A UNIFIED THEOREM WITH SOME APPLICATIONS TO GENERALIZATIONS OF G. REEB'S THEOREM

By

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Introduction

The main purpose of the present paper is to bring into unity several theorems in the field of differential geometry. Some new results are included. The relevant theorems are cited below.

Theorem A. (de Rham [1]). *Let M be a simply-connected complete Riemannian manifold and let $T(M)$ be the tangent bundle with the holonomy group as structural group.*

If the Whitney sum of vector sub-bundles of $T(M)$:

$$T_1(M) + \cdots + T_p(M)$$

can be reduced to $T(M)$, then M turns out to be a product of Riemanian manifolds $M_1, \cdots, M_p$ and a Euclidean space E^d, namely

$$M = M_1 \times \cdots \times M_p \times E^d.$$

Theorem B. (G. Reed [2] and J. Milnor [3]). *Let M be a compact differentiable manifold. If there exists a differentiable function with exactly two non-degenerate critical points over M, then M is homeomorphic to an n-sphere where n is the topological dimension of M.*

We shall state generalizations of this theorem at the end of this introduction.

Theorem C. (S. Kobayashi). (See [4]). *Let M be a complete Riemannian manifold. Then any Killing vector field defined over M generates a 1-parameter transformation group of isometries.*

Theorem D. (K. Nomizu [5]). *Let M be a simply-connected Riemannian manifold and let $\mathfrak{K}$ be the sheaf over M of germs of Killing vector fields. If $\dim \mathfrak{K}_x$ do not depend on x, then $\mathfrak{K}$ is a constant sheaf.*

We now sketch the outline of the present paper in what follows. The first chapter is intended for the proof of a tool theorem together with pertinent definitions. Let us state the main tool theorem precisely.

Let M be a (*not necessarily connected*) complete[1] Riemannian manifold and let Γ be a regular[2] and connected[3] Lie pseudogroup of local isometries. We denote the set of germs of local isometries belonging to Γ by S.

Main Tool Theorem. $(C(g), \alpha)$ *is a covering space of* $F_{\alpha(g)}$ *for each* $g \in S$, *where* $C(g)$ *is the connected component of* g *in* S *and* $F_{\alpha(g)}$ *that of* $\alpha(g)$ *in* M.

Let M be a locally connected and locally simply-connected Hausdorff space each component of which is compact. Let Γ be a connected and regular continuous pseudogroup of local transformations on M. In addition to these, we assume that the postulate of regular imbedness is valid for Γ. Then the same conclusion as above holds good[4].

In the second chapter we engage in some applications of the theorem stated above. In Sec. 1 we show that the theorem of identity is valid for affine maps of an affinely-connected manifold. In Sec. 2 an affine map of a Riemannian manifold is uniformly continuous with respect to the uniformity given by the metric. The present author has particular interest in whether or not a uniformity (canonical in a sense) can be introduced into an affinely-connected manifold in such a fashion that every affine map is uniformly continuous. This is one of the reason why we pay some attention to uniformity. In Sec. 3 some study of isometries in analytic Riemannian manifolds will be made. And there we prove the following theorem.

Theorem E. *Let* F_1 *and* F_2 *be simply-connected complete and analytic Riemannian manifolds of the same dimension. Then the existence of a local isometric map implies that of a global isometry. More precisely: Let f be an isometry of* $\alpha(f)$ *onto* $\beta(f)$, *where* $\alpha(f)$ *is an open set in* F_1 *and* $\beta(f)$ *in* F_2. *Then there exists an isometry* $\tilde{f}$ *of* F_1 *onto* F_2 *such that* $\tilde{f}|\alpha(f) = f$.

Sec. 4 is concerned with Lie pseudogroups and Theorem D is proved in this form:

Theorem F. *Let* M *be a simply-connected manifold and let* Γ *be*

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1) This means that each component is complete.

2) See Def 3.3, Chap. I and Proposition 4.2, Chap. II.

3) A continuous pseudogroup is called connected, if $S_x = \alpha^{-1}(x)$ are connected for all $x \in M$.

4) One of the applications is the Theorem 5.1, Chap. II where the postulate is automatically satisfied.

a regular Lie pseudogroup of local transfomations on M. Then the sheaf of germs of infinitesimal transformations belonging to Γ is a constant sheaf, provided that the theorem of identity holds good for Γ.

In Sec. 5 we prove Theorem A together with a preparatory theorem to the next section whose aim is generalizations of Theorem B. The results of this last section can be stated as two theorems each one of which includes the Reeb's classical theorem.

Let M be a Riemannian manifold. A differentiable function f defined over M is said to satisfy Lipschitz condition if and only if there exists a constant L over M such that for any vector field X

$$Xf = O \longrightarrow \left|X||\mathrm{df}||\right| \leq L ||X||^{5)}$$

where we note that since $||\mathrm{df}||$ is a scalar, X can operate on it.

Theorem G. *If there exists a differentiable function f satisfying Lipschitz condition and having only positive or negative difinite critical points over complete Riemannian M, then the number of critical points, denoted by $\mathfrak{n}(f)$, is equal to or smaller than 2 and*

(1) *M is homeomorphic to R^n for $\mathfrak{n}(f) = 1$,*

(2) *M is homeomorphic to S^n for $\mathfrak{n}(f) = 2^{6)}$,*

where n is the topological dimension of M.

Theorem H. *Let M be a compact differentiable manifold. Let f be a differentiable function defined except on a nowhere dense set F each point of which is removable in the sense stated in the last section. If all the critical points of f are positive or negative definite, then $\mathfrak{n}(f) \leq 2$, and*

(1) *M is homeomorphic to P^n for $\mathfrak{n}(f) = 1$,*

(2) *M is homeomorphic to S^n for $\mathfrak{n}(f) = 2^{6)}$.*

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5) It is interesting to note that this condition is satisfied by any differentiable function on compact M.

6) If $\mathfrak{n}(f) = 0$, then M is homeomorphic to a product manifold $M_{n-1} \times R$, where M_{n-1} is a suitable simply-connected manifold of dimension $n-1$.

CHAPTER I. THE MAIN TOOL THEOREM

1. Pseudogroups of local transformations

Let F_1 and F_2 be topological spaces[7]. Then a local bicontinuous map a is a homeomorphism of an open subset V on F_1 onto an open subset on F_2. V and $a(V)$ are called the domain of a and the range of a respectively. We denote them by $\alpha(a)$ and $\beta(a)$. We write $a\,|\,V'$ for the restriction on $V \cap V'$ of a, where V' is an open set on F_1. Let F_3 be another topological space and let b be a local bicontinuous map of an open set on F_2 into F_3. Then if $\beta(a) \cap \alpha(b) \neq 0$, we can define the product $b \cdot a$ on $\alpha(a) \subset a^{-1}(\alpha(b))$.

A *pseudogroup* of local bicontinuous maps (or local transformations) of a topological space M is by definition a collection Γ of local bicontinuous maps of M into itself which satisfies these:

(1) If $a, b \in \Gamma$ and $\beta(a) \cap \alpha(b) \neq 0$, then $b \cdot a \in \Gamma$.

(2) If $a \in \Gamma$, then $a^{-1} \in \Gamma$.

(3) If $a \in \Gamma$, then $a\,|\,U \in \Gamma$ for open U with $\alpha(a) \cap U \neq 0$.

(4) The identity of M onto itself belongs to Γ.

(5) If there exists an open covering $\{V_j\}_{j \in J}$ of $\alpha(a)$ such that for each j $a\,|\,V_j \in \Gamma$, then $a \in \Gamma$.

Let $\{F_\lambda\}_{\lambda \in \Lambda}$ be a family of connected, locally connected, locally simply-connected topological spaces. Define a space M by this:

$$M = \bigcup_{\lambda \in \Lambda} F_\lambda ,$$

and consider the topology of M as the one given by the totality of open sets of $F_\lambda, \lambda \in \Lambda$. Let Γ be a pseudogroup of local transformations on M. We denote by S the set of all the germs of elements of Γ and by $S_{\lambda, \mu}$ that of elements $a, a', \cdots$ with $\alpha(a)$, $\alpha(a'), \cdots \subset F_\lambda$ and $\beta(a), \beta(a'), \cdots \subset F_\mu$. We introduce the natural topology into S. The map that sends $\gamma_x a$ to $\gamma_{a(x)} a^{-1}$ is a homeomorphism of S onto itself that maps $S_{\lambda, \mu}$ onto $S_{\mu, \lambda}$.

We suppose now the following conditions be satisfied by Γ as we shall do throughout the first chapter.

(6) The condition of analyticity: Let $a, b \in \Gamma$ and $x \in \alpha(a) \cap \alpha(b)$. If there exists an open set V in $\alpha(a) \subset \alpha(b)$ such that

(1. 1) $\qquad\qquad a = b$ in V and $x \in \bar{V}$,

then $a = b$ in a neighborhood of x.

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7) By a space we always mean a Hausdorff one.

Further in this section we assume the following condition.

(7) For any $x \in M$ we can find a neighborhood $V(x)$ of x such that for any $a \in \Gamma$ with $x \in a(a)$ there exists $b \in \Gamma$ with $a(b) = V(x)$ and $a = b$ in $V(x) \subset a(a)$.

Remark. The condition (6) makes S a Hausdorff space and the condition (7) guarantees for each $x \in M$ the existence of a neighborhood which is evenly covered by a. It is interesting to note that (6) is satisfied not only by holomorphic maps of a complex manifold into a complex manifold, but also by several maps of other types, as will be shown in the sequel, and that our concern in this chapter is mainly directed to the question: *Under what conditions can* (7) *be proved*?

Let $g \in S_{\lambda,\mu}$ and let $C(g)$ denote the connected component of g in S. Suppose that $a(C(g)) = F_\lambda$. Then by means of (6) and (7) $(C(g), a)$ is a covering space of F_λ. Since F_λ has a universal covering space, denote it by $(\widetilde{F}_\lambda, \pi)$ and let $\widetilde{x}$ be a point over $a(g)$. Then there exists one and only one continuous map q_λ of $\widetilde{F}_\lambda$ onto $C(g)$ such that

$$\pi_\lambda = a \cdot q_\lambda \quad \text{and} \quad q_\lambda(\widetilde{x}_\lambda) = g \ .$$

We write $r_\lambda = \beta \cdot q_\lambda$. Then r_λ is a continuous map of $\widetilde{F}_\lambda$ into F_μ for some $\mu \in \Lambda$.

Definition 1.1. Two spaces F_λ and F_μ of the family are said to be *regularly connected* if there exists $g \in S_{\lambda,\mu}$ such that $C(g)$ covers F_λ by a and $C(g^{-1})$ F_μ (R. Hermann [6]).

Proposition 2.2 *Two regularly connected spaces of the family have homeomorphic universal coverings* (cf. [6], Theorem 2.1).

Proof. $C(g)$ and $C(g^{-1})$ with projection a are covering spaces of F_λ and F_μ respectively. $C(g)$ and $C(g^{-1})$ are homeomorphic to each other by the map that sends an element of S to its inverse. Hence they have homeomorphic universal coverings. These universal covering spaces coincide with those of F_λ and F_μ respectively. This completes the proof.

Remark. Let $(\widetilde{F}_\mu, \pi_\mu)$ be a universal covering space of F_μ and let x_μ be an arbitrary point over $\beta(g)$. Then there exists one and only one homeomorphism f of $\widetilde{F}_\lambda$ onto $\widetilde{F}_\mu$ with these conditions:

(1) $f(x_\lambda) = x_\mu \ . \ $

(2)   The following commutative diagram holds.

$$(1.2) \quad \begin{array}{ccc} \widetilde{F_\lambda} & \xrightarrow{\quad f \quad} & \widetilde{F_\mu} \\ q_\lambda \downarrow & & \downarrow \pi_\mu \\ C(\mathcal{G}) & \xrightarrow{\quad \beta \quad} & F_\mu \end{array}$$

Theorem 1. 1. *Let M be a topological space. Let Γ be a pseudogroup of local transformations on M which satisfies the condition of analyticity. Then any element of Γ admits only one way of extension to a connected open set in M, if possible. This can be more precisely as follows. Let $a, b \in \Gamma$ and $\alpha(a) = \alpha(b)$, for which we write D. Let D be connected. If there exists an open set V with $V \subset D$ such that $a(x) = b(x)$ for $x \in V$, then $a(x) = b(x)$ over the whole D, namely, $a = b$.*

Proof. Let V' be a maximal open set such that $V \subset V' \subset D$ and $a(x) = b(x)$ for $x \in V$. If $V' \neq D$, then $\partial V' = D \cap \overline{V'} - V' \neq 0$, because D is connected. Let $y \in \partial V$. Then by the means of the condition of analyticity we can find some open $V(y)$ with $y \in V(y)$ and with $a(x) = b(x)$ for $x \in V(y)$, where $V(y)$ is supposed to be contained in D. Hence $a(x) = b(x)$ for $x \in V' \cup V(y)$. Consequently we get $V(y) \subset V'$. This cotradiction completes the proof.

2. Foliation and uniformity

Let M be a topological space.

Definition 2. 1. A subspace imbedded in M is a pair of a connected topological space F and a continuous map ϕ of F into M such that each point of F has a neighborhood V with $\phi | V$ being a homeomorphism of V onto $\phi(V)$.

Definition 2. 2. A *foliation* in a topological space M is a family $\{(E_\mu, \phi_\mu)\}_{\mu \in \Lambda}$ of *subspaces imbedded* in M such that there passes through each $x \in M$ one and only one subspace of the family (which we frequently denote by F_x in brief).

Remark. Let V_μ be any open set in F_μ such that the imbedding ϕ_μ of F_μ is bicontinuous in it. The totality of $\phi_\mu(V_\mu)$ for such V_μ constitutes a basis of a new topology T' in M. Actually in [7] a foliation is defined by the pair of two topologies T and T' in M, where T is the *intrinsic* topology of M.

Now we make a brief review of the definition of uniformity. For the full details the reader must consult [8].

Let S be a set and 2^S the set of all the subsets of S. Then a filter $\mathfrak{F}$ is a subset of 2^S with these conditions:

1 $0 \notin \mathfrak{F}$ and $S \in \mathfrak{F}$.

2 If $S_1 \in \mathfrak{F}$ and $S_1 \subset S_2$, then $S_2 \in \mathfrak{F}$.

Now let M also be a set and consider $M \times M$. If σ_1, σ_2 are subsets of $M \times M$, the product and the inverse are defined as follows:

$$\sigma_1 \sigma_2 = \{(x, z) | (x, y) \in \sigma_1 \text{ and } (y, z) \in \sigma_2 \text{ for some } y \in M)\}$$

$$\sigma_1^{-1} = \{(x, y) | (y, x) \in \sigma_1\}.$$

Denote the diagonal set of $M \times M$ by Δ. Then a uniformity in M is by definition a structure given by a filter F of $M \times M$ with these conditions:

1 If $\sigma \in \mathfrak{F}$, then $\Delta \subset \sigma$.

2 For any $\sigma_1 \in \mathfrak{F}$ there exists $\sigma_2 \in \mathfrak{F}$ with $\sigma_2 \sigma_2 \subset \sigma_1$.

3 If $\sigma \in \mathfrak{F}$, then $\sigma^{-1} \in \mathfrak{F}$.

A uniformity on M is called complete if every Cauchy sequence or directed system is convergent. If M is a compact topological space, then M has a complete uniformity which is compatible with the topology of M and it is unique up to the uniform transformations.

Proposition 2. 1. *Let F_1 and F_2 be complete uniform spaces and let a be a uniformly bicontinuous map of an open set $\alpha(a) \subset F_1$ onto an open set $\beta(a) \subset F_2$. If $\alpha(a)$ is relatively compact in F_1, then $\beta(a)$ is so.*

Proof. Any relatively compact set in a complete uniform space is totally bounded and the image of a totally bounded set by a uniformly continuous map is totally bounded. Further a totally bounded set in a complete uniform space is relatively compact. Hence we get the proposition.

Definition 2. 3. Let M be a foliated space and let Γ is a pseudogroup of local transformations on M. Then Γ is said to be *compatible with the foliation* if for $a \in \Gamma$ $F_x = F_y$ implies $F_{a(x)} = F_{a(y)}$.

3. Continuous pseudogroups of local transformations

Let Γ be a pseudogroup of local transformations of a topological space M and let S denote the set of all the germs of elements of Γ. We can introduce a topology into S in the natural way. Then α (resp. β), a map assigning the domain (resp. the range) to each element of S, is a local homeomorphism of S onto M. We write $\alpha^{-1}(x)$ as S_x for each $x \in M$. Further let Δ be the set of the germs of the identities and *let $\tilde{x} \in \Delta$ be the germ of identities in the neighborhoods of x*. Then we have $\Delta \cap S_x = \tilde{x}$. By the map or the *cross-section* of S sending x to $\tilde{x}$ is M imbedded in S.

Definition 3. 1. A pseudogroup Γ of local transformations which satisfies the condition of analyticity is called a *continuous pseudogroup* if and only if

we can give to S one more topology of the following nature. Below the topology of S is assumed to be a new one.

(3. 1) S_x is open for each $x \in M$.

(3. 2)   The map to each $(g, h) \in \mathfrak{W}$ assigning $gh \in S$ is continuous, where $\mathfrak{M} = \{(g, h) | g, h \in S$ and $\beta(g) = \alpha(h)\}$.

(3. 3)   The map to each $(g, x) \in \mathfrak{Q}$ assigning $g(x) \in M$ is continuous, where $\mathfrak{Q} = \{(g, x) | g \in S$ and $\alpha(g) = x\}$.

(3. 4) S is a locally connected Hausdorff space.

The new topology will be referred to as *vertical* topology while the natural one as *horizontal*.

Definition 3. 2. A map φ of a product space $W \times V$ into S is said to be *compatible with the structure of S*, if and only if these two conditions are satisfied:

(1)   To each $g \in W$ φ takes $g \times V$ into one and only one component of S with respect to the horizontal topology.

(2)   To each $x \in V$ φ takes $W \times x$ into one and only one component of S with respect to the vertical topology.

In the present paper a map into S and with two variables g and x is called continuous if for each fixed g it is continuous with respect to x and if for each fixed x it is continuous with respect to g. It is noted that, if V and W are both connected and if φ is continuous, then φ is compatible.

We now add one more condition which will be of vital importance for the proof of our main tool theorem.

Let M be a foliated space[8]. Then the condition is that:

(3. 5)   To each $x \in M$ there exist an open neighborhood $V(x)$ of x in F_x and an open neighborhood $W(\tilde{x})$ of $\tilde{x}$ in S_x such that the inclusion of $W(\tilde{x})$ into S_x can be extended to a homeomorphism φ_x of the product space $W(\tilde{x}) \times V(x)$ into S so that it is compatible in the sense of Definition 3.2 and so that for $y \in V(x)$ $\varphi_x(\tilde{x}, y) = \tilde{y}$.

Definition 3. 3 (Regularity). A continuous pseudogroup of local transformations of a foliated space is called *regular* if and only if it satisfies the above condition (3. 5).

Let $\{V_i\}_{i \in I}$ be an open covering of a connected topological space K. By a chain of the nerve N we mean a sequence $i_1, i_2, \cdots, i_m$ with $(i_{k-1}, i_k) \in N$ for

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8)   All the time when we consider a foliated space, the continuous pseudogroups under consideration are assumed as the ones compatible with the foliation.

$l \leq k \leq m$. Then any pair (i, j) with $i, j \in I$ we can find a chain with the origin i and the end j (see A. Weil [9]).

Let us go back to a foliated space M and consider a regular continuous pseudogroup Γ of local transformations. Let K be a compact set on a leaf F. Then we can assign the map φ_x stated in the preceding section to each $x \in K$. Using the same notation as there, we can select a finite covering $V(x_i)$, $i = 1, \cdots, l$ of K. We write V_i (resp. W_i) instead of $V(x_i)$ (resp. $W(\tilde{x}_i)$). Let $g \in W_{i_0}$ and let N be the nerve of the covering. We say that g is *extendable along a chain* $(i_0, \cdots, i_k)$ of N if the connected component of g in S with respect to the horizontal topology meets $\varphi_{x_i}(W_i \times V_i)$ for every i belonging to the chain.

Let i_0 be fixed and let us use the symbol $C(g)$ for the component of g in S with respect to the horizontal topology. Further denote by $C^K(g)$ the total sum of $\varphi_{x_i}(W_i \times V_i) \cap C(g)$ such that i belongs to some chain with the origin i_0 along which g is extendable.

Lemma 3. 1. *There exists an open subset W in W_{i_0} such that if $g \in W$, then $C^K(g)$ covers K, i. e.,*

$$\alpha(C^K(g)) \supset K.$$

Proof. Assume the contrary. Then we can choose a directed system $\{g_j\}_{j \in J}$ of germs with $g_j \in W_{i_0}$ for all $j \in J$ so that $\alpha(C^K(g_j)) \not\supset K$ for all $j \in J$ and $\lim_J g_j = \tilde{x}_{i_0}$, where $\tilde{x}_{i_0}$ is the germ of identities in the neighborhoods of x_{i_0}. Select a point x_j from $C\alpha(C^K(g_j)) \cap K$. Then there exists an accumulation point $x \in K$ for $\{x_j | j \in J\}$. And we can find a chain $C = (i_0, \cdots, i_k)$ with $V_{i_k} \in x$. If W' is a suitable open set in W_{i_0}, then all $g \in W'$ are extendable along C. Hence for $g \in W'$ $\alpha(C^K(g)) \supset V_{i_k} \in x_j$ for some j with $j \geq j_0$, where j_0 is a suitable element of J. This is contrary to $C\alpha(C^K(g_i)) \in x_i$ for all $j \in J$. This completes the proof.

Proposition 3. 1. *Let M be a foliated space and let F be a leaf. Let Γ be a regular continuous pseudogroup of local transformations on M and S the set of germs of local transformation belonging to Γ. If V is a simply-connected and relatively compact open set on F and if x is a point of V, then by taking a suitable neighborhood W_x of x in S_x we can find a continuous and compatible map φ of $W_x \times V$ into M such that*

(1) $\varphi | \tilde{x} \times V$ *is trivial, i. e.* $\varphi(\tilde{x}, y) = \tilde{y}$ *for* $y \in V$.

and

(2) $\varphi | W_x \times x$ *is trivial, i. e.* $\varphi(v, x) = v$ *for* $v \in W_x$.

Proof. Let K be a compact set in F with $K \supset V$. We may suppose that K is connected, because any connected component is closed. Let g be a germ

belonging to S_x. Let $K(g)$ be a connected component of g in $\alpha^{-1}(K) \cap C^K(g)$, using the same notation as in the preceding. α gives rise to a local homeomorphism of $K(g)$ into K, which is denoted by the same letter α. If we take g from W in the preceding lemma, then α is surjective. It is clear that $V_i \cap K$ is evenly covered by α more precisely $\alpha|K(g)$. These imply that $(K(g), \alpha)$ is a covering space of K. Let $\widetilde{V}' = \alpha^{-1}(V) \cap K(g)$ and let $\widetilde{V}$ be a connected component of $\widetilde{V}'$. Then $(\widetilde{V}, \alpha)$ is also a covering space of V. On the other hand V is simply-connected. Hence $\widetilde{V}$ is homeomorphic to V by α. We shall denote this homeomorphism by α_g. We put:

(3. 6) $\qquad\qquad \varphi(g, y) = \alpha_g^{-1}(y)\,,$

Where (g, y) ranges over $W \times V$. Then φ is the map of the required nature. This completes the proof.

4. The proof of the main tool theorem

Let M be a locally connected and locally simply-connected topological space. We do not suppose it be connected. It can be considered as a foliated space the topologies of which coincide with each other. In this section we deal exclusively with continuous pseudogroups which are regular in the sense of Definition 3.3 and equi-uniform in the sense of the following definition.

Definition 4.1. A continuous pseudogroup Γ is called *equi-uniform* if all the maps that have relatively compact domain and that are the extensions[9] of local transformations of Γ have also relatively compact range.

Examples. (1) Pseudogroups of local uniformly bicontinuous maps in the case where the space M is a complete uniform space, especially a complete Riemannian manifold (see Sec. 2 of Chap. 2). (2) Pseudogroups of any kind in the case where all the leaves are compact.

Postulate of regular imbedness. Let x be any point of M. If we take sufficiently small $V(x)$ and $W(\widetilde{x})$, then $C(g) \subset \varphi_x(W(\widetilde{x}) \times V(x))$ is connected with respect to the natural topology, where we use the notation in (3.5) and g is any element of S.

In what follows we assume that regular and equi-uniform continuous Γ satisfies the above postulate unless stated otherwise.

Lemma 4.1. *Let $x \in M$ and let V be a connected, simply connected, and relatively compact open set in F_x with $x \in V$. Then there exists one and only one continuous map φ of $S_{x_0} \times V$ into S with these conditions:*

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9) By an extension we mean a continuous map f the domain of which has an open covering $\{U_i\}_{i \in I}$ with $f|U_i \in \Gamma$ for each $i \in I$.

(4. 1) $\varphi\,(h,\,x) = h$ for $h \in S_{x0}$ ..

(4. 2) $\varphi\,(x,\,y) = y$ for $y \in V$.

(4. 3) φ *is compatible in the sense of Definition* 3.2,

where $\tilde{x}$ is the germ of identities in neighborhoods of x.

   Proof.

   I. Uniqueness. Let φ_1 and φ_2 be two maps of the nature stated above. Since φ_1 and φ_2 are compatible,

$$\varphi_1(h,\,y),\;\;\varphi_2(h,\,y) \in S_{y0} \qquad \text{for any} \;\; y \in V.$$

Let $a_1(y) = \beta_0\varphi_1(h,\,y)$ and $a_2(y) = \beta_0\varphi_2(h,\,y)$. Then it follows that $\gamma_y a_1 = \varphi_1(h,\,y)$ and $\gamma_y\;a_2 = \varphi_2(h,\,y)$ for any $y \in V$. By (4. 1), $\gamma_x a_1 = \gamma_x a_2$. Then it follows from the theorem of identities that $a_1 = a_2$. Hence $\varphi_1(h,\,y) = \varphi_2(h,\,y)$ for any $y \in V$. This completes the proof of the uniqueness.

   II. Let H be an open set in S_{x0} and assume that there exists a compatible φ_H which sends $H \times V$ into S and which has the natures: (4.1) where S_{x0} is replaced by H, and (4.2), and (4.3).

   III. If $H \neq S_{x0}$, then let $h \in \partial H$ and write

$$H' = H + h .$$

Consider the connected subsets $S, S', \cdots$ of V with the property that $\varphi_H|H \times S, \varphi_H|H \times S', \cdots$ are extendable to $H' \times S$, $H' \times S', \cdots$ while preserving the above natures with S_{x0} replaced by H'. It is seen that among $S, S', \cdots$ there exists a maximal one with respect to the inclusion relation $\subset$. We denote it by V_0.

   IV. V_0 is open in V.

   Proof of IV. Denote the extension over $H' \times V_0$ of $\varphi_H|H \times V_0$ by the same symbol φ_H and let $y \in V_0$. We write $z = \beta_0\varphi(h_H,\,y)$. With the notation of Section 3, φ_z is a continuous map of $W(\tilde{z}) \times V(z)$ into S, where $V(z)$ and $W(\tilde{z})$ are connected neighborhoods of z and $\tilde{z}$ in F_z and S_{x0} respectively. Let f be an element of Γ belonging to the germ $\varphi_H(h,\,y)$ and let $(g,\,u) \in H \times (\alpha(f) \cap V)$. We define a map φ by

$$\varphi(g,\,u) = \gamma_u f\varphi_z\left(\varphi_H(h,\,y)^{-1}\varphi_H(g,\,y),\,f(u)\right) .$$

Then φ is a continuous and compatible map of $H' \times (\alpha(f) \cap V)$ into S. For any $g \in H$ we have

$$\varphi(g,\,y) = \gamma_y f\varphi_H(h,\,y)^{-1}\varphi_H(g,\,y) = \varphi_H(g,\,y).$$

It is noted here that we may suppose $\alpha(f) \cap V$ is connected. Then using the theorem of identity, we can get .

$$\varphi(g, u) = \varphi_H(g, u) \quad \text{in} \quad H \times (\alpha(f) \cap V).$$

Further by continuity we obtain the same equality for $(g, u) \in H' \times \alpha(f) \cap V_0$. If $\alpha(f) \cap V$ is not contained in V_0, then we can extend φ_H further. This is contrary to the definition of V_0. Hence $\alpha(f) \cap V \subset V_0$. This completes the proof of IV.

V. V_0 is closed in V.

Proof of V. Let $y \in \overline{V}_0 \cap V$ and further $y \notin V_0$. Since Γ is equi-uniform, the set: $\{\beta_0 \phi_H(h, u) | u \in V_0\}$ is relatively compact. Then we can find a directed system $\{y_j\}_{j \in J}$ such that the system itself tends to y from the inside of V_0 (this means that $y_j \in V_0$ for all $j \in J$) and such that the image system $\{\beta_0 \varphi_H(h, y_j)\}_{j \in J}$ also has a limit in $F_{\beta(h)}$. Denote the limit by z. We use the notation as in IV. It is seen from the postulate of regular imbedness that there exist a number j_0 in J and a neighborhood $W(h)$ of h in S_{x_0} so that

$$(4. 4) \qquad \varphi_H(h, y_j)^{-1} \varphi_H(g, y_j) \in \varphi_z(W(\tilde{z}) \times V(z)) \text{ for } j > j_0$$

$$\text{and for } g \in H' \cap W(h).$$

We can write $\varphi_z^{-1}(\varphi_H(h, y_j)^{-1} \varphi_H(g, y_j))$ as $(k, \beta \cdot \varphi_H(h, y_j))$. Then clearly k does not depend on the choice of y_j. We can, therefore, write it as $k(g)$. Then $k(g)$ is continuous with respect to variable g in $H' \cap W(h)$. Especially we have

$$\lim_{g \to h} k(g) = \tilde{z}.$$

Fix $g \in H \cap W(h)$ for a while and define the value at (h, y) of φ_H by

$$\varphi_H(h, y) = \varphi_H(g, y) k(g)^{-1}.$$

It is clear that $\beta \cdot \varphi_H(h, y) = z$. And we have

$$\lim_J \varphi_H(h, y_j) = \lim_J \varphi_H(g, y_j)_z \varphi(k(g), \beta \cdot \varphi_H(h, y_j))^{-1}$$

$$= \varphi_H(g, y) \varphi_z(k(g), z)^{-1} = \varphi_H(g, y) k(g)^{-1} = \varphi_H(h, y).$$

From this follows

$$\lim_{\substack{u \in V_0 \\ u \to g}} \varphi_H(h, u) = \varphi_H(h, y).$$

On the other hand we can see

$$\lim_{g \to h} \varphi_H(g, y) = \lim \varphi_H(g, y) k(g)^{-1} = \varphi_H(h, y).$$

because the definition of $\varphi_H(h, y)$ does not depend on the choice of g as seen from the above caluculation. Thus it is seen that we can extend φ_H to the region $H' \times (V + y)$. This is contrary to the definition of V_0. Hence

$$V_0 = \overline{V}_0 \cap V.$$

This completes the proof of V.

    VI.   Since V is connected, we can conclude

$$V = V_0.$$

In other words, we can find a continuous map of $H' \times V$ into S which has the conditions (4.1), (4.2) and (4.3), provided that in (4.1) S_{x_0} should be replaced by H'.

    VIII.   Consider the open subsets $H, H', \cdots$ of S_{x_0} with these two conditions:

1   $x \in H, H', \cdots$

2   There exist continuous maps $\varphi_H, \varphi_{H'}, \cdots$ of $H \times V, H' \times V, \cdots$ which satisfy (4.1), (4.2) and (4.3) with S_{x_0} being replaced by $H, H', \cdots$ respectively. Then there exists a maximal one among them with respect to the inclusion relation. Let H denote it for the economy of letters.

    -VIII.   Assume $H \neq S_{x_0}$ and let $h \in \partial H$. According to VI, we can extend the domain of φ_H to $(H + h) \times V$, continuously and preserving (4.1), (4.2), and (4.3), with S_{x_0} in (4.1) being replaced by $H + h$. Let $V' = \beta \cdot \varphi_H (h \times V)$. Then V' is relatively compact. Hence by Proposition 3.1 we can get a continuous map $\bar{\varphi}$ of $W(h) \times V$ into S such that

    (1) $\bar{\varphi}(h, y) = $ the germ of identities at $\varphi_H(h, y)$ for $y \in V$

    (2) $\bar{\varphi}(g, x) = h^{-1} g$ for $g \in W(h)$

where $W(h)$ is a connected open neighborhood of h in S_{x_0}. Consider the map φ defined by $\varphi(g, y) = \varphi_H(h, y) \bar{\varphi}(g, y)$. Then if $g \in W(h) \cap H$, $\varphi_H(g, x) = g = h h^{-1} g = \varphi(h, x) \bar{\varphi}(g, x) = \varphi(g, x)$. Therefore by the theorem of identity we obtain

$$\varphi_H(g, y) = \varphi(g, y) \qquad \text{for } (g, y) \in (W(h) \cap H) \times V.$$

This shows that φ_H can be extended to $(W(h) \cap H) \times V$, preserving the natures under consideration. This contradiction concludes

    IX. $H = S_{x_0}$.

This completes the proof of Lemma 4.1.

    Remark.   Let M be a complete (*not necessarily connected*) Riemannian manifold. Let Γ be a continuous pseudogroup of local isometric maps (see Sec. 1, Chap. 2). In this case *there is no need for the postulate of regular imbedness*. Remember that it was used only to guarantee (4.4). Consequently it suffices to show that (1.4) is verified without using this postulate. Let ε be a sufficiently small positive number. If we take sufficiently small $V'(z) \subset V(z)$, then the ε-neighborhood of any point $\in \beta \cdot \varphi_z(g, V'(z))$ where $g \in W(\tilde{z})$ is arbitrary is contained in $\beta \cdot \varphi_z(g, V(z))$. Take a number j_0 such that if $j_0 \leq j$, then $z_j \in V'(z)$

and y_j belong to the ε-neighborhood V_ε of y_{j_0}. Since $\varphi_H(h, y_{j_0})^{-1}\varphi_H(g, y_{j_0})$ converges to $\tilde{z}_{j_0}$ as g tends to h,

$$\varphi_H(h, y_{j_0})^{-1}\varphi_H(g, y_{j_0}) \in \varphi_z(k(g) \times V(z))$$

where $g \in W(h) \cap H$ for a sufficiently small neighborhood of h, $W(h)$. Let $f(u) = \beta \cdot \varphi_z(k(g), u)$. Let $z' = f(z)$. Then f is a local isometric map belonging to Γ. We write

$$\varphi_{z'}(k, u) = \varphi_z(k(g) k, f^{-1}(u)) .$$

We write $V(z')$ (resp. $W(\tilde{z}')$) for $f(V(z))$ (resp. $k(g)^{-1}W(\tilde{z})$). Then $\varphi_{z'}$ is a compatible map of $W(\tilde{z}') \times V(z')$ into S. Let g' be another element of $W(h) \cap H$. Then clearly

$$\varphi_H(g, y_{j_0})^{-1}\varphi_H(g', y_{j_0}) \in \varphi_{z'}(W(\tilde{z}') \times V(z')).$$

On the other hand $\beta \cdot \varphi_H(g, V) \subset \beta \cdot \varphi_z(k(g) \times V(z)$. It follows from this that we can define a map over V_ε by

$$\bar{\varphi}(\bar{y}) = \varphi_{z'}(k(g)^{-1}k(g')), \qquad \beta \cdot \varphi_H(g, \bar{y}) .$$

Then $\bar{\varphi}(y_{j_0}) = \varphi_H(g, y_{j_0})^{-1}\varphi_H(g', y_{j_0})$. Since $\alpha \cdot \bar{\varphi}(\bar{y}) = \beta \cdot \varphi_H(g, \bar{y})$, it follows from the theorem of identitiy that

$$\bar{\varphi}(\bar{y}) = \varphi_H(g, \bar{y})^{-1}\varphi_H(g', \bar{y}) .$$

Hence for $j \geqq j_0$,

$$\varphi_H(g, y_j)^{-1}\varphi_H(g', y_j) \in \varphi_{z'}(k(g)^{-1}k(g'), V(z')) .$$

If we make g' tend to h, then we can get

$$\varphi_H(g, y_j)^{-1}\varphi_H(g', y_j) \in \varphi_{z'}(k(g)^{-1}k(g'), V(z')).$$

Consequently we have

$$\varphi_H(h, y_j)^{-1}\varphi_H(g, y_j) \in \varphi_z(k(g), V(z')).$$

This completes the proof of our assertion.

It is noted that the same thing holds good for the case where Γ is a continuous pseudogroup of *local uniformly bicontinuous maps*.

Let Γ be a continuous pseudogroup of one of the two kinds in the statement of the tool theorem in the introduction. Then Γ is equi-continuous automatically. In the case of the second kind we assume that the condition of regular imbedness is satisfied. In both cases we can use the conclusion in Lemma 4.1 owing to the above remark. Now let us prove the theorem.

Proof of the main tool theorem. We divide the proof into the following two, in the first one of which we show that the projection α of $C(g)$ into

$F_{\alpha(g)}$ is surjective and in the second one of which we verify that there exists a neighborhood evenly covered by α for each $y \in F_{\alpha(g)}$.

I. $\alpha(C(g)) = F_{\alpha(g)}$.

Proof of I. Let $\{V_i\}_{i \in I}$ be a covering of $F_{\alpha(g)}$ by relatively compact and simply-connected open sets on $F_{\alpha(g)}$. We write $x = \alpha(g)$. Let N be the nerve of the covering. Take any point y of F_x. Then there exists a chain joining x and y. Let the chain be $(i_0, \cdots, i_m)$. First we assign a point $x_k \in V_{i_{k-1}} \cap V_{i_k}$ to each (i_{k-1}, i_k). Then by Lemma 4.1 we obtain maps φ'_k of $S_{x_k} \times V_{ik}$ into S which have the natures prescribed there. We put:

$$\varphi_k(g, z) = \varphi'_k(\varphi_{k-1}(g, x_k) z) \qquad \text{for} \quad (g, z) \in S_{x0} \times V_{ik}$$

consecutively. Then $\varphi_k(g, z) \in C(g)$ for $z \in V_{i_k}$. Especially we see $\varphi_m(g, y) \in C(g)$. Since $\alpha \cdot \varphi_m(g, y) = y$, this shows I.

II. Any relatively compact and simply-connected open set in F_x is evenly covered by α.

Proof of II. Let $y \in V$ and let $h, k \in S_{y0} \cap C(g)$. Then we get a map φ of $S_{y0} \times V$ into S by Lemma 4.1. We write:

$$f_h(z) = \beta \cdot \varphi(h, z) \quad \text{and} \quad f_k(z) = \beta \cdot \varphi(k, z) \qquad \text{for} \quad z \in V.$$

It follows from the identity theorem that if $\gamma_z f_h = \gamma_z f_k$ for some $z \in V$, then $f_h = f_k$. If $h \neq k$, then

$$\{\gamma_u f_h | u \in V\} \quad \text{and} \quad \{\gamma_u f_k | u \in V\}$$

are disjoint. If these sets are denoted by V_h and V_k, $\alpha | V_h$ and $\alpha | V_k$ are homeomorphisms. These complete the proof.

Remark. Let us consider the maps φ_y defined by

$$(4.5) \qquad\qquad \varphi_y(g) = \varphi(g, y) \qquad\qquad \text{for} \quad y \in V.$$

Then it is seen that φ_y is bijective for each $y \in V$. Let us prove it. Let $\bar{\varphi}$ be the map that is obtained by replacing x by y in Lemma 4.1. Then we get maps $\bar{\varphi}_z$ of S_{y0} into S_{x0} for $z \in V$ similarly to 4.5. By means of Lemma 4.1 (*uniqueness*)

$$\varphi | W \times V = \bar{\varphi} \cdot \varphi_y^{-1}$$

where $W = \varphi_y S_{x0}$ and φ_y^{-1} should be considered to be a map of $W \times V$ onto $S \times V$ in the natural way. Then we easily see:

$$(4.6) \qquad\qquad \varphi_x \cdot \varphi_y = \text{the identity of } S_{x0}.$$

Similarly:

(4. 7) $\qquad\qquad \varphi_y \cdot \varphi_x = $ the identity of S_{y0} .

(4.6) and (4.7) imply that φ_y and φ_x are the inverses of each other, and especially that they are surjective. This completes the proof. Hence if we denote $\bigcup\limits_{x \in M} S_{x0}$ by S_0, then φ is a map into S_0.

5. Some theorems

Let M be a connected foliated space the leaves of which have coverings by relatively compact and simply connected open sets. Let Γ be a regular and equi-uniform continuous pseudogroup of local transformations compatible with the foliation. Remember that S is the set of the germs of element of Γ. Let $x \in M$ and write F instead of F_x. Let $(\widetilde{F}, \pi)$ be a universal covering space of F. Further put $\widetilde{G} = S_{x0}$.

Proposition 5. 1. *Let* $\widetilde{S}$ *denote the product* $\widetilde{G} \times \widetilde{F}$. *Let* p *be the projection of* $\widetilde{S}$ *onto* F *defined by*

$$p(g, u) = \pi(u) \qquad \text{where} \quad (g, u) \in \widetilde{S} .$$

Then there exists one and only one continuous map φ *of* $\widetilde{S}$ *into* S *such that*

(5. 1) $\qquad\qquad \varphi(g, \bar{x}) = g \qquad\qquad$ *for any* $g \in \widetilde{G}$

(5. 2) $\qquad\qquad \alpha \cdot \varphi = p$

where $\bar{x}$ *is a fixed point over* x.

Proof. Take any g from $\widetilde{G}$ and $C(g)$ be the component of g in S with respect to the natural topology. Then by means of the main tool theorem $(C(g), \alpha)$ is a covering space of F. We can, therefore, find a continuous map φ_g of $g \times \widetilde{F}$ onto $C(g)$ with $\alpha \cdot \varphi_g = p | g \times F$. φ_g is uniquely determined by the condition

(5. 3) $\qquad\qquad \varphi_g(g, \bar{x}) = g .$

After all we get a continuous map φ by putting

$$\varphi(g, u) = \varphi_g(g, u) .$$

(5.1) follows from (5.3).

Let G be the set of elements $g \in \widetilde{G}$ with $\beta(g) \in F$. Then we can consider G as a group. For $g \in G$ there exists a homeomorphism $\tilde{g}$ with $\beta \cdot \varphi = \pi \cdot \tilde{g}$. Let $\tau(g, u) = \tilde{g}(u)$.

Theorem 5. 1. (G, τ) *is a transformation group acting on* $\widetilde{F}$ *and the diagram*

$$\varphi$$

$$G \times \widetilde{F} \xrightarrow{\hspace{4cm}} S$$

$$\tau \downarrow \hspace{5cm} \downarrow \alpha$$

$$\widetilde{F} \xrightarrow{\hspace{4cm}\pi\hspace{0.5cm}} F$$

is commutative.

Theorem 5. 2. $F_{\alpha(g)}$ *and* $F_{\beta(g)}$ *have homeomorphic universal covering spaces where g is any element of S.*

Theorem 5. 3. *Let L be the image of S_{x0} by β. If (S_{x0}, β) is a covering space of L, then $\widetilde{M}$ is homeomorphic to $\widetilde{S}_{x0} \times \widetilde{F}$.*

CHAPTER II. APPLICATIONS

1. The identity theorem for isometries

Let F be a Riemannian manifold with a Riemannian metric g. Then, as is well-known in classical theory (see [10]),· for each point $x \in F$ there exists a differentiable map k_x of $S_x^{n-1} \times I$ into F, what is called a *polar coordinate system* at x, such that

(1) if we fix $X \in S_x^{n-1}$, then $k_x(X, t)$ is a geodesic with canonical parameter t, ·

(2) $k_x(X, 0) = x$ for any $X \in S_x^{n-1}$,

where S_x^{n-1} $(n = \dim F)$ is the set of all the unit vectors at x and I an interval $]$-ε, $\varepsilon[$ with a certain positive number ε. Clearly k_x is injective on $S_x^{n-1} \times]0$, $\varepsilon[$ and it has the inverse. We write $k_x^{-1}(z) = (p_x(z), t)$ for $z \in k_x(S_x^{n-1} \times]0, \varepsilon[)$. Then p_x is a differentiable map which takes $k_x(S_x^{n-1} \times]0, \varepsilon[)$ onto S_x^{n-1}. It is noted that by setting

$$\widetilde{k}_x(Y, t) = k_x\left(\frac{Y}{\|Y\|}, \|Y\| t\right) \qquad \text{for any} \quad Y \in T_x$$

we can extend the domain of k_x to $(T_x - 0) \times]$-ε, $\varepsilon[$. *This map is also called a polar coordinate system.*

Let F_1 and F_2 be Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively. Let us consider a differentiable map f of F_1 into F_2 which is not necessarily injective and satisfies

(1. 1) $g_1(X, Y) = g_2(f_*(X), f_*(Y))$ for $X, Y \in T_x$

where x ranges over F_1 and f_* is the differential of f.

For such a map f we can see without difficulty the following facts.

(1) $f_*|T_x$ is injective for each $x \in F_1$.

(2) f has Jacobian matrices of maximal rank at all the points of F_1.

(3) If there exists such f of F_1 into F_2, then

$$\dim F_1 \leqq \dim F_2.$$

(4) f is a local diffeomorphism if $\dim F_1 = \dim F_2$.

By *isometries* we mean bidifferentiable maps satisfying (1.1). A connected open set in F_1 is also a Riemannian manifold. By *local isometric maps* we mean isometries of an arbitrary connected open subset of F_1 into F_2.

Lemma 1.1.   *The set of local isometric maps between F_1 and F_2 satisfies the condition of analyticity* (see Section 1, Chap. I).

Proof.   As is easily seen, it suffices to prove it in the case where

(1) f is a local isometric transformation of F_1, i.e. a local isometric map between F_1 and F_1,

(2) $f(x) = x$ for some $x \in \alpha(f)$,

(3) $f(z) = z$ is an open set U with $U \subset \alpha(f)$ and $\bar{U} \ni x$,

and the other map under consideration is the identity. Let us consider a polar coordinate system with origin x. Since f fixes all the geodesics through $z \in U$, we see

$$f_*(p_x(z)) = p_x(z) \qquad\qquad \text{for} \quad z \in U.$$

Stated in other words, f_* fixes $p_x(U)$, an open set in S_x^{n-1}. Hence $f_*|T_x$ must coincide with the identity of T_x. Consequently for any y in a neighborhood of x

$$f(y) = f(k_x(p_x(y), t)) = k_x f_*(p_x(y), t) = k_x(p_x(y), t) = y,$$

where $t =$ the distance between x and y. This completes the proof.

The theorem of identity[10] follows from Theorem 1.1, Chap. I with the help of this lemma[1].

Remark.   The proof of Lemma 1.1 holds good for local affine maps between an affinely-connected manifold and another affinely-connected manifold. Then the identity theorem is true for automorphisms of an affinely-connected manifold into an affinely-connected manifold.

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10) Theorem of identity: Let a and b be isometric (sesp. affine) maps of a connected Riemannian (resp. affinely-connected) manifold F_1 into a Riemannian (resp. affinely-connected manifold F_2. If there exists a point $x \in F_1$ such that $r_x a = r_x b$, then $a = b$ on the whole F_1.

2. Uniformities and affine transformations

Let F_1 and F_2 be Riemannian manifolds. Then F_1 and F_2 have the connections and the uniformities which are induced from the Riemannian metrics. If γ is a curve on F_1 whose origin is x and whose end is y, the parallel displacement along γ gives rise to an isomorphism of the vector space T_x onto the vector space T_y, which is also denoted by the same letter γ. The uniformity of F_1 (resp. F_2) will be denoted by $\mathfrak{U}_1$ (resp. $\mathfrak{U}_2$) in what follows.

Let f be a differentiable map of F_1 into F_2. The image curve of γ by f is denoted by $f(\gamma)$ in brief. f is called an affine map in case of f being a diffeomorphism of F_1 onto F_2 such that for an arbitrary γ

(2. 1) $f(\gamma)\cdot f_{*x}=f_{*y}\cdot \gamma,$

where f_{*x} is the restriction on T_x of the differential f_* and so on.

The following facts are well-known :

(2. 2) $\|\gamma(X)\| = \|X\|$ for $X \in T_x$.

(2. 3) $f\tilde{k}_x(X, t) = \tilde{k}_{f(x)}(f_*(X), t)$ for $X \in T_x$,

where $\tilde{k}_x$ and $\tilde{k}_{f(x)}$ are polar coordinate systems at x and $f(x)$ respectively.

Proposition 2. 1. *An affine map f of F_1 into F_2 is uniformly continuous with respect to $\mathfrak{U}_1$ and $\mathfrak{U}_2$, in other words, if $\sigma_2 \in \mathfrak{U}_2$, then we can find $\sigma_1 \in \mathfrak{U}_1$ such that*

$$f(\sigma_1) = \left\{(f(x),f(y))\,|\,(x, y) \in \sigma_1\right\} \subset \sigma_2 .$$

Proof. Let $\sigma_{1\delta}$ denote the set:

$$\left\{(x, y)|\ \text{the distance between } x \text{ and } y < \delta\right\}$$

and so on. Let σ_x denote the set:

$$\left\{z\,|\,(x, z) \in \sigma\right\}$$

and so on. Let $\sigma_{2*} \in U_2$ and $x \in F_1$. We can choose δ so that

$$\sigma_{2*f(x)} \supset \left\{\tilde{k}_{f(x)}(f_{*x}(X), t)\,|\,X \in S_x^{n-1} \text{ and } 0 \leqq t < \delta\right\}$$

Let y be any point on F_1 and join x to y by a curve γ and let $z \in \sigma_{1\delta y}$. Then $z = k_y(Y, t)$ for some $Y \in T_y$ with $\|Y\| = 1$ and with some t such that $0 \leqq t < \delta$. We write $\gamma^{-1}(Y) = X$. Then $X \in T_x$. We have

$$f(z) = f(k_y(Y, t)) = k_{f(y)}(f_*(Y), t) = k_{f(y)}(f_*(\gamma(X)), t)$$
$$= k_{f(y)}(f(\gamma)\cdot f_*(X), t) = k_{f(y)}(Z, t) ,$$

where Z is a vector of $T_{f(y)}$ with $\|Z\| = \|f_*(X)\| \leq \varepsilon/\delta$. Hence we see $f(z) \in \sigma_{2,f(y)}$, namely, $f(\sigma_{1\delta}) \subset \sigma_{2\varepsilon}$. This completes the proof.

3. Analytic Riemannian manifolds

Let F_1 and F_2 be analytic Riemannian manifolds. Then we have a well-known lemma.

Lemma 3. 1. *Let γ be a curve on F_1. Then any local isometric map of an open neighborhood of $\gamma(0)$ into F_2 can be extended along γ and this extension is unique, where F_2 is assumed to be complete.*

Let $S(F_1, F_2)$ denote the set of all the germs of local isometric maps between F_1 and F_2, and give the natural topology to it. Then the following lemma is a direct result of the preceding.
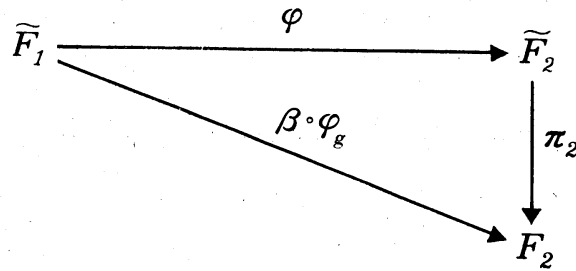
Lemma 3. 2. *Let γ be any curve on F_1, and let g be any germ of $S(F_1, F_2)$ with $\alpha(g) = \gamma(0)$. Then there exists one and only one curve $\tilde{\gamma}$ with origin g and with $\gamma = \alpha \cdot \tilde{\gamma}$.*

It is interesting to note that an arc-wise connected $\tilde{F}$ becomes a covering space of F if for any curve on F we can find one and only one curve that covers the curve on F and passes through any point over the origin of the curve on F.

Hence we see that, if $C(g)$ is the connected component of g in $S(F_1, F_2)$, Then $(C(g), \alpha)$ is a covering space of F_1. Suppose $(\tilde{F}_1, \pi_1)$ is a universal covering space of F_1 and $x = \alpha(g)$. Fix a point $\tilde{x}$ over x. Then there exists one and only one continuous map φ_g with a commutative diagram:
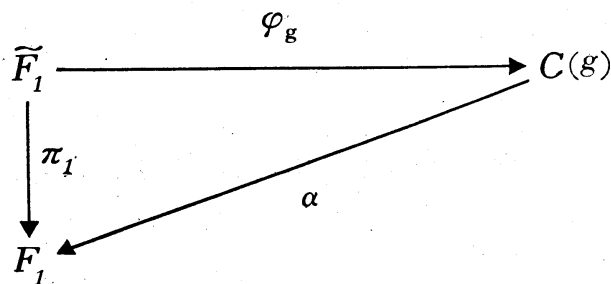
under the condition $\varphi_g(\tilde{x}) = g$. Let $y = \beta(g)$ and let $(\tilde{F}_2, \pi_2)$ be a universal covering space of F_2. Choose a point $\tilde{y}$ over y. Since $\beta \cdot \varphi_g$ is a continuous map of simply connected $\tilde{F}_1$ into F_2, we can find one and only one continuous map φ with a commutative diagram:

$$\widetilde{F}_1 \xrightarrow{\quad\varphi\quad} \widetilde{F}_2$$

with $\beta \circ \varphi_g$ diagonal and π_2 vertical to F_2.

under the condition $\varphi(\tilde{x}) = \tilde{y}$. Clearly φ is a local homeomorphism. Hence as a conclusion we can obtain the following theorem.

Theorem 3. 1. *Let $\widetilde{F}_1$ and $\widetilde{F}_2$ be simply connected analytic Riemannian manifolds of the same dimension. Let $\widetilde{F}_2$ be complete with respect to its Riemannian metric. If there exists an isometric map of an open set in $\widetilde{F}_1$ into $\widetilde{F}_2$. Then $\widetilde{F}_1$ can be immersed in $\widetilde{F}_2$.*

Consider now the case where F_1 is also complete. Then $(C(g^{-1}), a)$ is a covering space of F_2 where $C(g^{-1})$ is the component of g^{-1} in $S(F_1, F_2)$. Hence we can get Theorem E in the introduction by Proposition 1.2, Chap. I.

Let ξ be the germ of local Killing vector fields in a simply connected analytic complete Riemannian manifold F and X a Killing vector field belong to it. Then X generates a local 1-parameter group a_t $(0 \leq t < \varepsilon)$ of analytic local isometries. According to the above, each a_t can be extended to a global isometry of F. Therefore the local group a_t can be extended to a global 1-parameter group of isometries of F. Hence to each germ ξ we can assign a global Killing vector field of F. After all we can conclude it as follows.

Theorem 3. 2. (K. Nomizu [6]). *If $\widetilde{F}$ be a complete and simply connected analytic Riemannian manifold, then the sheaf $\widetilde{K}$ of germs of Killing vector fields is a constant sheaf.*

4. Lie pseudogroups

Let M be a differentiable manifold and let Γ be a *Lie pseudogroup* of local transformations (i. e. local bidifferentiable maps) on M. We follow the notation in Sec. 3, Chap. I. Then S is the set of germs of elements of Γ and S_{x_0} is the connected component of $S_x = a^{-1}(x)$ for each $x \in M$.

Definition 4. 1. A Lie pseudogroup Γ is of *finite dimension* if and only if for each $x \in M$ S_x is of finite dimension.

Let Γ be a Lie pseudogroup of finite dimension. Then we know these things :

(1) For each $x \in M$ S_{x_0} is a differentiable manifold.

(2) For each $x \in M$ the tangent vector space at x of S_{x_0} is the set of the germs at x of infinitesimal transformations belonging to Γ. We denote it by Θ_x and write $\Theta = \bigcup_{x \in M} \Theta_x$.

(3) (Θ, α, M) is a sheaf with Lie algebras Θ_x as stalks.

(4) Let $\xi \in \Theta_x$ and let X be a local vector field belonging to ξ. Consider a local 1-parameter group $\exp tX$. We write

$$\exp t\xi = \Upsilon_x \exp tX .$$

Then the following relation holds

(4. 1) $$\exp (st) \xi = \text{eup } t(s\xi) .$$

Proposition 4. 1. *If Γ is a Lie pseudogroup of finite dimension, then dim S_{x_0} is lower semi-continuous as a function of x.*

For the proof see [5].

Definition 4. 2. A Lie pseudogroup Γ is said to be of *constant dimension* if and only if dim S_{x_0} is continuous.

We can see that for each $x \in M$ there exists a neighborhood $V(x)$ such that $\Theta \mid V(x)$ is a constant sheaf, if Γ is of constant dimension.

Proposition 4. 2. *A Lie pseudogroup Γ is regular in the sense of Def. 4.3, Chap. I if and only if it is of constant dimension.*

Proof. Let us consider a sufficiently small neighborhood $W(\tilde{x})$ of $(\tilde{x})$ in S_{x_0} We put

$$\varphi_x(\exp t\xi, y) = \exp ti_x(\xi, y)$$

where i_x is the canonical map of the constant sheaf $\Theta_x \times V(x)$ onto $\Theta \mid V(x)$, and we assume $\exp t\xi \in W(\tilde{x})$. Suppose $\exp t\xi = \exp s\eta$. Then $\eta = \dfrac{t}{s} \xi$ and by (4.1)

$$\varphi_x(\exp s\eta, y) = \exp si_x(\eta, y) = \exp ti_x\left(\frac{s}{t} \eta, g \right)$$

$$= \exp ti_x(\xi, y) = \varphi_x(\exp t\xi, y) .$$

This φ_x is a map of $W(\tilde{x}) \times V(\tilde{x})$ into B and it has the nature required in (3.5), Chap. I. This completes the proof.

As easily seen, we can see from Proposition **4.1**, Chap. **I** that if V is a relatively compact and simply-connected open set in M, then $\Theta \mid V$ is a constant sheaf. Thus we can get Theorem F by the same technique as in the proof of the main tool theorem.

5. Local product spaces and de Rham's theorem

Let F_1 and F_2 be connected, locally connected, locally simply-connected Hausdorff spaces and let Γ'' be a pseudogroup of local transformations of $F_1 \times F_2$ which are compatible with the product structure, in other words, if $a \in \Gamma''$, then there exist local transformations of F_1 and F_2 which are denoted by a_1 and a_2 respectively such that

$$a(x, y) = (a_1(x), a_2(y)) \qquad \text{for} \quad x, y \in \alpha(a).$$

Definition 5.1. A topological space M is said to be a *local product space* if and only if there exists an open covering $\{V_i\}_{i \in I}$ such that for each $i \in I$ V_i is mapped onto $F_1 \times F_2$ by a homeomorphism f_i and such that $f_i \cdot f_j^{-1} \in \Gamma''$ for any pair $(i, j) \in N$ where N is the nerve of the covering.

Let us consider the totality of $f_i^{-1}(V_1 \times y)$ with any open set V_1 and any $y \in F_2$ (resp. of $f_i^{-1}(x \times V_2)$) with any open set V_2 in F_2 and any $x \in F_1$ where i runs through I. It defines a foliation, what will be called *canonical* in the present paper. We denote it by T_1 (resp. T_2). It will be referred to as *horizontal* (resp. *vertical*). The leaves of the foliations are also distinguished by the same terms.

From now on *we shall use the term "local transformations" in the meaning of local transformations with respect to the horizontal topology.*
Let us consider the local transformations of the type:

$$(5.1) \qquad f_i^{-1} \cdot p(y, \bar{y}) \cdot f_i, \qquad i \in I, \qquad y, \bar{y} \in F_2,$$

where $p(y, \bar{y})$ are the projections of $F_1 \times y$ onto $F_1 \times \bar{y}$ defined by $p(y, \bar{y})(x, y) = (x, \bar{y})$. It is seen that there exists the smallest pseudogroup of local transformations that contains those of the above type. We denote it by Γ_1. Γ_1 leaves the local product structure of M invariant.

We can also define the pseudogroup Γ_2 in exactly the same way.

Definition 5.2. The local product structure is called *equi-uniform* if and only if one of these two pseudogroups Γ_1 and Γ_2 is equi-uniform in the sense of Def. **4.1.** in Chap. 1 and satisfies the postulate of regular imbedness in Def. **3.3**, Chap. 1.

We here present some examples of the equi-uniform local product structures before going further.

1. If all the leaves of T_1 or T_2 are compact, then it is equi-uniform.

2. If all the leaves of T_1 or T_2 are metric spaces and if the maps of the type (1.1) are metric-preserving, then it is equi-uniform.

3. If all the leaves of T_1 or T_2 are Riemannian manifolds and if the

maps of the type (1.1) are affine transformations, then it is equi-uniform.

Let us denote the horizontal (resp. vertical) leaf through $x \in M$ by F_{1x} (resp. F_{2x}) and the set of all the germs of elements of Γ_1 (resp. Γ_2) that have domain x by S_{1x} (resp. S_{2x}). In what follows we take the case where the structure is equi-uniform with respect to Γ_1. It is seen directly from the definition of Γ_1 that if we take a sufficiently small neighborhood W in F_{2x} of $y \in F_{2x}$, we can attach a single element of S_{1y} to each $z \in W$. We can define by this map of W into S_{1y} a neighborhood system of y the germ of identities in the neighborhoods of y as the image of neighborhoods of y in W. Let g be any element of S_{1x} and let $\beta(g) = y$. Then we can define a neighborhood system of y in S_{1y} in the above way. Denote it by $\sum$. Then we can define a neighborhood system of g in S_{1x} by $g \sum$. It is easily seen that this introduces a topology into S_{1x} and that it is a locally connected Hausdorff one. We write $\bigcup_{x \in M} S_{1x}$ as S. Then S has two kinds of topology: The natural one as the set of germs and the above one. With respect to the latter one S_{1x} for each $x \in M$ is a connected component. We can easily get

Lemma 5.1. *The operations of S are continuous with respect to the above defined topology.*

Lemma 5.2. *Γ_1 and Γ_2 are regular in the sense of Def. 3.3, Chap. I.*

Lemma 5.3. *(S_{1x}, β) is a covering space of F_{2x}.*

Proof. Let g and h be the germs belonging to S_{1x} and with $\beta(g) = \beta(h) = y$. Let W be a neighborhood of y of the above stated nature. Then it is homeomorphic to a certain neighborhood of y, which will be denoted by the same letter W. Assume $gW \cap hW \neq 0$, which implies the existence of $k, l \in W$ such that $g^{-1}h = lk^{-1}$. lk^{-1} is the germ of the type (1.1) and it leaves y fixed. Hence $lk = y^{-1}$. This implies $g = h$. This completes the proof.

If M is a connected local product space all the horizontal leaves of which are compact, then Γ_1 is a regular, equi-uniform continuous pseudogroup satisfying the postulate of regular imbedness. Thus we get the following theorem.

Theorem 5.1. *Let M be a connected local product space all the horizontal leaves of which are compact. Then all the horizontal leaves have homeomorphic universal covering spaces. All the vertical leaves do so. Moreover the universal covering spece of M is a product space of that of the horizontal leaves and that of the vertical leaves.*

$$\widetilde{M} \approx \widetilde{F}_{1x} \times \widetilde{F}_{2x} \qquad \text{for any } x \in M.$$

Consider the case in which Γ_1 is a pseudogroup of local isometric maps. According to the remark in Sec. 4, Chap. I, there is no need for using the

postulate of regular imbedness.

Theorem 5. 2. *Let M be a connected complete local product Riemannian manifold. Then the same conclusion holds good likewise.*

As a result of this theorem we have the de Rham's decomposition theorem [1].

6. A generalization of a Reeb's theorem[11]

Let M be a compact differentiable manifold. Let f be a differentiable function defined over $M-F$ where F is a *nowhere dense* subset.

Definition 6. 1. A point x_0 of $M-F$ is said to be a non-degerate critical point of index k in case a system of coordinates $\varphi_1, \cdots, \varphi_n$ exists in a neighborhood of x_0, which satisfies these conditions:

(6. 1) $\qquad\qquad\qquad \varphi_i(x_0) = 0 , \qquad\qquad i = 1, 2, \cdots, n .$

(6. 2) $\qquad\qquad -\sum_{i=1}^{k} \varphi_i^2(x) + \sum_{j=k+1}^{n} \varphi_j^2(x) = f(x) - f(x_0) .$

Let K be the set of points where the differential f_* vanishes. We, further, assume that K is *nowhere dense*.

Definition 6. 2. A point x of $F \cup K$ is said to be *removable* if and only if there exists a function r of one real variable with these conditions:

1. The domain of r contains $f(V(x) \cap CF)$.
2. $r \cdot f$ has a differentiable extension over $V(x)$ which is free of critical points, in other words, the differential of which never vanishes,

where $V(x)$ is a sufficiently small neighborhood of x.

Prerequisites: *Of the functions which appear in this section, all the points of K are assumed to be removable or definite, where "definite" means "non-degenerate critical of index* 0 *or* n.

Examples of removable points. Let us consider a torus T^1 of dimension 1, *i. e.*, $\{\exp i\theta | \theta \in R\}$.

1. $\cos^2\theta$ is a function defined on it. The values $= 0$, $\pm\dfrac{1}{2}\pi, \pi$ give rise to critical points. But those corresponding to $\pm\dfrac{1}{2}\pi$ are removable. Actually $r = \sqrt{f}$ is a function of the desired nature.

2. Let F be the set composed of i and $-i$. Then $\tan\theta$ is a function

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11) See [11] and [12] where the same consequence is derived from a more geometrical problem.

defined over $T^1 - F$. But all the points of F are removable. Actually $r = \dfrac{1}{f}$ is a function of the desired nature.

Let x be ordinary (resp. removable). Then there exists a system of coordinates $\varphi_1, \cdots, \varphi_n$ in a neighborhood $V(x)$ of x such that

(6. 3) $\qquad\qquad\qquad \varphi_i(x) = 0 , \qquad\qquad i = 1, 2, \cdots, n ,$

(6. 4) $\qquad\qquad\qquad \varphi_n(y) = f(y) - f(x) \qquad$ for $y \in V(x).$

(resp. $\qquad\qquad \varphi_n(y) = r \cdot f(y) - r \cdot f(x))$

In $V(x)$, $\{y \,|\, f(y) = \text{const.}$ (resp. $r \cdot f = \text{const.})\}$ are submanifolds. The totality of such submanifolds constitute a basis of a foliation.

Lemma 6.1. *All the leaves of this foliation are compact.*

Proof. Let F be a leaf and let $x \in \bar{F}$. It suffices to show $x \in F$. We can take a sequence $\{x_i\}_{i=1,2,\cdots}$ such that

$$x_i \in F, \qquad i = 1, 2, \cdots \text{ and } \lim_{i \to \infty} x_i = x$$

By the assumption all the critical points, say, x_0, that are not removable have a system of coordinates about it with the conditions (6.1) and (6.2), from which we see that the closure of any leaf does not contain x_0. Consequently, x is not a non-removable critical point. Let $x \in F \cup K$. Then we can choose a system of coodinates with (6.3) and (6.4). First we note that for any $y, z \in F$

$$r \cdot f(y) = r \cdot f(z).$$

Hence if $x_i \in F_x$ for infinitely many i, then we can see that $(r \cdot f)_*$ vanishes at x. This is contrary to the definition of removability. Hence $x_i \in F_x$ for sufficiently large i Therefore $x \in F_{x_i} = F$. If $x \in M - F - K$, then the case is more simple than the preceding. Thus the proof has been completed.

Let us riemannianize M by any Riemannian metric. Let M be the complement of the set of all the non-removable critical points. Then by means of Theorem **5. 1,** we see that there exists a universal covering space of M which has the form $(S^{n-1} \times R, \pi)$, where π satisfies the conditions:

(1) It takes $u \times R$ onto an orthogonal trajectory to the leaves for each $u \in S^{n-1}$.

(2) It takes $S^{n-} \times t$ onto a leaf for each $t \in R$.

Let $u_0 \in S^{n-1}$ and let D be the Poincaré group of this covering space. By putting $d(u_0, t) = (d(u_0), d(t))$ where $d \in D$, D may be considered to be a transformation group acting on S^{n-1} or R.

We can see the following easily.

I.   If $d \in D$ has no fixed point in R, then d acts trivially, i.e., $d(t) = t$ for any $t \in R$.   As a result we have $d = e$, where e is the identity element of D.

II.   If $d \in D(d \neq e)$ has a fixed point t_0 in R, then $d(S^{n-1} \times t_0) = S^{n-1} \times t$. Actually d is a fixed point free transformation of $S^{n-1} \times t_0$.   Consequently a trajectory coming from the left goes back to the left.   We see that d has one and only one fixed point in R.   Then $d^2(t) = t$ for any $t \in R$.   Hence $d^2 = e$.

We can conclude the above in this way: If there exists at least one non-removable critical point, then $D = \{e\}$ or $D = \{e, \sigma\}$ where σ is an element idempotent and different from e.   (1) In the case where $D = \{e\}$, the manifold is homeomorphic to the n-sphere and the number of the non-removable critical points is 2.   (2) In the case where $D = \{e, \sigma\}$, the manifold is homeomorphic to the real projective space and the number of the non-removable critical points is 1.

Remark.   According to J. Milnor [4] "homeomophic" in the above can not be replaced by "diffeomorphic".

·   Finally we show that Theorem G is a generalization of Theorem B.   In other words.

Theorem 6. 1.   *Any differentiable function on a compact differentiable manifold satisfies Lipshitz condition* (see the introduction).

Actually this is a direct consequence of the following proposition.

Proposition 6. 1.   *Let M be an arbitrary differentiable manifold and let V be a relatively compact domain in M.   Let f be a differentiable function over M.   Then the following inequality holds in V:*

$$(6. 5) \qquad\qquad |X\|df\|| \leq L\|X\| \qquad \text{for any vector field } X,$$

where L is a constant over V.

Proof.   By means of schwartz's inequality

$$|X\|df\|| = |<X, d\|df\|>| \leq \|X\| \, \|d\|df\|\,\|.$$

If we denote the maximum of $\|d\|df\|\,\|$ in $\overline{V}$ by L, we get (6.1).

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