ON THE HILBERT TRANSFORM II

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CONTENTS

Chapter 1. Relevant theorems

§ 1. Relevant theorems of generalized harmonic analyses. We begin with several notations definitions and theorems which we shall quote from N. Wiener [13].

Definition 1. *We shall say that $f(x)$ belongs to the class W_2, if $f(x)$ is measurable and*

$$\int_{-\infty}^{\infty} \frac{|f(x)|^2}{1+x^2} dx < \infty . \tag{1.01}$$

Definition 2. *We shall say that $f(x)$ belongs to the class S_0, if $f(x)$ is measurable and exists*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(x)|^2 dx . \tag{1.02}$$

It is clear that

$$S_0 \subset W_2 . \tag{1.03}$$

For any function $f(x)$ of the class W_2, the Fourier-Wiener transform $s^f(u)$ is defined, that is

$$s^f(u)=\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right]f(x)\frac{e^{-iux}}{-ix}dx + \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}f(x)\frac{e^{-iux}-1}{-ix}dx. \tag{1.04}$$

Then we have by the Plancherel theorem [5] and the Wiener formula [4, 11].

Theorem A. *Let $f(x)$ be a function of the class S_0. Then*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2\,dt=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^f(u+\varepsilon)-s^f(u-\varepsilon)|^2\,du \tag{1.05}$$

in the sense that if either side of (1.05) *exists, the other exists and has the same value.*

Definition 3. *We shall say that $f(x)$ belongs to the class S, if $f(x)$ is measurable and*

$$\phi^f(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}\,dt \tag{1.06}$$

exists for every x.

Definition 4. *We shall say that $f(x)$ belongs to the class S', if $f(x)$ is measurable and $\phi^f(x)$ defined by* (1.06) *exists for every x and continuous over $(-\infty, \infty)$.*

Then we have

Theorem B. *If $f(x)$ belongs to the class S, then we have*

$$\phi^f(x)=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}|s^f(u+\varepsilon)-s^f(u-\varepsilon)|^2\,du. \tag{1.07}$$

for every x. Conversely let $f(x)$ belong to the class W_2 and that the limit of right hand side of (1.07) *exists for every x. Then $f(x)$ belongs to S and* (1.07) *hlods.*

Theorem C. *If $f(x)$ belongs to the class S', then we have* (1.07) *for every x. Conversely let $f(x)$ belong to the class W_2 and that the limit of right hand side of* (1.07) *exists for every x and continuous over $(-\infty, \infty)$. Then $f(x)$ belongs to S' and* (1.07) *holds.*

There is an important theorem due to N. Wiener,

Theorem D. *If $f(x)$ belongs to the class S, it will belongs to the class S', when and only when*

$$\lim_{A\to\infty}\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|s^f(u+\varepsilon)-s^f(u-\varepsilon)|^2\,du=0. \tag{1.08}$$

We add some theorems

Theorem E. *If $f(x)$ belongs to the class S', then there exists a real and monotone increasing function $\Lambda^f(u)$ such that*

$$(1.09) \qquad \phi^f(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iux}\, d\Lambda^f(u)$$

for every x and

$$(1.10) \qquad \Lambda^f(u)-\Lambda^f(0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi^f(x)\,\frac{e^{-iux}-1}{-ix}\,dx$$

for every u.

Theorem F. *If $f(x)$ belongs to the class S, then conclusions of Theorem E are true for a.e. x in* (1.09) *and every u in* (1.10).

Today we usually attain these theorems through S. Bochner's representation theorem using the notion of positive definite of function and Lévy's inversion formula [8]. However N. Wiener's original proof is also useful. From (1.06) we get

$$(1.11) \qquad |\phi^f(x)| \leqq \phi^f(0)$$

and $\phi^f(x)$ belongs to the class W_2. Therefore the Fourier-Wiener transform of $\phi^f(x)$ is defined for a.e. x. This is real value and can be defined such as to be monotone increasing. We shall denote this by $\sigma^f(u)$. Then we get for the class S'

$$(1.12) \qquad \phi^f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iux}\, d\sigma^f(u)$$

for every x and for the class S, (1.12) is true for a.e. x. Furthermore we get

$$(1.13) \qquad \sigma^f(u)-\Lambda^f(u)=\text{const.}$$

except over a null set. There he also proved

Theorem G. *If $f(x)$ belongs to the class S, then we have*

$$(1.14) \qquad \sigma^f(u)=\underset{\varepsilon\to 0}{\text{l.i.m.}}\left[\psi(\varepsilon)+\frac{2}{2\varepsilon\sqrt{2\pi}}\int_{0}^{u}|s^f(u+\varepsilon)-s^f(u-\varepsilon)|^2\,du\right]$$

over any finite range of u.

§ 2. **Relevant theorems of generalized Hilbert transforms.** For any function of the class W_2, the Hilbert transform does not necessarily exists [7, p. 177]. The introduce of modified definition was asked for this class. In compliance with this request, N. I. Achiezer [1, p. 128] introduced the following modified transform

$$H_1 f = \tilde{f}(x) = (x+i) \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} (-i \operatorname{sign} u)\, \psi(u) e^{ixu}\, du \tag{2.01}$$

where

$$\psi(u) = \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} \frac{f(t)}{t+i} e^{-iut}\, dt\,. \tag{2.02}$$

The autor [7, chapter 4] also introduced another modified one from the same idea which is equivalent for the class W_2. This is defined by the following formula

$$H_1 f = \tilde{f}(x) = \frac{(x+i)}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \frac{dt}{x-t}\,. \tag{2.03}$$

There we proved that this modified tranform well conserves properties which the original one has. These are

Theorem H. *Let $f(x)$ belong to W_2. Then its generalized Hilbert transform $\tilde{f}_1(x)$ exists for a.e. x and belongs to the same class and*

$$\int_{-\infty}^{\infty} \frac{|\tilde{f}_1(x)|^2}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{|f(x)|^2}{1+x^2} dx\,. \tag{2.04}$$

Theorem I. *Under the hypothesis of Theorem H.*

$$H_1^2 f = -f \tag{2.05}$$

for a.e. x.

Let us put

$$C_1(z, f) = \frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \frac{dt}{t-z} \tag{2.06}$$

$$P_1(z, f) = \frac{z+i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \frac{y\, dt}{(t-x)^2+y^2} \tag{2.07}$$

and

$$\tilde{P}_1(z, f) = -\frac{z+i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \frac{(t-x)\, dt}{(t-x)^2+y^2} \tag{2.08}$$

then between these formulas there is a relation

$$2C_1(z, f) = P_1(z, f) + i\tilde{P}_1(z, f)\,. \tag{2.09}$$

Definition 5. $\mathfrak{H}^2(0, \infty)$ *is the class of analytic function $f(z)$ in the half-plane $y>0$ such that*

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 dx < const. \quad (0<y<\infty). \tag{2.10}$$

Then we have furthermore

Theorem J. *Let $g(x)$ belong to the class W_2. If we put*

(2.11) $$f_1(z)=2C_1(z, g), \quad z=x+iy$$

then we have

(2.12) $$\lim_{y\to 0} f_1(z)=g(x)+i\tilde{g}_1(x)$$

for a.e. x as an angular limit. If we put

(2.13) $$f_1(x)=g(x)+i\tilde{g}_1(x)$$

then we have

(2.14) $$f_1(z)=C_1(z, f_1)=P_1(z, f_1)$$

and $f_1(z)/(z+i)$ belongs to the class $\mathfrak{H}^2(0, \infty)$.

Theorem K. *Let $f_1(z)$ be an analytic function such that $f_1(z)/(z+i)$ belongs to $\mathfrak{H}^2(0, \infty)$. Then there exists*

(2.15) $$\lim_{y\to 0} f_1(z) = f_1(x)$$

as an angular limit and $f_1(x)$ belongs to the class W_2. If we write

(2.16) $$\mathfrak{R} f_1(x) = g(x)$$

then

(2.17) $$\mathfrak{I} f_1(x) = \tilde{g}_1(x)$$

and thus

(2.18) $$f_1(x) = g(x)+i\tilde{g}_1(x) .$$

Furthermore $f_1(z)$ is represented by its Cauchy integral (2.06) *and its Poisson integral* (2.07) *respectively and we have*

(2.19) $$\lim_{y\to 0}\int_{-\infty}^{\infty}\frac{|f_1(z)-f_1(x)|^2}{1+x^2}dx=0 .$$

§ 3. Unified theorems of generalized Fourier transforms and generalized Hilbert transforms. Let $g(x)$ belong to the class W_2. Then $g_1(x)$ also does to W_2 and the Fourier-Wiener transform of $\tilde{g}_1(x)$ can be defined. We shall denote this by $s^{\tilde{g}_1}(u)$. Then in the previous paper [7, chap. 5] we have proved

Theorem L. *Let $g(x)$ be real or complex valued function and belong to the class W_2. Then we have for any given positive number $\varepsilon>0$,*
(i) *if $|u|>\varepsilon$, then*

(3.01) $$s^{\tilde{g}_1}(u+\varepsilon)-s^{\tilde{g}_1}(u-\varepsilon)=(-i\operatorname{sign} u)\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}$$

and

(ii) *if* $|u|<\varepsilon$, *then*

(3.02)
$$s^{g_1}(u+\varepsilon)-s^{g_1}(u-\varepsilon)=i\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}+2r_1^g(u+\varepsilon)+2r_2^g(u+\varepsilon),$$

where

(3.03)
$$r_1^g(u)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(t)}{t+i}\;\frac{e^{-iut}-1}{-it}\,dt$$

(3.04)
$$r_2^g(u)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(t)}{t+i}\,e^{-iut}\,dt\,.$$

Here we remark that the limit operation in (3.03) *is taken over the interval* $(-\varepsilon, \varepsilon)$.

Theorem M. *Let* $g(x)$ *be real or complex valued function and belong to the class* W_2. *Let us put*

(3.05)
$$f_1(x) = g(x)+i\tilde{g}_1(x)$$

and let us denote the Fourier-Wiener transform of $f_1(x)$ *by* $s^{f_1}(u)$ *then we have for any given positive number* $\varepsilon>0$,

(i) *if* $|u|>\varepsilon$, *then*

(3.06)
$$s^{f_1}(u+\varepsilon)-s^{f_1}(u-\varepsilon)=(1+\text{sign}\,u)\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}$$

and

(ii) *if* $|u|<\varepsilon$, *then*

(3.07)
$$s^{f_1}(u+\varepsilon)-s^{f_1}(u-\varepsilon)=2ir_1^g(u+\varepsilon)+2ir_2^g(u+\varepsilon),$$

where $r_1^g(u)$ *and* $r_2^g(u)$ *are defined by* (3.03) *and* (3.04) *respectively.*

Setting these results as the base of aruments we have

Theorem N. *Let* $g(x)$ *be real or complex valued function and belong to the class* S_0. *Let us suppose that*

(K_1)
$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du=0$$

and

(K_2) *there exists a constant* a^g *such that*

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{0}^{2\varepsilon}\left|\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(t)}{t+i}\,e^{-iut}\,dt-\sqrt{\frac{\pi}{2}}\,a^g\right|^2du=0\,.$$

Then its generalized Hilbart transform of order 1, $\tilde{g}_1(x)$ *does also to the same class* S_0 *and we have*

(3.08)
$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{g}_1(t)|^2\,dt=|a^g|^2+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)|^2\,dt$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^{\tilde{g}_1}(u+\varepsilon)-s^{\tilde{g}_1}(u-\varepsilon)|^2\,du \tag{3.09}$$

$$=|a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du\,.$$

If $g(x)$ is of real valued, then this limit equals to

$$|a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_{0}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du\,. \tag{3.09$'$}$$

Theorem O. *Let $g(x)$ be a real valued function and belong to S_0. Let us suppose that conditions* (K_1) *and* (K_2) *are satisfied. Then $f_1(x)$ defined by* (3.05) *also does to the same class S_0 and we have*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_1(t)|^2\,dt \tag{3.10}$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)|^2\,dt+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{g}_1(t)|^2\,dt$$

$$=|a^g|^2+2\lim_{T\to\infty}\int_{-T}^{T}|g(t)|^2\,dt$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^{f_1}(u+\varepsilon)-s^{f_1}(u-\varepsilon)|^2\,du \tag{3.11}$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du+\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^{\tilde{g}_1}(u+\varepsilon)-s^{\tilde{g}_1}(u-\varepsilon)|^2\,du$$

$$=|a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_{0}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du\,.$$

We observe that if $g(x)$ is of real valued, then we have

$$s^g(-u+\varepsilon)-s^g(-u-\varepsilon)=\overline{s^g(u+\varepsilon)-s^g(u-\varepsilon)}\,. \tag{3.12}$$

Theorem P. *Let $g(x)$ be a real or complex valued function and belong to the class S. Let us suppose that conditions* (K_1) *and* (K_2) *are satisfied. Then $\tilde{g}_1(x)$ does also to the same class S and we have*

$$\phi^{\tilde{g}_1}(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{g}_1(x+t)\overline{\tilde{g}_1(t)}\,dt \tag{3.13}$$

$$=|a^g|^2+\phi^g(x)$$

$$=|a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du\,.$$

If $g(x)$ is of real valued, then this limit equals to

$$(3.13)' \qquad |a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_0^\infty \cos ux\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du\,.$$

Theorem Q. *Let $g(x)$ be real or complex valued function and belong to the class S'. Let us suppose that conditions* (K_1) *and* (K_2) *are satisfied. Then $\tilde{g}_1(x)$ does also to the same class S' and we have* (3.13). *In particular if $g(x)$ is of real valued, this limit equals to* (3.13)$'$.

Now it is natural to ask the following proposition

Let $g(x)$ be a real valued function of the class S. Let us suppose that conditions (K_1) and (K_2) are satisfied. Then $f_1(x)$ defined by (3.05) also does to the same class S and we have

$$(3.14) \qquad \phi^{f_1}(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(x+t)\overline{f_1(t)}\,dt$$
$$=|a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_0^\infty e^{iux}\,|s^g(u+s)-s^g(u-\varepsilon)|\,du\,.$$

This is an open question (c.f. [7, chap. 5, Theorems 56 and 57]). But we shall obtain

Theorem 1. *Let $g(x)$ be a real valued function of the class S'. Let us suppose that conditions* (K_1) *and* (K_2) *are satisfied. Then $f_1(x)$ defined by* (3.05) *also does to the same class S' and we have* (3.14).

Proof of Theorem 1. It is enough to prove that the existence of the following limit

$$(3.15) \qquad \lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_0^\infty \cos ux\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du$$

contains that of

$$(3.16) \qquad \lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_0^\infty \sin ux\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du\,,$$

and this is continuous at $x=0$. By the existence of Theorem D, it is enough to prove that

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_0^A \sin ux\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du$$

exists for appropriate A's belonging to an indefinitely increasing sequence. This is obtained from Theorem G and the Paley-Wiener lemma ([10, pp. 134–5]). The continuity of this limit function at $x=0$ is obvious.

Let $f(x)$ be almost periodic function in a sense of Besicovitch of order 2. We shall denote this by B_2-almost periodic function. Let us put

$$M_2(|f(t)|)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|^2\,dt\,. \tag{3.17}$$

Then this asserts that to every $\eta>0$ there corresponds a relatively dense set of real number τ such that

$$M_2(|f(t+\tau)-f(t)|)<\eta\,. \tag{3.18}$$

It is equivalent that to every $\eta>0$ there corresponds the Bochner-Fejér polynomial

$$\sigma^f_{B_p}(x)=\sum d^B_n c_n e^{i\lambda_n x} \tag{3.19}$$

or

$$\sigma^f_{\binom{n_1,n_2,\cdots,n_p}{\beta_1,\beta_2,\cdots,\beta_p}}(x)=\sum\Big(1-\frac{|\nu_1|}{n_1}\Big)\Big(1-\frac{|\nu_2|}{n_2}\Big)\cdots\Big(1-\frac{|\nu_p|}{n_p}\Big)c_n e^{i\lambda_n x} \tag{3.20}$$

where

$$\lambda_n=\frac{\nu_1}{n_1}\beta_1+\frac{\nu_2}{n_2}\beta_2+\cdots+\frac{\nu_p}{n_p}\beta_p\,, \tag{3.21}$$

such that

$$M_2(|f(t)-\sigma^f_{B_p}(t)|)<\eta\,. \tag{3.22}$$

Then we have in the previous paper [7, chap, 5].

Theorem R. *Let $g(x)$ be a real or complex valued measurable function over $(-\infty,\infty)$. Let us suppose that the condition* (K₁) *is satisfied. Then the necessary and sufficient condition* (K₁) *for $\tilde{g}_1(x)$ to be B_2-almost periodic is that the condition* (K₂) *is satisfied for the constant term of $\tilde{g}_1(x)$. If the associated Fourier series of $g(x)$ is*

$$g(x)\sim\sum{}' c_n e^{i\lambda_n x} \tag{3.23}$$

then

$$\tilde{g}_1(x)\sim a^g+\sum{}'(-i\,\mathrm{sign}\,\lambda_n)c_n e^{i\lambda_n x}\,, \tag{3.24}$$

where the prime means that the summation does not contain the constant term.

In almost periodic functions, the three function $g(x)$, $\tilde{g}_1(x)$ and $f_1(x)$ can be approximated by the Bochner-Fejér sequence which are constructed from the same base. Therefore if $g(x)$ and $f_1(x)$ are both B_2-almost periodic, then $\tilde{g}_1(x)$ also does and we obtain

Theorem S. *Let $g(x)$ be a real valued measurable function over $(-\infty,\infty)$. Then the necessary and sufficient condition for $f_1(x)$ to be B_2-almost periodic is that the condition* (K₂) *is satisfied for the constant* *term of $\tilde{g}_1(x)$. If the associated Fourier series of $g(x)$ is presented by* (3.23), *then*

$$f_1(x) \sim ia^g + 2\sum_{\lambda_n>0} c_n e^{i\lambda_n x} . \tag{3.25}$$

Chapter 2. Generalized harmonic analyses in the complex domain

4. Generalized harmonic analyses in the half-plane. Now Paley-Wiener pointed out that every theory of harmonic analyses of functions of arguments in the real domain has an associated theory of functions of arguments complex domain. They proved the following

Theorem T. *Let $f(z)$ be analytic over $a \leqq x \leqq b$ and*

$$\int_{-A}^{A} |f(x+iy)|^2 dy = O(A) \tag{4.01}$$

uniformly in x over $a \leqq x \leqq b$. Let $f(a+iy)$ and $f(b+iy)$ both belong to S as a function of y. Then $f(x+iy)$ belongs to S' over $a \leqq x \leqq b$ as a fuction of y.

Their proof is very ingenious. We shall give a direct proof by our method. Both depend the same sauce of idea. They proved in the vertical strip domain but we consider firstly in the upper half-plane.

Let $g(x)$ be real or complex valued measurable function of W_2. Let us put

$$f_1(z) = 2C_1(z, g) . \tag{4.01}$$

Then by Theorems J and K, the Fourier-Wiener transform of $f_1(x)$ is defined for every $y>0$ as a function of x $(z=x+iy)$. We shall denote this by $s^{f_1}(u, y)$. Then we can state the following theorem

Theorem 2. *Let $g(x)$ be a real or complex valued function and belong to W_2. Then for any given positive number ε, we have*

(i) *if $|u|>\varepsilon$, then*

$$\begin{aligned} &s^{f_1}(u+\varepsilon, y) - s^{f_1}(u-\varepsilon, y) \\ &= (1+\operatorname{sign} u)e^{-yu}[\{s^g(u-\varepsilon) - s^g(u-\varepsilon)\} + r^g(u, y, \varepsilon)] \end{aligned} \tag{4.03}$$

where

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{|u|>\varepsilon} |r^g(u, y, \varepsilon)|^2 du = 0 \tag{4.04}$$

for every $y>0$,

(ii) *if $|u|<\varepsilon$, then*

$$s^{f_1}(u+\varepsilon, y)-s^{f_1}(u-\varepsilon, y) = 2ir_1^g(u+\varepsilon)+2ir_2^g(u+\varepsilon)+2ir_3^g(u+\varepsilon, y)\,, \tag{4.05}$$

where

$$r_1^g(u) = \underset{B\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} \frac{g(t)}{t+i} \frac{e^{-iut}-1}{-it}\, dt \tag{4.06}$$

$$r_2^g(u) = \underset{B\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} \frac{g(t)}{t+i} e^{-iut}\, dt \tag{4.07}$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{|u|<\varepsilon} |r_3^g(u+\varepsilon, y)|^2\, du = 0 \tag{4.08}$$

for every $y>0$.

Here we remark that in (4.07) the limit operation is taken over $(-\varepsilon, \varepsilon)$.

Lemma 2_1. *We have*

$$\underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \frac{2\sin\varepsilon t}{t} e^{-iut}\, dt = \sqrt{2\pi}\, \chi_\varepsilon(u) \tag{4.09}$$

where $\chi_\varepsilon(u)$ *is the characteristic function on* $(-\varepsilon, \varepsilon)$.

Lemma 2_2. *We have*

$$\underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \frac{e^{-iut}}{s-z}\, dt = i\frac{(1+\text{sign}\, u)}{2} \sqrt{2\pi}\, e^{-i(s-iy)u} \tag{4.10}$$

where $z=x+iy$, $y>0$.

From these two lemmas we get

Lemma 2_3. *We have*

$$\underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \frac{2\sin\varepsilon t}{t} \frac{e^{-iut}}{s-z}\, dt \qquad (z=t+iy,\ y>0) \tag{4.11}$$

$$= \begin{cases} \sqrt{2\pi}\, ie^{-i(s-iy)u} \dfrac{e^{i(s-iy)\varepsilon}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}\,, & u>\varepsilon \\ \sqrt{2\pi}\, ie^{-i(s-iy)u} \dfrac{e^{i(s-iy)u}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}\,, & -\varepsilon\leqq u\leqq\varepsilon \\ 0\,, & u<-\varepsilon\,. \end{cases}$$

Lemma 2_4. *Under the hypotheses of Theorem 2, if we put*

$$g_B(t) = \begin{cases} g(t)\,, & \text{if } |t|\leqq B \\ 0\,, & \text{if } |t|>B\,, \end{cases} \tag{4.12}$$

then for any given positive number ε, *we get*

(i) *if* $|u|>\varepsilon$, *then*

$$\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}C_1(z, g_B)\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt \tag{4.13}$$

$$=\frac{(1+\text{sign}\,u)}{2}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}g(s)\frac{e^{i\varepsilon(s-iy)}-e^{-i\varepsilon(s-iy)}}{i(s-iy)}e^{-iu(s-iy)}\,ds$$

(ii) *if* $|u|<\varepsilon$, *then*

$$\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}C_1(z, g_B)\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt \tag{4.14}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}g(s)\frac{e^{i(s-iy)u}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}e^{-iu(s-iy)}\,ds+\frac{i}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\,ds\,.$$

These four lemmas are proved in the previous paper [7, chap. 5].

Proof of Theorem 2. (i) *the case of* $|u|>\varepsilon$. We decompose the kernel of integral of right-hand side of (4.13) as follows

$$\frac{e^{i\varepsilon(s-iy)}-e^{-i\varepsilon(s-iy)}}{i(s-iy)}=\frac{2\sin\varepsilon s}{s-iy}+\frac{e^{i\varepsilon s}(e^{\varepsilon y}-1)}{i(s-iy)}-\frac{e^{-i\varepsilon s}(e^{-\varepsilon y}-1)}{i(s-iy)}$$

$$=\frac{2\sin\varepsilon s}{s}+\frac{iy}{s-iy}\,\frac{2\sin\varepsilon s}{s}+\frac{e^{i\varepsilon s}(e^{\varepsilon y}-1)}{i(s-iy)}-\frac{e^{-i\varepsilon s}(e^{-\varepsilon y}-1)}{i(s-iy)}$$

$$=\frac{2\sin\varepsilon s}{s}+K_{01}(s, y, \varepsilon)+K_{02}(s, y, \varepsilon)-K_{03}(s, y, \varepsilon)\,,\quad\text{say.}$$

Let us put

$$r^g_{0i}(u, y, \varepsilon)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}g(s)K_{0i}(s, y, \varepsilon)e^{-ius}\,ds\,,\quad (i=1, 2, 3)\,.$$

Then as for $r^g_{01}(u)$,

$$r^g_{01}(u)=(iy)\,\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s-iy}\,\frac{2\sin\varepsilon s}{s}e^{-ius}\,ds$$

and applying the Plancherel theorem, we get

$$\frac{1}{\varepsilon}\int_{|u|>\varepsilon}|r^g_{01}(u, y, \varepsilon)|^2\,du\leqq\varepsilon\int_{-\infty}^{\infty}\left|\frac{g(s)}{s-iy}\right|^2\left(\frac{2\sin\varepsilon s}{s}\right)^2ds$$

$$\leqq 4\varepsilon\int_{-\infty}^{\infty}\frac{|g(s)|^2}{s^2+y^2}\,ds$$

Thus we get

$$\frac{1}{\varepsilon}\int_{|u|>\varepsilon}|r^g_{01}(u, y, \varepsilon)|^2\,du=O(\varepsilon)\,, \tag{4.15}$$

for every $y>0$, as $\varepsilon\to 0$. Next for $r^g_{02}(u)$,

$$r_{02}^g(u, y, \varepsilon)=(e^{\varepsilon y}-1)\ \underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s-iy}e^{-i(u-\varepsilon)s}\,ds$$

and applying the Plancherel theorem again we get

$$\frac{1}{\varepsilon}\int_{|u|>\varepsilon}|r_{02}^g(u, y, \varepsilon)|^2\,du \leqq \frac{(e^{\varepsilon y}-1)^2}{\varepsilon}\int_{-\infty}^{\infty}\frac{|g(s)|^2}{s^2+y^2}\,ds\,.$$

If we remark that $e^{\varepsilon y}-1=O(\varepsilon)$ as $\varepsilon\to 0$, we get

$$\frac{1}{\varepsilon}\int_{|u|>\varepsilon}|r_{02}^g(u, y, \varepsilon)|^2\,du=O(\varepsilon)\,. \tag{4.16}$$

for every $y>0$, as $\varepsilon\to 0$. By the similar manner

$$\frac{1}{\varepsilon}\int_{|u|>\varepsilon}|r_{03}^g(u, y, \varepsilon)|^2\,du=O(\varepsilon)\,. \tag{4.17}$$

Thus if we put

$$r_0^g(u)=r_{01}^g(u)+r_{02}^g(u)+r_{03}^g(u) \tag{4.18}$$

and if we remark that

$$\underset{B\to\infty}{\text{l.i.m.}}\, C_1(z, g_B)\frac{\sin\varepsilon t}{t}=C_1(z, g)\frac{2\sin\varepsilon t}{t} \tag{4.19}$$

which is guaranteed by Theorems J and K, then from (4.15)$\sim$(4.18) the first part of Theorem 2 is established.

(ii) *the case* $|u|\leqq\varepsilon$. We rewrite the kernel of integral of right-hand side of (4.14) as follows

$$\frac{e^{i(s-iy)u}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}e^{-i(s-iy)u}+\frac{i}{s+i}$$

$$=\frac{e^{isu}-e^{-isu}}{is}e^{-isu}+\frac{i}{s+i}+\left\{\frac{e^{i(s-iy)u}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}e^{-i(s-iy)u}-\frac{e^{isu}-e^{-is\varepsilon}}{is}e^{-isu}\right\}$$

$$=\frac{i}{s+i}\,\frac{e^{-i(u+\varepsilon)s}-1}{-is}+\frac{i}{s+i}e^{-i(u+\varepsilon)s}+\left\{\frac{e^{-(s-iy)(u+\varepsilon)}-1}{-i(s-iy)}-\frac{e^{-is(u+\varepsilon)}-1}{-is}\right\}$$

$$=iK_1(s, u+\varepsilon)+iK_2(s, u+\varepsilon)+K_3(s, u+\varepsilon, y)\,,\quad\text{say.}$$

Then we get

$$r_1^g(u+\varepsilon)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}g(s)K_1(s, u+\varepsilon)\,ds$$

$$r_2^g(u+\varepsilon)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}g(s)K_2(s, u+\varepsilon)\,ds$$

and if we get

$$r_3^g(u+\varepsilon, y)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B} g(s)K_3(s, u+\varepsilon, y)\,ds$$

then the remaining part to prove is

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon} |r_3(u+\varepsilon, y)|^2\,du=0\,. \tag{4.20}$$

For this purpose we decompose $K_3(s, u+\varepsilon, y)$ as follows

$$K_3(s, u+\varepsilon, y)=\frac{iy(1-e^{-i(u+\varepsilon)s})-se^{-i(u+\varepsilon)s}-se^{-i(u+\varepsilon)(s-iy)}}{is(s-iy)}$$

$$=\frac{y(1-e^{-i(u+\varepsilon)s})}{s(s-iy)}+\frac{(1-e^{-uy})e^{-i(u+\varepsilon)s}}{i(s-iy)}$$

and we put

$$r_3^g(u+\varepsilon, y)=-i(1-e^{-uy})\,\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s-iy}\,e^{-i(u+\varepsilon)s}\,ds$$

$$+iy\,\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s-iy}\,\frac{e^{-i(u+\varepsilon)s}-1}{-is}\,ds$$

$$=-ir_{31}^g(u+\varepsilon, y)+ir_{32}^g(u+\varepsilon, y)\,.$$

Then applying the Plancherel theorem to $r_{31}^g(u)$ we get

$$\frac{1}{\varepsilon}\int_{|u|<\varepsilon} |r_{31}^g(u+\varepsilon, y)|^2\,du \leqq \frac{(1-e^{-\varepsilon y})^2}{\varepsilon}\int_{-\infty}^{\infty}\frac{|g(s)|^2}{s^2+y^2}\,ds$$

and

$$\frac{1}{\varepsilon}\int_{|u|<\varepsilon} |r_{31}^g(u+\varepsilon, y)|^2\,du=O(\varepsilon)\,, \tag{4.21}$$

for every $y>0$, as $\varepsilon\to 0$.

As for $r_{32}^g(u)$ we can write

$$r_{32}^g(u+\varepsilon, y)=y\int_0^{u+\varepsilon} dv\,\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\,e^{-ivs}\,ds$$

and if we put

$$\hat{g}_1(v)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\,e^{-ivs}\,ds\,. \tag{4.22}$$

Then we get

$$\frac{1}{\varepsilon}\int_{|u|<\varepsilon} |r_{32}^g(u+\varepsilon, y)|^2\,du \leqq \frac{y^2}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left|\int_0^{u+\varepsilon}\hat{g}_1(v)\,dv\right|^2 du$$

$$\leqq 2\varepsilon y^2\int_0^{2\varepsilon} |\hat{g}_1(v)|^2\,dv$$

and

$$\frac{1}{\varepsilon}\int_{|u|<\varepsilon} |r_{32}^g(u+\varepsilon, y)|^2\,du = o(\varepsilon) \tag{4.23}$$

for every $y>0$ as $\varepsilon\to 0$. Thus we obtain (4.20) from (4.21) and (4.23) and the second half-part of Theorem 2 is established.

Using this results as the base of arguments we shall obtain following theorems.

Theorem 3. *Let $g(x)$ be a real valued function of S_0. Let us suppose that*

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon} |s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du = 0 \tag{K$_1$}$$

(K_2) *there exists a constant a^g such that*

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_0^{2\varepsilon}\left|\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}e^{-ius}\,ds-\sqrt{\frac{\pi}{2}}\,a^g\right|^2 du = 0$$

and

$$\underset{\varepsilon\to 0}{\text{l.i.m.}}\frac{1}{\varepsilon\sqrt{2\pi}}\int_0^{u} |s^g(v+\varepsilon)-s^g(v-\varepsilon)|^2\,dv \tag{L$_0$}$$

exists over any finite range of u.
Then $f_1(z)$ $(z=x+iy,\ y>0)$ defined by (4.02) belongs to S_0 for every $y>0$ as a function of x and we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_1(z)|^2\,dt \quad (z=x+iy,\ y>0) \tag{4.24}$$

$$= |a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_0^{\infty}|e^{-uy}\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}|^2\,du\,.$$

Lemma 3_1. *Let $g(x)$ be a real-valued measurable function of W_2. Then*

$$\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}|r_1^g(u+\varepsilon)|^2\,du = o(\varepsilon) \tag{4.25}$$

as $\varepsilon\to 0$.

Proof of Lemma 3. This is obtained by the same arguments which we obtain (4.23).

We shall quote the lemma due to Paley-Wiener [10, pp. 134–5]. This is

Lemma 3_2 *(Paley-Wiener). Let $f_\varepsilon(x)$ be a family of functions which are measurable and square integrable. Let*

$$(4.26) \qquad \underset{\varepsilon\to\varepsilon_0}{\text{l.i.m.}} f_\varepsilon(x) = f(x)$$

over any finite range of x. Let us suppose that $f_\varepsilon(x)$ are monotone. Then we can definite its limit function $f(x)$ such that

$$(4.27) \qquad \lim_{\varepsilon\to\varepsilon_0} f_\varepsilon(x) = f(x)$$

for a.e. x.

From Lemmas 3_1 and 3_2 we get immediately.

Lemma 3_3. *Let $g(x)$ be a real valued function of W_2. Let* (L_0) *be satisfied. Then we can define such that*

$$(L) \qquad \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_0^A |s^g(u+\varepsilon) - s^g(u-\varepsilon)|^2 du$$

exists for appropriate A's belongs to an indefinitely increasing sequence.

Lemma 3_4. *Let $g(x)$ be a real-ualued function of S_0. Let us suppose that the condition* (L_0) *is satisfied. Then there exists*

$$(4.28) \qquad \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon} \int_0^\infty |e^{-uy}\{s^g(u+\varepsilon) - s^g(u-\varepsilon)\}|^2 du$$

for every $y>0$.

Proof of Lemma 3_4. From the fact that $g(x)$ belongs to S_0, there exists

$$(4.29) \qquad \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^\infty |s^g(u+\varepsilon) - s^g(u-\varepsilon)|^2 du$$

and

$$(4.30) \qquad \lim_{A\to\infty} \overline{\lim_{\varepsilon\to 0}} \frac{1}{2\pi\varepsilon} \int_A^\infty |e^{-uy}\{s^g(u+\varepsilon) - s^g(u-\varepsilon)\}|^2 du$$

$$\leqq \lim_{A\to\infty} \lim_{\varepsilon\to 0} \frac{e^{-2Ay}}{2\pi\varepsilon} \int_{-\infty}^\infty |s^g(u+\varepsilon) - s^g(u-\varepsilon)|^2 du$$

$$= 0 .$$

Therefore it is sufficient to prove that the following limit

$$(4.31) \qquad \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon} \int_0^A |e^{-uy}\{s^g(u+\varepsilon) - s^g(u-\varepsilon)\}|^2 du$$

exists for appropriate A's belonging to an indefinitely increasing sequence. That is obtained from Lemma 3_3.

Proof of Theorem 3. Combining these lemmas and the Wiener formula we get Theorem 3 immediately.

Theorem 4. *Let $g(x)$ be a real valued measurable function of S.*

Let us suppose that condition (K_1) *and* (K_2) *of Theorem 4 are satisfied. Then* $f_1(z)$ ($z=x+iy$) *defined by* (4.02) *belongs to* S' *for every* $y>0$ *as a function of* x, *and we have*

$$\phi_1^f(x, y)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(z+x)\overline{f_1(z)}\,dt \tag{4.32}$$

$$=|a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_0^{\infty} e^{iux}\,|e^{-uy}\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}|^2\,du$$

where $z=t+iy$, $y>0$.

Lemma 4. *Let* $g(x)$ *be a real valued function of* S. *Then the condition* (L_0) *is satisfied.*

Proof of Lemma 4_1. Since $g(x)$ is of real valued we get the relation (3.12) and the condition (L_0) is deduced from Theorem G.

Lemma. 4_2. *Under the hypothesis of Lemm* 4_1, *we get for any real or complex number* w *such as* $|w|=1$,

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_0^{A}(2+we^{-iux}+\overline{w}e^{iux})\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du \tag{4.33}$$

exists for every x *and appropriate* A*'s belonging to an inefinitely increasing sequence.*

Proof of Lemma 4. This is obtained from Lemmas 3_3 and 4_1.

Lemma 4_3. *Under the hypothesis of Lemma* 4_2, *the following limit*

$$\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_0^{\infty}(2+we^{-iux}+\overline{w}e^{iux})\,|e^{-uy}\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}|^2\,du \tag{4.34}$$

exists for every $y>0$ *and every* x.

Proof of Lemma 4_4. This is obtained from Lemma 4_2 by the same arguments which we attain Lemma 3_4 from Lemma 3_3.

Proof of Theorem 4. Let $s^{f_1}(u, y)$ be the Fourier-Wiener transform of $f_1(z)$ defined by (4.02). Then from Theorem 2 and conditions (K_1) and (K_2) we get for any real or complex number w such as $|w|=1$,

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}(2+we^{-iux}+\overline{w}e^{iux})\,|s^{f_1}(u+\varepsilon, y)-s^{f_1}(u-\varepsilon, y)|^2\,du \tag{4.35}$$

$$=\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_{-0}^{\infty}(2+we^{-iux}+\overline{w}e^{iux})\,|e^{-yu}\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}|^2\,du$$

$$+|a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}(2+we^{-iux}+\overline{w}e^{iux})\,du\,,$$

for every $y>0$ and every x. The existence of this limit is guaranteed by Lemma 4_3. Therefore

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_1(z+x)+wf_1(z)|^2\,dt \quad (z=t+iy,\ y>0) \tag{4.36}$$

exists and equals to (4.35) (c.f. [13, p. 158]). Here if we put $w=\pm 1$, $\pm i$ and taking the linear combination of these formulas appropriately we get (4.32). We get therefore that $f_1(z)$ belongs to S. Now that $f_1(z)$ belongs to S' is obtained from Theorem D. Because we get from the first half part of Theorem 2,

$$\begin{aligned}&\frac{1}{4\pi\varepsilon}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|s^{f_1}(u+\varepsilon,y)-s^{f_1}(u-\varepsilon,y)|^2\,du\\ &\leqq\frac{1}{\pi\varepsilon}\int_{A}^{\infty}|e^{-yu}\{s^g(u+\varepsilon)-s^g(u-\varepsilon\}|^2\,du\\ &\leqq\frac{e^{-2Ay}}{\pi\varepsilon}\int_{0}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du\,.\end{aligned}$$

This reads

$$\lim_{A\to\infty}\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|s^{f_1}(u+\varepsilon,y)-s^{f_1}(u-\varepsilon,y)|^2\,du=0\,. \tag{4.37}$$

Thus Theorem 4 is proved completely.

We shall add two more theorems for the sake of completeness which are obtained combining Theorems J, K, 3 and 4.

Theorem 5. *Let $f_1(z)$ $(z=x+iy)$ be an analytic function in $y>0$ such that $f_1(z)/(z+i)$ belong to $\mathfrak{H}^2(0,\infty)$. Let the real part of its limit function $g(x)$ be real valued, belong to S_0 and satisfy the conditions* (K_1), (K_2) *and* (L_0). *Then $f_1(z)$ belongs to S_0 for every $y>0$ as a function of x and (4.24) is true.*

Theorem 6. *Let $f_1(z)$ $(z=x+iy)$ be an analytic function in $y>0$ such that $f_1(z)/(z+i)$ belong to $\mathfrak{H}^2(0,\infty)$. Let the real part of its limit function $g(x)$ be real valued, belong to S and satisfy the conditions* (K_1) *and* (K_2). *Then $f_1(z)$ belongs to S' for every $y>0$ as a function of x and (4.32) is true.*

Combining Theorems J and 2 we get immediately.

Theorem 7. *Let $g(x)$ be real or complex valued measurable function of W_2. Let us put $f_1(x)=g(x)+i\tilde{g}_1(x)$. Let $s^{f_1}(u)$ be the Fourier-Wiener transform of $f_1(x)$. Then we have for any given positive number ε,*
(i) *if $|u|>\varepsilon$, then*

$$\begin{aligned}&s^{f_1}(u+\varepsilon,y)-s^{f_1}(u-\varepsilon,y)\\ &=e^{-yu}\{s^{f_1}(u+\varepsilon)-s^{f_1}(u-\varepsilon)\}+(1+\text{sign}\,u)e^{-yu}r_0^g(u,y,\varepsilon)\,,\end{aligned} \tag{4.38}$$

where the remainder term $r_0^g(u)$ *satisfies*

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon} |r_0^g(u, y, \varepsilon)|^2\, du=0 \tag{4.39}$$

for every $y>0$,

(ii) *if* $|u|<\varepsilon$, *then*

$$\begin{aligned} & s^{f_1}(u+\varepsilon, y)-s^{f_1}(u-\varepsilon, y) \\ &= \{s^{f_1}(u+\varepsilon)-s^{f_1}(u-\varepsilon)\}+2ir_3^g(u+\varepsilon, y)\,, \end{aligned} \tag{4.40}$$

where the remainder term $r_3^g(u)$ *satisfies*

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon} |r_3^g(u+\varepsilon, y)|^2\, du=0\,. \tag{4.41}$$

Then corresponding to Theorems 3, 4, 5 and 6 we get the following

Theorem 8. *Let* $g(x)$ *be real or complex valued measurable function. Let* $f_1(x)$ *defined by* $g(x)+i\tilde{g}_1(x)$ *belong to* S_0. *Let us suppose that*

$$\underset{\varepsilon\to 0}{\text{l.i.m.}}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-u}^{u} |s^{f_1}(v+\varepsilon)-s^{f_1}(v-\varepsilon)|^2\, dv \tag{L_0'}$$

exists over any finite range of u. *Then* $f_1(z)$ $(z=x+iy)$ *defined by* $2C_1(z, g)$ *belongs to* S_0 *for every* $y>0$ *as a function of* x *and we have*

$$\begin{aligned} & \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |f_1(z)|^2\, dx \\ &= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\infty} |e^{-uy}\{s^{f_1}(u+\varepsilon)-s^{f_1}(u-\varepsilon)\}|^2\, du\,. \end{aligned} \tag{4.42}$$

Theorem 9. *Let* $g(x)$ *be a real or complex valued measurable function. Let* $f_1(x)$ *belong to* S. *Then* $f_1(z)$ *belongs to* S' *for every* $y>0$ *as a function of* x *and we have*

$$\begin{aligned} & \phi^{f_1}(x, y)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(z+x)\overline{f_1(z)}\, dt \qquad (z=t+iy) \\ &= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\infty} e^{iuy}\, |e^{-uy}\{s^{f_1}(u+\varepsilon)-s^{f_1}(u-\varepsilon)\}|^2\, du\,. \end{aligned} \tag{4.43}$$

Theorem 10. *Let* $f_1(z)$ $(z=x+iy)$ *be an analytic function such that* $f_1(z)/(z+i)$ *belong to* $\mathfrak{H}^2$. *Let us suppose that its limit function* $f_1(x)$ *satisfy the condition* (L_0'). *Then* $f_1(z)$ *belongs to* S' *for every* $y>0$ *as a function of* x *and we have* (4.42).

Theorem 11. *Let* $f_1(z)$ $(z=x+iy)$ *be an analytic function such that*

$f_1(z)/(z+i)$ belong to $\mathfrak{H}^2$. Let us suppose that its limit function $f_1(x)$ belong to S. Then $f_1(z)$ belongs to S' for every $y>0$ as a function of x and we have (4.43).

We observe that

$$s^{f_1}(u+\varepsilon)-s^{f_1}(u-\varepsilon)=0 \qquad \text{if } u<-\varepsilon,$$

and the condition (L_0') asserts in fact the existence of the following limit

$$\underset{\varepsilon\to 0}{\text{l.i.m.}}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{u}|s^{f_1}(v+\varepsilon)-s^{f_1}(v-\varepsilon)|^2\,dv\,.$$

We also observe that conditions (K_1) and (K_2) are contained into the part of $(-\varepsilon, \varepsilon)$ of $s^{f_1}(u)$ and do not appear.

5. Generalized harmonic analyses in the strip domain. The purpose of this section is to establish the parallel theory with the previous section in the strip domain. Let us begin with the following theorem

Theorem 12. *Let $f(z)$ $(z=x+iy)$ be analytic in the strip domain $a<y<b$ and satisfy*

$$\int_{-\infty}^{\infty}\frac{|f(x+iy)|^2}{1+x^2}dx<\text{constant}. \qquad (a<y<b). \tag{5.01}$$

Then we can find the boundary functions at $y=a$ and b. If we denote these by $f(x+ia)$ and $f(x+ib)$ respectively. Then we have

(i)

$$\lim_{y\to a+}f(x+iy)=f(x+ia) \tag{5.02}$$

$$\lim_{y\to b-}f(x+iy)=f(x+ib) \tag{5.03}$$

as an angular limit.

(ii) *$f(x+ia)$ and $f(x+ib)$ both belong to W_2 and*

$$\lim_{y\to a+}\int_{-\infty}^{\infty}\frac{|f(x+iy)-f(x+ia)|^2}{1+x^2}dx=0 \tag{5.04}$$

$$\lim_{y\to b-}\int_{-\infty}^{\infty}\frac{|f(x+iy)-f(x+ib)|^2}{1+x^2}dx=0 \tag{5.05}$$

(iii) *$f(z)$ is represented as the difference of analytic functions $y>a$ and $y<b$ restectively. That is, for any real number c such as*

$$c<a<b\,, \tag{5.06}$$

we have

(5.07)
$$f(z)=\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t+ia)}{t+i(a-c)}\frac{dt}{(t+ia-z)}$$
$$-\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{f(x+ib)}{t+i(b-c)}\frac{dt}{(t+ib-z)}$$
$$=f^{+}(z,a)-f^{-}(z,b) \qquad \text{say.}$$

Then $f^{+}(z,a)/(z-ic)$ *and* $f^{-}(z,b)/(z-ic)$ *belongs to the class* $\mathfrak{H}^2$ *over* $y>a$ *and* $y<b$ *respectively.*

To prove this theorem we also quote another lemma due to Paley-Wiener [10, p. 5].

Lemma 12_1 (*Paley-Wiener*). *Let* $f(\sigma+it)$ *be a function of the complex variable* $s=\sigma+it$, *which is analytic in and on the boundary of the strip* $-\lambda\leqq\sigma\leqq\mu$, *and let*

(5.08)
$$\int_{-\infty}^{\infty}|f(\sigma+it)|^2dt<const. \qquad (-\lambda\leqq\sigma\leqq\mu).$$

Then we get

(5.09)
$$f(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\mu+iy)}{\mu+iy-s}dy-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(-\lambda+iy)}{-\lambda+iy-s}dy.$$

This lemma can be stated under somewhat relaxed condition. We state this in the horizontal strip domain.

Lemma 12_2. *Let* $f(z)$ *be function of the complex variable* $z=x+iy$ *which is analytic in the strip* $a<y<b$ *and let*

(5.10)
$$\int_{-\infty}^{\infty}|f(x+iy)|^2dx<const. \qquad (a<y<b).$$

Then we can find two functions $f(x,a)$ *and* $f(x,b)$ *which belong to* L_2 *and* $f(z)$ *is represented as follows.*

(5.11)
$$f(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(t,a)\frac{dt}{t+ia-z}-\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(t,b)\frac{dt}{t+ib-z}.$$

Proof of Lemma 12_2. From Lemma 12_1, for any given positive number $\varepsilon>0$, we get

$$f(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(t+ia+i\varepsilon)\frac{dt}{t+ia+i\varepsilon-z}-\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(t+ib-i\varepsilon)\frac{dt}{t+ib-i\varepsilon-z}$$
$$=\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(t+ia+i\varepsilon)\frac{dt}{t+ia-z}-\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(t+ib-i\varepsilon)\frac{dt}{t+ib-z}+O(\varepsilon).$$

Here taking an appropriate sequence (ε_n) tending to 0 and applying the F. Riesz theorem (c.f. S. Banach [2, p. 130]) to the above formula, we find $f(t, a)$ and $f(t, b)$ in L_2 as the weak limit and we get (5.11)

Lemma 12$_3$. *Under the hypotheses of Theorem 12, we can find two functions $f(x, a)$ and $f(x, b)$ which belong to W_2 and $f(z)$ is represented as follows*

$$f(z)=\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t, a)}{t+i(a-c)}\frac{dt}{t+ia-z}-\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t, b)}{t+i(b-c)}\frac{dt}{t+ib-z}.$$

Proof of Lemma 12$_3$. Taking any real number c such as $c<a<b$ and applying Lemma 12$_2$ to $f(z)/z-ic$ then we get Lemma 12$_3$ immediately.

Proof of Theorem 12. From Lemma 12$_3$, $f(z)$ is represented as the difference of analytic function over $y>a$ and $y<b$ respectively. If we write these as follows

$$f(z)=f^+(z, a)-f^-(z, b).$$

Then $f^+(z, a)/(z-ic)$ and $f^-(z, b)/(z-ic)$ belong to the class $\mathfrak{H}^2$ over $y>a$ and $y<b$ respectively. Therefore applying the results of §2, there exist boundary function of $f(z)$ in the strong sense. If we denote these to $f(x+ia)$ and $f(x+ib)$, then these belong to W_2 and (5.04) and (5.05) are satisfied. Thus we get

(5.12) $$f(x, a)=f(x+ia) \quad \text{and} \quad f(x, b)=f(x+ib),$$

for a.e. x. The remaining parts are obvious.

Let us write

(5.13) $$\tilde{f}_1(x+ia)=\frac{x-ic}{\pi}\int_{-\infty}^{\infty}\frac{f(t+ia)}{t+i(a-c)}\frac{dt}{x-t}$$

(5.14) $$\tilde{f}_1(x+ib)=\frac{x-ic}{\pi}\int_{-\infty}^{\infty}\frac{f(t+ib)}{t+i(b-c)}\frac{dt}{x-t}.$$

Then we get from (5.07) and (5.12) as follows

(5.15) $$f(x+ia)=\frac{1}{2}\{f(x+ia)+i\tilde{f}_1(x+ia)\}-f^-(x+ia, b)$$

(5.16) $$f(x+ib)=f^+(x+ib, a)+\frac{1}{2}\{f(x+ib)-\tilde{f}_1(x+ib)\}$$

respectively. Now $f(x+ia)$, $\tilde{f}_1(x+ia)$ and $f^+(z, a)$ all belong to W_2 as a function of x. Thus the Fowier-Wiener transform of these are difined. We shall denote these to $s^f(u, a)$, $s^{\tilde{f}_1}(u, a)$ and $s^{f^+}(u, y)$ respectively. Then repeating the same arguments as Theorem 2 we get

Theorem 13. *Under the hypotheses of Theorem 12, for any given positive number ε, we get for every $y>a$,*

(i) *if* $|u|>\varepsilon$, *then*

(5.17)
$$s^{f^+}(u+\varepsilon, y)-s^{f^+}(u-\varepsilon, y)$$
$$=\frac{(1+\text{sign } u)}{2}e^{-(y-a)u}[\{s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a)\}+r_0^f(u, y, a, c, \varepsilon)]$$

where

(5.18)
$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|>\varepsilon}|r_0^f(u, y, a, c, \varepsilon)|^2 du=0$$

(ii) *if* $|u|<\varepsilon$, *then*

(5.19)
$$s^{f^+}(u+\varepsilon, y)-s^{f^+}(u-\varepsilon, y)$$
$$=ir_1^f(u+\varepsilon, a)+ir_2^f(u+\varepsilon, a)+ir_3^f(u+\varepsilon, y, a, c),$$

where

(5.20)
$$r_1^f(u+\varepsilon, a)=(a-c)\ \underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(t+ia)}{t+i(a-c)}\ \frac{e^{-iut}-1}{-it}$$

(5.21)
$$r_2^f(u+\varepsilon, a)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(t+ia)}{t+i(a-c)}e^{-iut}\,dt$$

(5.22)
$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|r_3^f(u+\varepsilon, y)|^2 du=0\,.$$

Similarly we shall denote the Fourier-Wiener transform $f(t+ib)$ $\tilde{f}_1(t+ib)$ and $f^-(z, b)$ in Theorem 12 by $s^f(u, b)$, $s^{\tilde{f}_1}(u, b)$ and $s^{f^-}(u, y)$ respectively. Then we get

Theorem 14. *Under the hypotheses of Theorem 12, for any given positive number* ε, *we get for every* $y<b$

(i) *if* $|u|>\varepsilon$, *then*

(5.23)
$$s^{f^-}(u,+\varepsilon, y)-s^{f^-}(u-\varepsilon, y)$$
$$=(-1)\frac{(1-\text{sign } u)}{2}e^{(b-y)u}[\{s^f(u+\varepsilon, b)-s^f(u-\varepsilon, b)\}+r_0^f(u, y, b, c, \varepsilon)],$$

where

(5.24)
$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|r_0^f(u, y, b, c, \varepsilon)|^2 du=0$$

(ii) *if* $|u|<\varepsilon$, *then*

(5.25)
$$s^{f^-}(u+\varepsilon, y)-s^{f^-}(u-\varepsilon, y)$$
$$=ir_1^f(u-\varepsilon, b)+ir_2^f(u-\varepsilon, b)+ir_3^f(u-\varepsilon, y, b, c),$$

where

$$(5.26)\qquad r_1^f(u-\varepsilon, b)=(b-c)\ \underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(t+ib)}{t+i(b-c)}\,\frac{e^{-iut}-1}{-it}$$

$$(5.27)\qquad r_2^f(u-\varepsilon, b)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(t+ib)}{t+i(b-c)}e^{-iut}\,dt$$

and

$$(5.28)\qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|\,r_3^f(u-\varepsilon, y, b, c)\,|^2\,du=0\,.$$

For the proof of this theorem, instead of Lemma 2_2, we use the followings

Lemma 14_1. *We have*

$$(5.29)\qquad \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{e^{-iut}}{s-\bar{z}}\,dt=(-1)i\frac{(1-\text{sign}\,u)}{2}e^{-iu(s+iy)}$$

where $\bar{z}=x-iy,\ y>0$.

From Lemma 2 and 14_1, we get

Lemma 14_2. *We have*

$$(5.30)\qquad \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\,\frac{e^{-iut}}{s-\bar{z}}\,dt\qquad(\bar{z}=x-iy,\ y>0)$$

$$=\begin{cases}0\,, & u>\varepsilon\\ -\sqrt{2\pi}\,ie^{-i(s+iy)u}\dfrac{e^{i(s+iy)\varepsilon}-e^{i(s+iy)u}}{i(s+iy)}\,, & -\varepsilon\leqq u\leqq\varepsilon\\ -\sqrt{2\pi}\,ie^{-i(s+iy)u}\dfrac{e^{i(s+iy)\varepsilon}-e^{-i(s+iy)\varepsilon}}{i(s+iy)}\,, & u<-\varepsilon\,.\end{cases}$$

In the last we shall denote by $s^f(u, y)$ the Fourier-Wiener transform of $f(z)$. If we interprete $f(x+ia)$ and $f(x+ib)$ into the right-hand side of (5.08) and (5.09) respectively, then from Theorems 13 and 14, elementary but somewhat complicated calculations lead the followings

Theorem 15. *Under the hypotheses of Theorem 12, for any given positive number* ε, *and every* y *such as* $a<y<b$, *we have*

(i) *if* $|u|>\varepsilon$, *then*

$$(5.31)\qquad s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)$$
$$=e^{-(y-a)u}\{s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a)\}+\delta_0(u, y, a, b, c, \varepsilon)\,,$$

where

$$(5.32)\qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|>\varepsilon}|\,\delta_0(u, y, a, b, c, \varepsilon)\,|^2\,du=0\,.$$

(ii) *if* $|u|<\varepsilon$, *then*

(5.33)
$$s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)$$
$$=i\{s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a)\}+\delta_1(u, y, a, b, c, \varepsilon),$$

where

(5.34)
$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|\delta_1(u, y, a, b, c, \varepsilon)|^2\,du=0.$$

Theorem 16. *Under the hypotheses of Theorem 12, for any positive number* ε, *and every* y *such as* $a<y<b$, *we have*

(i) *if* $|u|>\varepsilon$, *then*

(5.35)
$$s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)$$
$$=e^{(b-y)u}\{s^f(u+\varepsilon, b)-s^f(u-\varepsilon, b)\}+\eta_0(u, y, a, b, c, \varepsilon),$$

where

(5.36)
$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{|u|<\varepsilon}|\eta_0(u, y, a, b, c, \varepsilon)|^2\,du=0.$$

(ii) *if* $|u|<\varepsilon$, *then*

(5.37)
$$s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)$$
$$=\{s^f(u+\varepsilon, b)-s^f(u-\varepsilon, b)\}+\eta_1(u, y, a, b, c, \varepsilon),$$

where

(5.38)
$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{|u|<\varepsilon}|\eta_1(u, y, a, b, c, \varepsilon)|^2\,du=0.$$

Now we get immediately the following theorem

Theorem (*Paley-Wiener*). *Under the hypotheses of Theorem 12, we have for every* y *such as* $a<y<b$,

(5.39)
$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\{s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)\}$$
$$-e^{-(y-a)u}\{s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a)\}|^2\,du=0$$

and

(5.40)
$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\{s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)\}$$
$$-e^{(b-y)u}\{s^f(u+\varepsilon, b)-s^f(u-\varepsilon, b)\}|^2\,du=0.$$

Corresponding to Theorem 3, we get

Theorem 17. *Let* $f(z)$ $(z=x+iy)$ *be analytic over* $a<y<b$ *and satisfy* (5.01). *Let its boundary functions* $f(x+ia)$, $f(x+ib)$ *both belong to the class* S_0. *Let us suppose that*

$$(\mathrm{L}_0)\qquad \begin{cases} \underset{\varepsilon\to 0}{\mathrm{l.i.m.}}\dfrac{1}{4\pi\varepsilon}\displaystyle\int_0^u |s^f(v+\varepsilon, a)-s^f(v-\varepsilon, a)|^2\,dv \\ \underset{\varepsilon\to 0}{\mathrm{l.i.m.}}\dfrac{1}{4\pi\varepsilon}\displaystyle\int_0^u |s^f(v+\varepsilon, b)-s^f(v-\varepsilon, b)|^2\,dv \end{cases}$$

exist over any finite range of u. Under these assumptions, for any y such as $a<y<b$, $f(z)$ *belongs to the class* S_0 *and we have*

$$(5.41)\qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x+iy)|^2\,dx$$
$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)|^2\,du$$
$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|e^{-(y-a)u}\{s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a)\}|^2\,du$$
$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|e^{(b-y)u}\{s^f(u+\varepsilon, b)-s^f(u-\varepsilon, b)\}|^2\,du\,.$$

Similarly corresponding to Theorem 4, we get

Theorem (*Paley-Wiener*). *Let* $f(z)$ $(z=x+iy)$ *be analytic over* $a<y<b$ *and satisfy* (5.01). *Let its boundary functions* $f(x+ia)$ *and* $f(x+ib)$ *both belong to the class S. Then for any y such as* $a<y<b$, $f(z)$ *belongs to the class* S' *and we have*

$$(5.42)\qquad \phi^f(x, y)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t+iy+x)\overline{f(t+iy)}\,dt$$
$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}|s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)|^2\,du$$
$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}|e^{-(y-a)u}\{s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a)\}|^2\,du$$
$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}|e^{(b-y)u}\{s^f(u+\varepsilon, b)-s^f(u-\varepsilon, b)\}|^2\,du\,.$$

Proof of theorem. From (5.39) and (5.40) we get

$$(5.44)\qquad \lim_{A\to\infty}\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)|^2\,du=0\,.$$

Therefore it is sufficient to show that the following limit

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-A}^{A}e^{iux}|s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)|^2\,du$$

exists and equals to

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-A}^{A} e^{iux}\,|\,e^{-(y-a)u}\{s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a)\}\,|^2\,du$$

for all A's belonging to an appropriate increasing sequence indefinitely. This is obvious since $f(x+ia)$ belongs to the class S. The last relation of (5.42) is also derived by the similar manner from the fact that $f(x+ib)$ belongs to the class S. That $f(z)$ belong to the class S' is obtained from (5.43) and Theorem D. Thus the proof is completed.

6. Analytic almost periodic functions of the Besicovitch class. As an application of the results of preceding sections, we shall prove some theorems for almost periodic function. Firstly corresponding to Theorem R we shall prove

Theorem 18. *Let $g(x)$ be a real valued measurable function defined on $(-\infty, \infty)$. Let $g(x)$ be B_2-almost periodic and satisfy the conditions* (K_1) *and* (K_2). *Then $f_1(z)$ defined by $2C_1(z, g)$ is also B_2-almost periodic for every $y>0$ as a function of x. If the associated Fourier series of $g(x)$ is*

$$g(x) \sim {\sum}' a_n e^{i\lambda_n x} \tag{6.01}$$

then

$$f_1(z) \sim ia^g + {\sum}' (1+\operatorname{sign}\lambda_n)a_n e^{i\lambda_n z} \tag{6.02}$$

where the prime means that the summation does not contain the constant term.

Proof of Theorem 18. From Lemma 2_1, we get

$$\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} ia^g\,\frac{2\sin\varepsilon t}{t}\,e^{-iut}\,dt = i\sqrt{2\pi}\,a^g\chi_\varepsilon(u)\,. \tag{6.03}$$

Since $g(x)$ is B_2-almost periodic, for any arbitarily small positive number η, there corresponds the Bochner-Fejér polynomial

$$\sigma^g_{B_p}(x) = {\sum}' d^B_n a_n e^{i\lambda_n x} \tag{6.04}$$

such that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |\,g(t)-\sigma^g_{B_p}(t)\,|^2\,dt<\eta\,. \tag{6.05}$$

Here we remark that $\sigma^g_B(t)$ does not contant term from the fact that the condition (K_1) is satisfied. Let us fix this polynomial and let us put

$$\sigma^{f_1}_{B_p}(z) = {\sum}' (1+\operatorname{sign}\lambda_n)d^B_n a_n e^{i\lambda_n z} \qquad (z=x+iy,\ y>0)\,. \tag{6.06}$$

Furthermore let us denote by $s^\sigma(u, y)$ and $s^\sigma(u)$, the Fourier-Wiener transform of $\sigma_B^{f_1}(z)$ and $\sigma_B^g(x)$ respectively. Then by (6.03) we get

(6.07) $$s^\sigma(u+\varepsilon, y)-s^\sigma(u-\varepsilon, y)=\sum{}' (1+\text{sign}\,\lambda_n)d_n^B a_n e^{-\lambda_n y}\chi_\varepsilon(u-\lambda_n)$$

(6.08) $$s^\sigma(u+\varepsilon)-s^\sigma(u-\varepsilon)=\sum{}' d_n^B a_n \chi_\varepsilon(u-\lambda_n)\,.$$

If we observe that

$$(e^{-\lambda_n y}-e^{-uy})\chi_\varepsilon(u-\lambda_n)=0\{e^{-\lambda_n y}(1-e^{-\varepsilon y})\chi_\varepsilon(u-\lambda_n)\}$$
$$=0(\varepsilon)\cdot\chi_\varepsilon(u-\lambda_n)\,,\quad \varepsilon\to 0\,.$$

Then we get if $|u|>\varepsilon$,

(6.09) $$s^\sigma(u+\varepsilon, y)-s^\sigma(u-\varepsilon, y)$$
$$=(1+\text{sign}\,u)e^{-uy}\{s^\sigma(u+\varepsilon)-s^\sigma(u-\varepsilon)\}+0(\varepsilon)\cdot\sum\chi_\varepsilon(u-\lambda_n)\,.$$

On the other hand if $|u|<\varepsilon$, since $\sigma^{f_1}(z)$ does not cotain costant term, we get

(6.10) $$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\,s^\sigma(u+\varepsilon, y)-s^\sigma(u-\varepsilon, y)\,|^2\,du=0\,.$$

From (4.05) of Theorem 2 and the condition (K_2) we get

(6.11) $$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,\{s^{f_1}(u+\varepsilon, y)-s^{f_1}(u-\varepsilon, y)\}-i\sqrt{2\pi}\,a^g\,|^2\,du=0\,.$$

Thus from (6.09) to (6.11) we obtain

(6.12) $$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,\{s^{f_1}(u+\varepsilon, y)-s^{f_1}(u-\varepsilon, y)\}$$
$$-i\sqrt{2\pi}\,a^g-\{s^\sigma(x+\varepsilon, y)-s^\sigma(u-\varepsilon, y)\}\,|^2\,du$$
$$=\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_0^{\infty}|\,e^{-yu}[\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}-\{s^\sigma(u+\varepsilon)-s^\sigma(u-\varepsilon)\}]\,|^2\,du$$
$$\leqq\lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_{-\infty}^{\infty}|\,\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}-\{s^\sigma(u+\varepsilon)-s^\sigma(u-\varepsilon)\}\,|^2\,du\,.$$

Thus we obtain

(6.13) $$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\,f_1(z)-\{ia^g+\sigma_B^{f_1}(z)\}\,|^2\,dt\quad (z=t+iy,\ y>0)$$
$$\leqq\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\,g(t)-\sigma_B^g(t)\,|^2\,dt\,.$$

Hence by (6.05) (6.13) we have the required result

(6.14) $$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\,f_1(z)-\{ia^g+\sigma_{B_p}^{f_1}(z)\}\,|^2\,dt<\eta\,.$$

The B_2-almost periodicity of $f_1(z)$ is thus established. The remaining part is obvious (c.f. also [7, chap. 5, p. 217]).

We now consider analytic almost periodic functions in strip domain. Firstly we obtain from Theorems 14 and 15,

Theorem 19. *Under the hypotheses of Theorem 12, for any given positive number* $\varepsilon>0$ *and every* y *such as* $a<y<b$, *we get*

(i) *if* $|u|>\varepsilon$, *then*

(6.15)
$$e^{-(y-a)u}\{s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a)\} = e^{(b-y)u}\{s^f(u+\varepsilon, b)-s^f(u-\varepsilon, b)\}+\xi_0(u, y, a, b, c, \varepsilon),$$

where

(6.16)
$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|\xi_0(u, y, a, b, c, \varepsilon)|^2\,du=0\,;$$

and

(ii) *if* $|u|<\varepsilon$, *then*

(6.17)
$$s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a) = s^f(u+\varepsilon, b)-s^f(u-\varepsilon, b)+\xi_1(u, y, a, b, c, \varepsilon),$$

where

(6.18)
$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|\xi_1(u, y, a, b, c, \varepsilon)|^2\,du=0\,.$$

From this we have

Theorem 20. *Let* $f(z)$ $(z=x+iy)$ *be analytic over* $a<y<b$ *and*

$$\int_{-\infty}^{\infty}\frac{|f(x+iy)|^2}{1+x^2}\,dx<\text{const.} \qquad (a<y<b)\,.$$

Let its boundary functions $f(x+ia)$ *and* $f(x+ib)$ *both* B_2-*almost periodic. Let their associatied Fourier series be*

(6.19)
$$f(x+ia)\sim a_0+\sum a_n e^{i\lambda_n(x+ia)}$$

(6.20)
$$f(x+ib)\sim b_0+\sum b_n e^{i\mu_n(x+ib)}\,.$$

Then we have

(6.21)
$$a_0=b_0$$

and

(6.22)
$$\lambda_n=\mu_n,\qquad a_n=b_n\quad (n=1, 2, \cdots)\,.$$

As a simple application of the Wiener formula we get

Lemma 20₁. *Let* $f(x)$ *be any function of the class* W_2. *We shall denote by* $s^f(u)$ *the Fourier-Wiener transform of* $f(x)$. *Let* $f(x)$ *be* B_2-

almost periodic and its associated Fourier series be

(6.23) $$f(x)\sim c_0+\sum c_n e^{i\lambda_n x}.$$

Then we have for any real number λ

(6.24) $$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s^f(u+\varepsilon)-s^f(u-\varepsilon)\}\,du$$
$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t)e^{-i\lambda t}\,dt.$$

In particular we have

(6.25) $$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda_n-\varepsilon}^{\lambda_n+\varepsilon}\{s^f(u+\varepsilon)-s^f(u-\varepsilon)\}\,du=c_n\qquad (n=1,2,\cdots).$$

If we observe that from (6.16) and (6.18) we get for any real number $\lambda\neq 0$,

$$\frac{1}{2\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\xi_0(u,y,a,b,c,\varepsilon)\,du=o(1)$$

and

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\xi_1(u,y,a,b,c,\varepsilon)\,du=o(1)$$

as $\varepsilon\to 0$ respectively. Then we obtain immediately Theorem 20. From this theorem we understand that $f(x+ia)$ and $f(x+ib)$ can be approximated by the Bochner sequences which are constricted from the same base. Now we can prove the following

Theorem 21. *Under the hypotheses of Theorem 20, $f(z)$ is also B_2-almost periodic and if we denote its associated Fourier series as follows*

(6.26) $$f(z)\sim c_0+\sum c_n e^{i\nu_n z}\qquad (z=x+iy).$$

Then we have

(6.27) $$c_0=a_0=b_0$$

(6.28) $$\nu_n=\lambda_n=\mu_n,\qquad c_n=a_n=b_n\quad (n=1,2,\cdots).$$

Proof of Theorem 21. The proof can be done by running on the line of Theorem R. But we consider the positive part and negative part of spectrum separately. Since $f(x+ia)$ and $f(x+ib)$ are both B_2-almost periodic, for any given positive number η, there correspond the Bochner-Fejér polynomials

(6.29) $$\sigma^f_{B_p}(x,a)=a_0+\sum d^B_n a_n e^{i\lambda_n(x+ia)}$$

(6.30) $$\sigma^f_{B_p}(x, b) = b_0 + \sum d^B_n b_n e^{i\lambda_n(x+ib)}$$

where $a_0 = b_0$, such that

(6.31) $$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} | f(t+ia) - \sigma^f_{B_p}(t, a) |^2 dt < \eta$$

(6.32) $$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} | f(t+ib) - \sigma^f_{B_p}(t, b) |^2 dt < \eta$$

respectively. Here we observe that $\sigma^f_B(t, a)$ and $\sigma^f_B(t, b)$ are constructed from the same base and so in (6.29) and (6.30) we can take the same d^B_n. Let us fix these polynomials. Let us put

(6.33) $$\sigma^f_{B_p}(x, y) = c_0 + \sum d^B_n c_n e^{i\lambda_n(x+iy)}$$

where

(6.34) $$c_0 = a_0 = b_0$$

and

(6.35) $$c_n = \begin{cases} a_n, & \text{if } \lambda_n > 0 \\ b_n, & \text{if } \lambda_n < 0. \end{cases}$$

Let us denote by $s^\sigma(u, a)$, $s^\sigma(u, b)$ and $s^\sigma(u, y)$, the Fourier-Wiener transform of $\sigma^f_B(x, a)$, $\sigma^f_B(x, b)$ and $\sigma^f_B(x, y)$ respectively. Then we have for sufficiently small positive number ε,

(a) if $u > \varepsilon$, then

(6.36) $$s^\sigma(u+\varepsilon, y) - s^\sigma(u-\varepsilon, y)$$
$$= e^{-(y-a)}\{s^\sigma(u+\varepsilon, a) - s^\sigma(x-\varepsilon, a)\} + O(\varepsilon)\cdot \sum_{\lambda_n>0} \chi_\varepsilon(u-\lambda_n)$$

(b) if $u < -s$, then

(6.37) $$s^\sigma(u+\varepsilon, y) - s^\sigma(u-\varepsilon, y)$$
$$= e^{(b-y)u}\{s^\sigma(u+\varepsilon, b) - s^\sigma(u-\varepsilon, b)\} + O(\varepsilon)\cdot \sum_{\lambda_n>0} \chi_\varepsilon(u-\lambda_n),$$

where $\chi_\varepsilon(u)$ is the characteristic function over $(-\varepsilon, \varepsilon)$ and the summations are taken over the λ_n containing in polynomials to be fixed respectively,

(c) if $-\varepsilon < u < \varepsilon$, then

(6.38) $$s^\sigma(u+\varepsilon, y) - s^\sigma(u-\varepsilon, y)$$
$$= \{s^\sigma(u+\varepsilon, a) - s^\sigma(u-\varepsilon, a)\} = \sqrt{2\pi}\, c_0 \chi_\varepsilon(u) = \sqrt{2\pi}\, a_0 \chi_\varepsilon(u)$$

by Lemma 2_1 and (6.34).

On the other hand from (6.31) and (6.32) we get

(6.39) $$\lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} | \{s^f(x+\varepsilon, a) - s^f(u-\varepsilon, a)\}$$
$$- \{s^\sigma(u+\varepsilon, a) - s^\sigma(u-\varepsilon, a)\} |^2 du < \eta$$

$$(6.40)\qquad \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\{s^f(u+\varepsilon, b)-s^f(u-\varepsilon, b)\} -\{s^\sigma(u+\varepsilon, b)-s^\sigma(u-\varepsilon, b)\}|^2 du<\eta .$$

Therefore from (5.31) of Theorem 14, (6.36) and (6.39) we get

$$(6.41)\qquad \overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{\infty}|\{s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)\} -\{s^\sigma(u+\varepsilon, y)-s^\sigma(u-\varepsilon, y)\}|^2 du$$

$$=\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{\infty}|e^{-(y-a)u}[\{s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a)\} -\{s^\sigma(u+\varepsilon, a)-s^\sigma(u-\varepsilon, a)\}]|^2 du$$

$$\leqq\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\{s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a)\} -\{s^\sigma(u+\varepsilon, a)-s^\sigma(u-\varepsilon, a)\}|^2 du<\eta .$$

Similarly from (5.35) of Theorem 15, (6.37) and (6.40) we get

$$(6.42)\qquad \overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{-\varepsilon}|\{s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)\} -\{s^\sigma(u+\varepsilon, y)-s^\sigma(u-\varepsilon, y)\}|^2 du<\eta .$$

In the last we have

$$(6.43)\qquad \overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\{s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)\} -\{s^\sigma(u+\varepsilon, y)-\varepsilon^\sigma(u-\varepsilon, y)\}|^2 du=0 .$$

Because the left-hand side of (6.43) does not surpass than the following

$$2^3\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\{s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)\} -\{s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a)\}|^2 du$$

$$+2^3\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\{s^f(u+\varepsilon, a)-s^f(u-\varepsilon, a)\} -\{s^\sigma(u+\varepsilon, a)-s^\sigma(u-\varepsilon, a)\}|^2 du$$

$$+2^3\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\{s^\sigma(u+\varepsilon, y)-s^\sigma(u-\varepsilon, y)\} -\{s^\sigma(u+\varepsilon, a)-s^\sigma(u-\varepsilon, a)\}|^2 du ,$$

and here the first term of right-hand side vanish by (5.33) of Theorem 15, the second term vanish by (6.39) and third term vanish by (6.38). Thus we get (6.34). Therefore we obtain

(6.44) $$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\{s^f(u+\varepsilon, y)-s^f(u-\varepsilon, y)\} -\{s^\sigma(u+\varepsilon, y)-s^\sigma(u-\varepsilon, y)\}|^2\,du<2\eta$$

or

(6.45) $$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(t+iy)-\sigma^f_{B_p}(t, y)|^2\,dt<2\eta\,.$$

The B_2-almost periodicity of $f(z)$ is proved. The remaining part is to prove (6.28) and this is obtained from (6.35) or an application of Lemma 20_1 to Theorems 14 and 15.

7. Analytic almost periodic functions of the Bohr-Stepanoff class. We take part in almost periodic function in a sense of Stepanoff of order 1. This asserts that to every $\varepsilon>0$ there corresponds a relatively dense set of real number τ such that

(7.01) $$\sup_{-\infty<x<\infty}\int_{x}^{x+1}|f(t+\tau)-(t)|\,dt<\varepsilon\,.$$

Then E. H. Linfoot [9] proved the following

Theorem U. *If for two different values of σ_1, σ_2 of σ, the series $\sum a_\wedge e^{i\wedge s}$ $(s=\sigma+it)$ is the Fourier series of the S_1-almost periodic function, then for $\sigma_1<\sigma<\sigma_2$ it is the Dirichlet series of an analytic function, in (σ_1, σ_2) and uniformly almost periodic in $[\sigma_1, \sigma_2]$.*

Here (σ_1, σ_2) denotes the open strip domain $\sigma_1<\sigma<\sigma_2$ and the symbol $[\sigma_1, \sigma_2]$ means every strip interior to (σ_1, σ_2). The uniformly almost periodicity of Bohr's sense in $[\sigma_1, \sigma_2]$ asserts that to every $\varepsilon>0$ there corresponds a relatively dense set of real number τ such that

(7.03) $$|f(s+i\tau)-f(s)|<\varepsilon$$

for every s in $[\sigma_1, \sigma_2]$. Concerning to this we shall prove the following theorem in a half-plane.

Theorem 22. *Let $g(x)$ be a real or complex valued measurable function of W_2. Let $f_1(x)$ defined by $g(x)+i\tilde{g}_1(x)$ be S_1-almost periodic. Then $f_1(z)$ defined by $C_1(z, f_1)$ is uniformly almost periodic in $0<y<\infty$. If the associated Fourier series with $f_1(x)$ is*

(7.04) $$f_1(x)\sim c_0+\sum c_n e^{i\lambda_n x}$$

then

(7.05) $$f(z)\sim c_0+\sum c_n e^{i\lambda_n z}\,. \qquad (z+x+iy)$$

Lemma 22_1. *We have*

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+t^2}\,\frac{y}{(t-x)^2+y^2}\,e^{-iut}\,dt=\frac{ye^{-|u|}e^{-|yu|}e^{-iux}}{x^2+(y+1)^2} \tag{7.06}$$

and in particular

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+t^2}\,\frac{y}{(t-x)^2+y^2}\,dt=\frac{1+y}{x^2+(y+1)^2}\,. \tag{7.07}$$

Lemma 22_2. *We have*

$$\frac{z+i}{\pi}\int_{-\infty}^{\infty}\frac{1}{t+i}\,\frac{y}{(t-x)^2+y^2}\,e^{iux}\,dt=\begin{cases}e^{iuz}, & u>0\\ 1, & u=0\\ 0, & u<0,\end{cases} \tag{7.08}$$

where $z=x+iy$.

Lemma 22_3. *If we put*

$$p(t)=a_0+\sum_{n=1}^{N}a_n e^{i\lambda_n t}\qquad(\lambda_n>0,\ n=1,2,\cdots,N) \tag{7.09}$$

then we have

$$\begin{aligned}p_1(z)&=\frac{z+i}{\pi}\int_{-\infty}^{\infty}\frac{p(t)}{t+i}\,\frac{y}{(t-x)^2+y^2}\,dt \\ &=a_0+\sum a_n e^{i\lambda_n z}\qquad\qquad(z=x+iy).\end{aligned} \tag{7.10}$$

Lemma 22_4. *Under the hypotheses of Theorem 22, we have*

$$|f_1(z)|\leqq 2\sqrt{2\pi}\,\frac{(1+y)^{\frac{5}{2}}}{y^2}\sup_{-\infty<x<\infty}\int_x^{x+1}|f_1(t)|\,dt\,. \tag{7.10°}$$

Proof of Lemma 22_4. We have

$$|f_1(z)|\leqq\frac{|z+i|}{\pi}\sum_{n=-\infty}^{\infty}\int_n^{n+1}\frac{|f(t)|}{\sqrt{1+t^2}}\,\frac{y}{(t-x)^2+y^2}\,dt\,.$$

If $x+n\leqq t\leqq x+n+1$ and $n\leqq-2$ or $n\geqq1$, then

$$\frac{1}{\sqrt{1+t^2}}\leqq\frac{1}{\sqrt{1+(x+n)^2}}\leqq\frac{2}{\sqrt{1+(x+n+1)^2}}\leqq\frac{2}{\sqrt{1+t^2}}$$

and

$$\frac{2}{(t-x)^2+y^2}\leqq\frac{y}{n^2+y^2}\leqq\frac{4y}{(n+1)^2+y^2}\leqq\frac{4y}{(t-x)^2+y^2}\,.$$

Thus we get

$$\begin{aligned}&\Big(\sum_{n=-\infty}^{-2}+\sum_{n=1}^{\infty}\Big)\int_{x+n}^{x+n+1}\frac{|f_1(t)|}{\sqrt{1+t^2}}\,\frac{y}{(t-x)^2+y^2}\,dt\\ &\leqq2\Big(\sum_{n=-\infty}^{-2}+\sum_{n=1}^{\infty}\Big)\int_{x+n}^{x+n+1}\frac{1}{\sqrt{1+t^2}}\,\frac{y\,dt}{(t-x)^2+y^2}\sup_{-\infty<x<\infty}\int_x^{x+1}|f_1(t)|\,dt\end{aligned}$$

$$\leqq 8\sqrt{2\pi}\left\{\int_{-\infty}^{\infty}\frac{1}{\sqrt{1+t^2}}\frac{y\,dt}{(t-x)^2+y^2}\sup_{-\infty<x<\infty}\int_x^{x+1}|f_1(t)|\,dt\,.\right.$$

By Lemma 22_1, the last formular equals to

$$8\sqrt{2\pi}\frac{(1+y^2)^{\frac{1}{2}}}{z+i}\sup_{-\infty<x<\infty}\int_x^{x+1}|f_1(t)|\,dt\,.$$

If $n=-1$ or 0, then $x-1<t<x+1$ and we get

$$\frac{y}{1+y^2}<\frac{y}{(t-x)^2+y^2}<\frac{1}{y}$$

and that

$$\int_{x-1}^{x+1}\frac{|f_1(t)|}{\sqrt{1+t^2}}\frac{y\,dt}{(t-x)^2+y^2}\leqq\frac{2}{\sqrt{1+x^2}}\frac{1}{y}\sup_{-\infty<x<\infty}\int_{x-1}^{x+1}|f_1(t)|\,dt$$

$$\leqq\frac{8(1+y^2)}{y^2}\int_{x+1}^{x+1}\frac{1}{\sqrt{1+t^2}}\frac{y}{(t-x)^2+y^2}dt\sup_{-\infty<x<\infty}\int_{x+1}^{x+1}|f_1(t)|\,dt$$

Combining these estimations we get (7.11). We notice that the order of y of multiple constant seems to be not the best possible one but for our purpose we do not require the further details.

Proof of Theorem 22. By Theorem K, $f_1(z)$ defined by (4.02) can be represented by its Poisson integral of order 1, that is

$$f_1(z)=\frac{z+i}{\pi}\int_{-\infty}^{\infty}\frac{f_1(t)}{t+i}\frac{y\,dt}{(t-x)^2+y^2}\,. \tag{7.12}$$

Then uniform almost periodicity is proved as follows. Since $f_1(t)$ is S_1-almost periodic, for any positive number η there corresponds a trigonometrical polynomial $p(t)$, which does not contain the negative spectrum, such as

$$\sup_{-\infty<x<\infty}\int_x^{x+1}|f_1(t)-p(t)|\,dt<\eta\,. \tag{7.13}$$

Then by Lemma 22_3 and 22_4, $p_1(z)$ defined by (7.10) is also trigonometrical polynomial and

$$|f_1(z)-p_1(z)|\leqq A(y)\sup_{-\infty<x<\infty}\int_x^{x+1}|f_1(t)-p(t)|\,dt\leqq A(y)\eta\,. \tag{7.14}$$

That is $f_1(z)$ is approximated by trigonometrical polynomial in arbitrary scale. Thus the uniform almost periodicity of $f_1(z)$ is proved. If we take as $p(t)$ the Bochner-Fejér polynomial then (7.14) reads (7.05) immediately. Thus the Theorem 22 is established completely.

Correction. Added on June 1, 1960.

(1) On the Hilbert transform I, Journ. Faculrt Sci. Hokkaido Univ. 14 (1959), 153–224.

1. p. 157. In Lemma A_2, we add the following condition

$$\int_X \phi(|f_n|)\,d\mu < K \qquad (n=1,2,\cdots) \tag{2.16$_1$}$$

2. pp. 158–159. In Theorems A, B and C, the proposition

"*In particular the operation T can be uniquely extended to the whole space*"

is replaced by

"*In particular if the operation T is linear, then it can be uniquely extended to the whole space*".

3. p. 158. In Theorem A, we assume that $\phi(u)$ is a continuous increasing function with $\phi(0)=0$. This can be relaxed as follows.

The $\phi(u)$ is a continuous function and satisfies

(i)$_1$ *$\phi(u)>0$ for all $u>0$ and $\phi(0)=0$*

(i)$_2$ *$\phi(u)$ increase at the neighbourhood of $u=0$ and $u=\infty$ and*

(i)$_3$ *in the intermediate interval $\phi(u)$ is of bounded variation.*

Typical case of $\phi(u)$ is

$$\phi(u)=u^p(1+\log^+ u)^\alpha,$$

where $p>1$ and α is any real number.

4. p. 200. The $f(x)$ in the left-hand side of (14.22) is replaced by $\tilde{f}(x)$.

5. p. 209. The

$$\lim_{T\to\infty}\int_{-T}^{T}|g(t)|^2\,dt$$

in the right-hand side of (17.04) is replaced by

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)|^2\,dt.$$

6. Recentely the author receieved some papers from Prof. H. Kober. In these, there are important studies related to the Hilbert transform. These are published about the year 1941 to 1946. The author thanks to him.

The basic theorem of this arguments are as follows.

Theorem ([*15, p. 440, Lemma 2*]). *Let $w=(i-z)(i+z)^{-1}$. Then the function $f(z)$ belongs to $\mathfrak{H}_p$ if, and only if, the function* $(1+w)^{-2/p}\phi(w)$

belongs to H_p, where $\phi(w)=F(z)$.

Theorem ([*14, p. 50, Corollary 2*]). *The sequence*

$$\{\pi^{-1/2}(i-t)^n(i+t)^{-n-1}\} \qquad (n=0, \pm1, \pm2,\cdots)$$

is a complete orthogonal and normal system with respect to

$$L^p(-\infty, \infty) \qquad (1\leqq p<\infty).$$

7. p. 181, Lemma 22$_4$. Let $F(\zeta)$ be defined on a Jordan curve C in the complex plane. Various writers have treated the problem of representing $F(\zeta)$ in the form $F(\zeta)=F_1(\zeta)+F_2(\zeta)$; $F_j(\zeta)$ $(j=1, 2)$ are required to be the limit-functions of functions $F_j(z)$ $(z=x+iy;\ z\to\zeta,\ \zeta$ on $C)$ which are analytic in the interior or exterior of C, respectively.

H. Kober treated of the case $C=(-\infty, \infty)$ and $F(x)(1+x^2)^{-1}\in L_1(-\infty, \infty)$. For this purpose, he introduced the modified Hilbert transform

$$\mathfrak{K}F=\mathfrak{K}(F(t);x)=\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}F(t)\left(\frac{1}{t-x}-\frac{t}{t^2+1}\right)dt$$

Concerning to this, he proved the following three lemmas and applied to the solution of the generalied Stieltjes problem.

Theorem ([*16, pp. 415–6, Lemma 9*]). *If $(z+i)^{-1}F(z)\in\mathfrak{H}_1$ or $(z+i)^{-2}F(z)\in\mathfrak{H}_1$, respectively, then*

$$F(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(t)\,dt}{t-z} \quad or \quad F(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}F(t)\left(\frac{1}{t-z}-\frac{1}{t+i}\right)dt$$

for $y>0$, while the integrals vanish identically for $y<0$.

Theorem ([*16, p. 416, Lemma 10*]). *The function $(t+i)^{-j}F(t)$ $(j=1, 2)$ is the limit-function of an element of $\mathfrak{H}_1$, if, and only if, $(1+|t|)^{-j}F(t)\in L_1(-\infty, \infty)$ and $\mathfrak{H}F=iF$ or $\mathfrak{K}F=i(F-a_0)$, respectively, where*

$$a_0=\pi^{-1}\int_{-\infty}^{\infty}(1+t^2)^{-1}F(t)\,dt$$

and

Theorem ([*16, p. 415, Lemma 11*]). *If* (i) $(1+|t|)^{-j}F(t)$ $(j=1, 2)$ *and* (ii) $(1+|x|)^{-1}\mathfrak{H}F$ *or* $(1+x^2)^{-1}\mathfrak{K}F$; *respectively, belong to* $L_1(-\infty, \infty)$, *then*

$$\mathfrak{H}^2F=-F(x)\ (j=1) \quad or \quad \mathfrak{K}^2F=-F(x)+a_0\ (j=2).$$

Here

(i) *The last formular of first Theorem is just of ours* (c.f. p. 192, (13.03)).

(ii) *The first part of last Theorem is also obtained from our theorem* (c.f. p. 194, Theorem 40).

(2) On the Hilbert transform II.

8. p. 112, Theorem 12. Let a ond b be finite real numbers $(a<b)$ and $0<p\leqq\infty$. Let S be the region $-\infty<x<\infty$, $a<y<b$, and let $S_p(a, b)$ be the set of functions which are analytic in S and satisfy the condition

$$||F(x+iy)||_p \leqq A_p \qquad (a<y<b),$$

where A_p does not depend on y.

Then the H. Kober [17, p. 24, Theorem 13] also proved the completely analosous results as those of E. Hille-J. D. Tamarkin in the half-plane. The method of his proof is based the approximation by integral functions in the complex domain as similar to his previous papers. If we apply this theorem to the function $f(z)/(z+ic)$ $(c<a<b)$ then we get our Theorem 12 and moreover some new results.

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