, \quad 0<|q^*|<C|q|^{\nu}.$$

Then if $\kappa>\mu+\nu$, *there exists a natural number* N *depending only on* $\alpha, \mu, \nu, C, P_1, \cdots, P_m, Q_1, \cdots, Q_n$, *such that*

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{|q|^{\kappa}}$$

has no solution p, q *in* $K[t]$ *with* $(p, q)=1$ *and*

$$\max(\deg p, \deg q)>N.$$

We can prove this theorem in almost the same way as in the proof of Theorem 2, (i), on the basis of a slightly modified form of Lemma 6.

As to the mixed approximation to algebraic functions by rational functions, we obtain:

Theorem 5. *Suppose that the equation*

$$a_0x^n+a_1x^{n-1}+\cdots+a_n=0 \qquad (a_0a_n\neq 0),$$

where $a_i=a_i(t)\in K[t]$ $(0\leqq i\leqq n)$, *has a root* $\alpha=\alpha(t)$ *in* $K\langle t^{-1}\rangle$, *a root* $\alpha_1=\alpha_1(t)$ *in* $K\langle\tau_1\rangle, \cdots,$ *a root* $\alpha_s=\alpha_s(t)$ *in* $K\langle\tau_s\rangle$, $\tau_1, \cdots, \tau_s$ *being distinct primary irreducible polynomials in* $K[t]$. *Then, if* $\kappa>2$, *there exists a natural number* N *depending only on* $a_0, a_1, \cdots, a_n, \tau_1, \cdots, \tau_s, \kappa$, *such that the inequality*

$$\min\left(1, \left|\alpha-\frac{p}{q}\right|\right)\prod_{j=1}^{s}\min(1, |p-q\alpha_j|_{\tau_j})<|p, q|^{-\kappa}$$

has no solution $p=p(t)$, $q=q(t)\neq 0$ *in* $K[t]$ *with* $(p, q)=1$ *and*

$$\max(\deg p, \deg q)>N.$$

This is a partial generalization of Theorem 2 and its proof can be carried out in a similar manner, making use of Lemmas 5 and 6.

11. Applications. Again, let K denote a field of characteristic 0.

As an easy application of Theorem 2 we may mention the following

Theorem 6. *Let* $F(x, y)$ *be a binary form of degree* $n\geqq 3$, *without multiple factors, whose coefficients belong to* $K[t]$. *Let* $G(x, y)$ *be any polynomial of total degree* $<n-2$ *with coefficients in* $K[t]$ *which has no common factor with* $F(x, y)$. *Then there exists an integer* $N>0$ *depending only on* F *and* G, *such that the equation*

$$F(x, y)=G(x, y)$$

has no solution $x=x(t)$, $y=y(t)$ *in* $K[t]$ *with* $(x, y)=1$ *and*

$$\max(\deg x, \deg y)>N.$$

To prove this, we apply Theorem 2, (i), taking account of an extended valuation of $|\ \ |$ on $K(t)$ to an appropriate finite algebraic extension over $K(t)$.

The following theorem is an immediate consequence of Theorem 5:

Theorem 7. *Let $F(x, y)$ be a binary form of degree $n\geqq 3$, without multiple factors, whose coefficients belong to $K[t]$. Let $\tau_1, \cdots, \tau_s$ be distinct primary irreducible polynomials in $K[t]$ and let $H(p, q)$ denote the highest power-product of $\tau_1, \cdots, \tau_s$ which divides $F(p, q)$, where $p=p(t)$, $q=q(t)$ are polynomials in $K[t]$. Then, if $\kappa>2$, there exists an integer $N>0$ depending only on F, $\tau_1, \cdots, \tau_s$ and κ such that the inequality*

$$\left|\frac{F(p, q)}{H(p, q)}\right|<|p, q|^{n-\kappa}$$

has no solution $p=p(t)$, $q=q(t)$ in $K[t]$ with $(p, q)=1$ and

$$\max(\deg p, \deg q)>N .$$

Now, let $\alpha=\alpha(t)$ be an element of $K\langle t^{-1}\rangle$ and write

$$\alpha=\sum_{i=0}^{\infty} c_i t^{l-i} ,$$

where l is a non-negative integer. We put

$$\{\alpha\}=\sum_{i=l+1}^{\infty} c_i t^{l-i} .$$

Then, as an easy consequence of Theorem 4, we obtain

Theorem 8. *Let $\alpha=\alpha(t)\neq 0$ be any element of $K\langle t^{-1}\rangle$ algebraic over $K(t)$. Let $A=A(t)$, $B=B(t)$ be polynomials in $K[t]$ having no factor in common, such that $|A|>|B|>1$, and let ε be an arbitrarily small positive number. Then the inequality*

$$\left|\left\{\alpha\cdot\left(\frac{A}{B}\right)^s\right\}\right|<e^{-\varepsilon s}$$

is satisfied by at most a finite number of positive integers s.

This is an analogue for rational functions of a theorem of Mahler [3]. To prove Theorem 8, apply Theorem 4 with

$$\mu=1-\delta , \quad \nu=0 , \quad C=|\alpha|^{\delta}+1 ,$$

$$\kappa=1-\delta+\frac{1}{2}\varepsilon(\log|A|)^{-1}>\mu+\nu ,$$

where $\delta=\log|B|/\log|A|$, so that $0<\delta<1$. Here $P_1, \cdots, P_m$ and $Q_1, \cdots, Q_n$ are distinct irreducible factors of B and A, respectively, and

$$p^* = \alpha \cdot \left(\frac{A}{B}\right)^s - \left\{\alpha \cdot \left(\frac{A}{B}\right)^s\right\}, \quad q^* = 1 .$$

Note that $|p^*| > 0$ for all sufficiently large s.

References

[1] J. W. S. CASSELS: An introduction to Diophantine approximation, Cambridge Tracts in Math. and Math. Phys., No. 45, Cambridge (1957). Chap. VI, Lemma 5.

[2] K. MAHLER: On a theorem of Liouville in fields of positive characteristic, Canadian J. Math., 1 (1949), 397–400.

[3] ———: On the fractional parts of the powers of a rational number (II), Mathematika, 4 (1957), 122–124.

[4] D. RIDOUT: Rational approximations to algebraic numbers, Mathematika, 4 (1957), 125–131.

[5] K. F. ROTH: Rational approximations to algebraic numbers, Mathematika, 2 (1955), 1–20.

[6] S. UCHIYAMA: On the Thue-Siegel-Roth theorem, III, Proc. Japan Acad., 36 (1960), 1–2.

Department of Mathematics,
Hokkaidô University

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