SOME THEOREMS ON GALOIS THEORY OF SIMPLE RINGS

By

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Recently, in [7], we have succeeded in constructing Galois theory of simple rings under the assumption that the extension R/S considered is hereditarily Galois (h-Galois) and locally finite. However, we have believed that [7, Theorem 2.1] and [7, Theorem 3.1] should be stated under more desirable assumptions. One of the purposes of the present paper is to give a settlement to this problem. Concerning [7, Theorem 2.1], one will see that the assumption that R/S is locally finite can be excluded from those assumed there (Theorem 1). On the other hand, as was shown in [7, Lemma 3.4], if R/S is h-Galois and locally finite then $\mathfrak{G}R_r$ is dense in $\mathrm{Hom}_{S_l}(R, R)$. In §**1**, one will see also that if R/S is locally finite and $\mathfrak{G}R_r$ is dense in $\mathrm{Hom}_{S_l}(R, R)$ then the fundamental theorems in Galois theory of finite dimension hold still for regular intermediate rings of R/S left finite over S (Theorems 2 ond 3). And, if R/S is locally finite, $\mathfrak{G}R_r$ dense in $\mathrm{Hom}_{S_l}(R, R)$, and $V_R(V_R(S'))$ is simple for each regular intermediate ring S' of R/S with $[S' : S]_l < \infty$, then [7, Theorem 3.1] is still valid even for a regular intermediate ring R' of R/S (Theorem 6). The proof of this improvement will be given in §**3**. §**2** is devoted exclusively to the treaty of algebraic Galois extensions, which is our second purpose. In fact, Theorem 4 may be regarded as a complete extension of [2, Theorem 3] to simple rings as well as an improvement of [7, Lemma 1.9]. §**2** contains also a sharpening of [7, Lemma 1.10] (Theorem 5).

Throughout the present paper, $R = \sum_1^n De_{ij}$ be a simple ring, where e_{ij}'s are matrix units and $D = V_R(\{e_{ij}\text{'s}\})$ is a division ring. And S be always a simple subring of R (containing 1 of R), $\mathfrak{G}$ the group of all the S-(ring) automorphisms of R. Further, we set $C = V_R(R)$, $Z = V_S(S)$, $V = V_R(S)$, $H = V_R(V)$ and $C_0 = V_V(V)$. As to general notations and terminologies used here we follow the previous paper [7].

1. Now, we shall begin our study with the following lemma.

Lemma 1. *Let S be a regular subring of R, and $\mathfrak{H}$ a subgroup of $\mathfrak{G}$ containing $\tilde{V}$. If T is an intermediate ring of R/S left finite over S such that R is T-R-irreducible, and $T' = J(\mathfrak{H}(T, R))$, then* $\infty > [(\mathfrak{H}|T)R_r : R_r]_r$

$=[(\mathfrak{H}|T')R_r : R_r]_r$.

Proof. By [7, Lemma 1.3] and [7, Lemma 1.5], $(\mathfrak{H}|T)R_r=\sum_1^t{}_\oplus(\tilde{V}\sigma_i|T)R_r$ for some σ_i's$\in\mathfrak{H}$. As $V_R(T)=V_R(T')$, [7, Lemma 1.4 (iii)] will yield $\infty>[(\tilde{V}\sigma_i|T)R_r : R_r]_r=[V: V_R(T)]_r=[V: V_R(T')]_r=[(\tilde{V}\sigma_i|T')R_r : R_r]_r$. Further, by [7, Lemma 1.3 (iv)], $\sum_1^t(\tilde{V}\sigma_i|T')R_r=\sum_1^t{}_\oplus(\tilde{V}\sigma_i|T')R_r$. Combining these facts mentioned above, it follows that $[(\mathfrak{H}|T)R_r : R_r]_r\leqslant[(\mathfrak{H}|T')R_r : R_r]_r$. Now, if $[(\mathfrak{H}|T)R_r : R_r]_r<[(\mathfrak{H}|T')R_r : R_r]_r$ then there exists some $\tau\in\mathfrak{H}$ such that $\tau|T'\notin\sum_1^t{}_\oplus(\tilde{V}\sigma_i|T')R_r$. While, $\tau|T\in\sum_1^t(\tilde{V}\sigma_i|T)R_r$ implies $\tau|T=\sigma_j\tilde{v}|T$ with some σ_j and $\tilde{v}\in\tilde{V}$ by [7, Corollary 1.1]. And then, noting that $\tau'=\tau(\sigma_j\tilde{v})^{-1}\in\mathfrak{H}(T)$, we obtain $\tau|T'=\tau'\sigma_j\tilde{v}|T'=\sigma_j\tilde{v}|T'\in\sum_1^t(\tilde{V}\sigma_i|T')R_r$, which is a contradiction.

Now, by making use of the same method as in the proof of [7, Theorem 2.1], we can prove the following theorem which contains evidently [7, Theorem 2.1].

Theorem 1. *Let S be a regular subring of R, $\mathfrak{H}$ a subgroup of $\mathfrak{G}$ containing $\tilde{V}$, and $\mathfrak{H}R_r$ dense in* $\mathrm{Hom}_{S_l}(R,R)$. *If T is an intermediate ring of R/S left finite over S such that R is T-R-irrducible, then $J(\mathfrak{H}(T),R)=T$.*

Proof. Suppose on the contrary $T'=J(\mathfrak{H}(T),R)\supsetneqq T$. Then, we can find an S-left submodule M of T' such that $[T:S]_l<[M:S]_l<\infty$. Since $\mathrm{Hom}_{S_l}(M,R)=\mathfrak{H}R_r|M=((\mathfrak{H}|T')R_r)|M$ by our assumption, there holds $[M:S]_l=[\mathrm{Hom}_{S_l}(M,R):R_r]_r=[((\mathfrak{H}|T')R_r)|M:R_r]_r$. Hence, we obtain $[(\mathfrak{H}|T)R_r : R_r]_r=[T:S]_l<[M:S]_l\leqslant[(\mathfrak{H}|T')R_r : R_r]_r$, which contradicts Lemma 1.

If a division ring R is Galois over S, $\mathfrak{H}R_r$ is dense in $\mathrm{Hom}_{S_l}(R,R)$ for any subgroup $\mathfrak{H}$ of $\mathfrak{G}$. Consequently, we obtain the following corollary, that is [4, Theorem 1] itself. And, one should remark here that the proof of Theorem 1 is notably easier than that of [4, Theorem 1] given in [4].

Corollary 1. *Let a division ring R be Galois over S, $\mathfrak{H}$ a subgroup of $\mathfrak{G}$ containing $\tilde{V}$ with $J(\mathfrak{H},R)=S$. If T is an intermediate ring of R/S left finite over S, then $J(\mathfrak{H}(T),R)=T$.*

To prove Theorems 2 and 3, the following generalization of [6, Lemma 4] will be needed.

Lemma 2. *Let R be locally finite over a regular subring S, and $\mathfrak{G}R_r$ dense in* $\mathrm{Hom}_{S_l}(R,R)$. *If S' is an intermediate simple ring of R/S with $[S':S]_l<\infty$ then R is S'-R-completely reducible. In particular, if additionally $V_R(S')$ is a division ring then R is S'-R-irreducible.*

Proof. Let M be an arbitrary minimal S'-R-submodule of R such that the composition series of M as R-right module is of the shortest length among minimal S'-R-submodules of R. Then, $M=S'aR$ with non-zero $a\in M$. Now, we set $S^*=S'[a, \{e_{ij}\text{'s}\}]$, which is a regular subring of R left finite over S. And, as R is evidently S^*-R-irreducible, by [7, Lemma 1.3], $\mathrm{Hom}_{S_l}(S^*, R) = (\mathfrak{G}|S^*)R_r = \sum_1^s \oplus (\sigma_i|S^*)R_r$ with some σ_i's$\in \mathfrak{G}$, where one may remark that each $(\sigma_i|S^*)R_r$ is S_r^*-R_r-irreducible. Accordingly, the S_r^*-R_r-submodule $\mathrm{Hom}_{S'_l}(S^*, R)$ of $\mathrm{Hom}_{S_l}(S^*, R)$ is completely reducible: $\mathrm{Hom}_{S'_l}(S^*, R) = \sum_1^{s'} \oplus \mathfrak{M}_j$, and each $\mathfrak{M}_j$ is S_r^*-R_r-isomorphic to $(\sigma'_j|S^*)R_r$ for some $\sigma'_j \in \{\sigma_j\text{'s}\}$. If $\mathfrak{M}_j \ni \varepsilon_j \leftrightarrow \sigma'_j|S^* \in (\sigma'_j|S^*)R_r$ under the above isomorphism, then $x_r^*\varepsilon_j \leftrightarrow x_r^*(\sigma'_j|S^*) = (\sigma'_j|S^*)(x^*\sigma'_j)_r \leftarrow \varepsilon_j(x^*\sigma'_j)_r$ for each $x^* \in S^*$. And so, we have $x_r^*\varepsilon_j = \varepsilon_j(x^*\sigma'_j)_r$, whence it follows $x^*\varepsilon_j = (1\varepsilon_j)\cdot(x^*\sigma'_j)$. Hence, $\mathfrak{M}_j = \varepsilon_j R_r = (\sigma'_j u_{jl}|S^*)R_r$ with $u_j = 1\varepsilon_j$. Since $\mathfrak{M}_j$ is contained in $\mathrm{Hom}_{S'_l}(S^*, R)$, it will be easy to see that $M_j = (S'a)\mathfrak{M}_j$ is an S'-R-submodule of R. Moreover, there holds $M_j = (S'a)\sigma'_j u_{jl} R_r = u_j \cdot (S'a)\sigma'_j \cdot R = u_j \cdot (S'a \cdot R\sigma_j'^{-1})\sigma'_j = u_j \cdot (S'aR)\sigma'_j = u_j \cdot M\sigma'_j$, whence it follows $[M_j|R]_r = [u_j \cdot (M\sigma'_j)|R]_r \leqslant [M\sigma'_j|R]_r = [M|R]_r$ where $[M|R]_r$ means the length of the composition series of the R-right module M. Recalling here that $[M|R]_r$ is chosen to be the least, we see that the S'-R-submodule M_j is 0 or S'-R-irreducible. Finally, noting that R is $S'_l \cdot \mathrm{Hom}_{S'_l}(R, R)$-irreducible, there holds $R = (S'a)\,\mathrm{Hom}_{S'_l}(R, R) = (S'a)\,(\mathrm{Hom}_{S'_l}(R, R)|S^*) = (S'a)\mathrm{Hom}_{S'_l}(S^*, R) = (S'a)\sum_1^{s'}\mathfrak{M}_j = \sum_1^{s'} M_j$, which proves evidently the complete reducibility of R as an S'-R-module. The latter half of our theorem is a direct consequence of the fact that $V_{\mathrm{Hom}(R,R)}(S'_l \cdot R_r) = (V_R(S'))_l$.

Now, we can prove the next theorem:

Theorem 2. *Let R be locally finite over a regular subring S, $\mathfrak{H}$ a subgroup of $\mathfrak{G}$ containing $\tilde{V}$, and $\mathfrak{H}R_r$ dense in $\mathrm{Hom}_{S_l}(R, R)$. If T is a regular intermediate ring of R/S left finite over S then $J(\mathfrak{H}(T), R) = T$, and in particular R is Galois over S.*

Proof. Let $V_R(T) = \sum Wc_{pq}$, where c_{pq}'s are matrix units and $W = V_{V_R(T)}(\{c_{pq}\text{'s}\})$ is a division ring. Then $T' = T[\{c_{pq}\text{'s}\}] = \sum Tc_{pq}$ is a simple ring left finite over S and $V_R(T') = W$ is a division ring, whence R is T'-R-irreducible by Lemma 2. And so, in virtue of Theorem 1, there holds $J(\mathfrak{H}(T'), R) = T'$. Moreover, if we set $V' = \sum C'c_{pq}$ with the center C' of T' then V' is a simple subring of V and $J(\tilde{V}', T') = V_{T'}(V') = V_{T'}(\{c_{pq}\text{'s}\}) = T$. Hence, we obtain $J(\mathfrak{H}(T), R) = T$.

Further, Lemma 2 enables us to apply the same arguments as in the proofs of [7, Lemma 3.11] and [7, Corollary 3.7] to see the following theorem that contains [7, Corollary 3.7] obviously.

Theorem 3. *Let R be locally finite over a regular subring S, $\mathfrak{H}$ a subgroup of $\mathfrak{G}$ containing $\tilde{V}$ and $\mathfrak{H}R_r$ dense in $\mathrm{Hom}_{S_l}(R, R)$. If σ is an S-(ring) isomorphism of a regular intermediate ring T of R/S left finite over*

S onto a regular subring of R, then σ is contained in $\mathfrak{H}|T$.

Here, as careful readers realize at once, in the proofs of Theorems 1–3 no theorems from Galois theory of finite dimension are needed. And, [8, Theorem 5] and [8, Theorem 6] may be regarded as special cases of Theorem 2 and Theorem 3 respectively.

2. If R is left algebraic over a simple subring S, and r a regular element of R, then $r^{-1}\in S[r]$. In fact, as R is left algebraic over a division subring E of S, $[E[r]:E]_l<\infty$, whence it will be easy to see that $r^{-1}\in E[r]\subseteq S[r]$. As an easy consequence of this fact, we see that if R is left algebraic over S then each intermediate ring of $R/S[\{e_{ij}\text{'s}\}]$ is a regular subring of R. In what follows, the above remark will be used often without mention.

The next two lemmas, Lemmas 3 and 4, have been given in [3]. However, for the sake of completeness, we shall present here their proofs those are rather easier than the earlier ones.

Lemma 3. *Let R/S be outer Galois, and T an intermediate ring of R/S left finite over S. If R is T-R-irreducible then $\sharp\{x\mathfrak{G}\}<\infty$ for each $x\in T$.*

Proof. As $\infty>[T:S]_l=[\mathrm{Hom}_{S_l}(T,R):R_r]_r$, by [7, Lemma 1.3], we have $(\mathfrak{G}|T)R_r=\sum_{1\oplus}^t(\sigma_i|T)R_r$ with some σ_i's in $\mathfrak{G}$. And so, if σ is an arbitrary element of $\mathfrak{G}$ then $\sigma|T=\sigma_j|T$ for some σ_j by [7, Corollary 1.1]. Hence, it follows that $\sharp\{x\mathfrak{G}\}=\sharp\{x\sigma_1,\cdots,x\sigma_t\}$ for $x\in T$.

Lemma 4. *If R/S is outer Galois and left algebraic then it is locally finite.*

Proof. If $S\subseteq C$ then $C=V=R$ and our assertion is obviously true. And so, in what follows, we shall assume that $S\not\subseteq C$.

Case I: $n=1$. As R is evidently S-R-irreducible, $S[x]$-R-irreducible for each $x\in R$. And so, $\mathfrak{G}$ is locally finite by Lemma 3. Now, our assertion is clear by [10, (**a***)].

Case II: $n>1$. Let a be an arbitrary element of $S\backslash C$. By [6, Lemma 7 (i)], there exists such a regular element $r\in R$ that $a\tilde{r}=\sum_1^n d_{ij}e_{ij}$ with $d_{1n}=1$ and $d_{in}=0$ $(i\geqslant 2)$. R is then outer Galois and left algebraic over $S\tilde{r}$ and $S\tilde{r}\not\subseteq C$. To our end, it suffices to prove that $R/S\tilde{r}$ is locally finite. And so, without loss of generality, we may assume from the beginning that S contains $a=\sum_1^n d_{ij}e_{ij}$ with $d_{1n}=1$ and $d_{in}=0$ $(i\geqslant 2)$. If D (and so R) is finite, there is nothing to prove. And so, we may restrict further our attention to the case where D is infinite. Now, let Q be the set of all $q\in R$ with the property that each intermediate (simple) ring T of $R/S[q,\{e_{ij}\text{'s}\}]$ is $S[q]$-T-irreducible. By

[6, Lemma 8], $t_{xi}=e_{nn-1}+\cdots+xe_{ii-1}+\cdots+e_{21}\in Q$ for each non-zero $x\in D$. Accordingly, if $x\in D$ is neither 0 nor 1 then $e_{ii-1}=t_{xi}-t_{x-1i}$ and $xe_{ii-1}=t_{1i}-t_{1-xi}$ are both contained in $S[Q]$ $(2\leqslant i\leqslant n)$, whence we obtain $De_{nj}\subseteq S[Q]$ $(n>j\geqslant 1)$ and $De_{i1}\subseteq S[Q]$ $(n\geqslant i>1)$. Further, $De_{1j}=aDe_{nj}\subseteq S[Q]$ $(n>j\geqslant 1)$, whence it follows $e_{nn}=1-\sum_1^{n-1}e_{k1}e_{1k}\in S[Q]$. And then, $e_{1n}=ae_{nn}\in S[Q]$. Now, it is easy to see that e_{ij}'s and D are contained in $S[Q]$, that is, $S[Q]=R$, whence $\mathfrak{G}$ is locally finite by Lemma 3. Accordingly, R/S is locally finite again by [10, (**a***)].

Corollary 2. *If R is Galois and left algebraic over S and $[V:C]<\infty$ then V/Z and $H[V]/S$ are locally finite.*

Proof. At first, as $S[V]=S\times_Z V$, V/Z is algebraic. We set here $V=\sum_1^t C_0 v_k$, where $v_1=1$ and $v_i v_j=\sum c_{ijk}v_k$ $(c_{ijk}\in C_0)$. Let $\{u_1,\cdots,u_s\}$ be a finite subset of V and write $u_h=\sum c_{hk}v_k$ $(c_{hk}\in C_0)$. Evidently, $C^*=Z[\{c_{ijk}\text{'s}\},\{c_{hk}\text{'s}\}]$ is a subfield of C_0 finite over Z. Hence, $V^*=\sum_1^t C^*v_k$ is an intermediate ring of $V/Z[u_1,\cdots,u_s]$ and finite over Z. This means evidently that V/Z is locally finite. Next, as the simple ring H is outer Galois and left algebraic over S, H/S is locally finite by Lemma 4. If F is an arbitrary finite subset of $H[V]$, there exist finite subsets $F_1\subseteq H$ and $F_2\subseteq V$ such that $S[F_1,F_2]\supseteq S[F]$. We set here $S[F_1]=\sum_1^p Sx_i$ and $Z[F_2]=\sum_1^q Zy_j$. Then, $S[F]\subseteq S[F_1,F_2]=S[F_1]\cdot Z[F_2]=\sum_1^p\sum_1^q Sx_iy_j$, whence it follows $[S[F]:S]_l<\infty$.

Corollary 3. *If R is outer Galois and left algebraic over S, then for each finite subset F of R there exists an element a such that $S[F]=S[a]$.*

Proof. As R/S is locally finite by Lemma 4, [10, (**b***)] implies that $\mathfrak{G}$ is l.f.d. Accordingly, our assertion is a consequence of [6, Theorem 2].

Lemma 5. *Let R/S be Galois, T an intermediate ring of R/S left finite over S such that R is T-R-irreducible. If H is a simple ring left algebraic over S, and $T'=J(\mathfrak{G}(T),R)$ then $[T'\cap H:S]<\infty$.*

Proof. By Lemma 1, $(\mathfrak{G}|T')R_r=\sum_{1\oplus}^t(\sigma_i|T')R_r$ with some σ_i's$\in\mathfrak{G}$. And so, in virtue of [7, Corollary 1.1], there holds $\mathfrak{G}|T'=\cup_1^t(\tilde{V}\sigma_i|T')$, whence $\mathfrak{G}|T'\cap H=\cup_1^t(\tilde{V}\sigma_i|T'\cap H)=\{\sigma_1|T'\cap H,\cdots,\sigma_t|T'\cap H\}$. Consequently, it follows $\#(\mathfrak{G}|T'\cap H)<\infty$. Recalling here that H is outer Galois and locally finite over S by Lemma 4 and $J(\mathfrak{G}|H,H)=S$, [7, Lemma 1.8] yields at once $[T'\cap H:S]<\infty$.

Lemma 6. *Let R be Galois and left algebraic over S, $[V:C]<\infty$, T an intermediate ring of R/S left finite over S such that R is T-R-irreducible, and $T'=J(\mathfrak{G}(T),R)$.*

(*i*) *$H[T]$ $(=H[T'])$ and T' are simple rings, and $H[T]$ is outer Galois and locally finite over T'.*

(ii) $[T':S]_l<\infty$, *and so* $H[T]/S$ *is locally finite.*

Proof. (i) Since R is (inner) Galois and finite over H and $H[T]$-R-irreducible, $H[T]$ is a simple ring by [6, Lemma 3]. And so, $H[T]=V_R(V_R(H[T]))=V_R(V_R(T))\supseteq T'$, whence $H[T]=H[T']$. We set here $\mathfrak{T}=\mathfrak{G}(T)$. Then, $\mathfrak{T}|H[T]$ and $\mathfrak{T}|H$ are automorphism groups of $H[T]$ and H respectively, and $J(\mathfrak{T}|H[T],H[T])=T'$, $J(\mathfrak{T}|H,H)=T'\cap H$. Further, as H/S is locally finite by Lemma 4, $\mathfrak{G}(H/S)$ is locally finite by [7, Lemma 1.6], whence $\mathfrak{T}|H$ is locally finite, that is, $\#\{F'\mathfrak{T}\}<\infty$ for each finite subset F' of H. Let the simple ring $H[T]$ be represented as a complete matrix ring over a division subring of $H[T]$ with matrix units $\{f_{ij}\text{'s}\}$. If F is an arbitrary finite subset of $H[T]$, there exists a finite subset E of H such that $F\cup\{f_{ij}\text{'s}\}\subseteq T[E]$. Since $\#\{E\mathfrak{T}\}<\infty$, $T^*=T'[E\mathfrak{T}]$ is $\mathfrak{T}$-normal and $\#(\mathfrak{T}|T^*)<\infty$. Moreover, noting that $H[T]\supseteq T^*\supseteq\{f_{ij}\text{'s}\}$, we see that T^* is a simple ring. And, as $H[T]=V_R(V_R(T))\supseteq T^*\supseteq T$, $\mathfrak{T}|T^*$ is a finite outer group. Hence, $T'=J(\mathfrak{T}|T^*,T^*)$ is a simple ring and $[T^*:T']=\#(\mathfrak{T}|T^*)<\infty$, which proves that $H[T]$ is outer Galois and locally finite over T'.

(ii) We shall use the same notations as in (i), and in particular we set $F=\{d_{hk}\text{'s}\}$ (matrix units of H). As H/S is locally finite by Lemma 4, $\mathfrak{G}(H/S)$ is l.f.d. by [10 (**b***)]. Accordingly, every intermediate ring of H/S is simple by [5, Theorem 1.1]. And, this fact will be used often in the sequel.

Since $T^*\cap H$ is $\mathfrak{T}$-normal and $\#(\mathfrak{T}|T^*\cap H)\leqslant\#(\mathfrak{T}|T^*)<\infty$, $\mathfrak{T}|T^*\cap H$ is a finite outer group. Hence, $[T^*\cap H:T'\cap H]=\#(\mathfrak{T}|T^*\cap H)<\infty$. On the other hand, $[T'\cap H:S]<\infty$ by Lemma 5. Consequently, there holds

$$(*)\qquad [T^*\cap H:S]=[T^*\cap H:T'\cap H]\cdot[T'\cap H:S]<\infty\,.$$

We set here $\mathfrak{T}^*=\mathfrak{G}(T^*)=\mathfrak{T}(T^*)$, $H^*=T^*\cap H$, $K^*=V_{H^*}(\{d_{hk}\text{'s}\})$ and $K=V_H(\{d_{hk}\text{'s}\})$ (a division ring). Then, as is shown in (i), $[T^*:T']<\infty$ and $H[T]$ is outer Galois and locally finite over T'. Since $H[T]$ is T^*-$H[T]$-irreducible and $\mathfrak{T}|H[T]$ is a Galois group of $H[T]/T'$, [7, Corollary 2.1] enables us to see that $J(\mathfrak{T}^*,R)=J(\widetilde{V_R(H[T])},R)\cap J(\mathfrak{T}^*,R)=J(\mathfrak{T}^*|H[T],H[T])=T^*$, whence one will readily see that $\mathfrak{T}^*|K$ is a Galois group of K/K^*. We shall prove here that $[T^*:K^*]_l\leqslant[R:K]_l$. To this end, suppose $\{t,\cdots,t_m\}\subseteq T^*$ is linearly left independent over K^* but linearly dependent over K. Then, without loss of generality, we may assume that $t_1=\sum_2^{m'}x_it_i$ $(x_i\in K)$ is a non-trivial relation of the shortest length and $x_2\in K\backslash K^*$. Recalling that $\mathfrak{T}^*|K$ is a Galois group of K/K^*, there exists some $\sigma\in\mathfrak{T}^*$ witn $x_2\sigma\neq x_2$. Accordingly, we obtain $0=t_1-t_1\sigma=\sum_2^{m'}(x_i-x_i\sigma)t_i$ and $x_i-x_i\sigma\in K$. This contradiction proves evidently $[T^*:K^*]_l\leqslant[R:K]_l=[R:H]_l\cdot[H:K]<\infty$. Combining this with $(*)$, we obtain eventually $[T':S]_l\leqslant[T^*:S]_l=[T^*:H^*]_l\cdot[H^*:S]\leqslant$

$[T^*:K^*]_l\cdot[T^*\cap H:S]<\infty$.

Now, by the light of Lemmas 4, 6 we can prove the following theorem:

Theorem 4. *If R is Galois and left algebraic over S and $[V:C]<\infty$ then R/S is locally finite.*

Proof. In case $S\subseteq C$, our assertion is clear by Corollary 2. Thus, in what follows, we shall assume $S\not\subseteq C$.

Case I: $n=1$. Since $H\not\subseteq C$, $R=H[d]$ with some $d\in R$ by [6, Corollary 2]. Then, $T=S[d]$ is left finite over S, $R=H[T]$, and R is T-R-irreducible. Hence, $R=H[T]$ is locally finite over S by Lemma 6.

Case II: $n>1$. By the same reason as in the proof of Lemma 4, without loss of generality, we may assume that S contains $a=\sum_1^n d_{ij}e_{ij}$ with $d_{1n}=1$ and $d_{in}=0$ $(i\geqslant 2)$. We set $R'=H[\{e_{ij}\text{'s}\}]=\sum_1^n D'e_{ij}$ with $D'=V_{R'}(\{e_{ij}\text{'s}\})$ (a division ring), and distinguish further two cases:

(1) $D'=C$. Since $[R:C]=[R:R']\cdot[R':D']<\infty$, $[S:Z]<\infty$ by [9, Lemma]. The field $Z[C]$ contained in the center of V is locally finite over Z by Corollary 2. If F is an arbitrary finite subset of C, $\mathfrak{G}|Z[F\mathfrak{G}]$ and $\mathfrak{G}|(Z\cap C)(F\mathfrak{G})$ are (outer) Galois groups of $Z[F\mathfrak{G}]/Z$ and $(Z\cap C)(F\mathfrak{G})/Z\cap C$ respectively. Further, to be easily seen, they are of same finite order. And so, we have $Z[F\mathfrak{G}]=Z\times_{Z\cap C}(Z\cap C)(F\mathfrak{G})$. We have proved therefore that $Z[C]=Z\times_{Z\cap C}C$, whence it follows particularly $[Z:Z\cap C]=[Z[C]:C]\leqslant[R:C]<\infty$. Since V is locally finite over Z (Corollary 2) and $D=V_R(R')$ is contained in V, D is evidently locally finite over $Z\cap C$. Accordingly, $R/Z\cap C$ is locally finite. Combining this with $[S:Z\cap C]=[S:Z]\cdot[Z:Z\cap C]<\infty$, our assertion will be easily seen.

(2) $D'\supsetneqq C$. Since R is Galois and finite over H, it will be easy to see that D is Galois and finite over D' not contained in C. Hence, by [6, Corollary 2], $D=D'[d]$ with some non-zero d in D. And then, $R=R'[d]$ of course. We set here $u=de_{21}+\sum_3^n e_{ii-1}$. Then, $T=S[u]=S[a,u]$ is left finite over S, and R is T-R-irreducible by [6, Lemma 8]. And so, $H[T]$ is a simple ring by Lemma 6 (i). Since $u^{k-1}au^{n-1}=d^2e_{k1}$ is a non-zero element of $H[T]\cap e_{kk}R$ $(k=1,\cdots,n)$, $H[T]$ contains $e_{11},\cdots,e_{nn}$ by [6, Lemma 5]. It follows therefore $e_{1n}=e_{11}ae_{nn}\in H[T]$ and $d=(u+e_{1n})^n\in H[T]$, whence $e_{k1}\in H[T]$ $(k=1,\cdots,n)$. Accordingly, $V_R(H[T])\subseteq V_R(\{e_{11},e_{21},\cdots,e_{n1}\})=D$, which implies $H[T]=V_R(V_R(H[T]))\supseteq V_R(D)\supseteq\{e_{ij}\text{'s}\}$. And moreover, $H[T]=H[u,\{e_{ij}\text{'s}\}]=H[d,\{e_{ij}\text{'s}\}]=R'[d]=R$. Hence, $R=H[T]$ is locally finite over S by Lemma 6.

Theorem 3 enables us to restate [7, Conclusion 2.1] and [7, Corollary 4.2] under the same assumptions as in [1, VII. §6]:

Corollary 4. *Let R be Galois and left algebraic over S and $[V:C]<\infty$.*

(*i*) *For each regular intermediate rings R_1, R_2 of R/S, every S-ring isomorphism ρ of R_1 onto R_2 can be extended to an automorphism of R.*

(*ii*) *For each regular intermediate ring R' of R/S, R/R' is $\mathfrak{G}(R')$-locally Galois.*

(*iii*) *There exists a 1–1 dual correspondence between closed $(*)$-regular subgroups of $\mathfrak{G}$ and regular intermediate rings of R/S, in the usual sense of Galois theory.*

Next, we shall improve [7, Lemma 1.10].

Lemma 7. *Let R be Galois and left 2-algebraic over S, $[V:C_0]<\infty$, and Q the set of all $q\in R$ with the property that each intermediate (simple) ring T of $R/S[q, \{e_{ij}\text{'s}\}]$ is $S[q]$-T-irreducible. If $S[Q]$ coincides with R and there exists an element $r'\in Q$ such that $R'=V_R(V_R(S[r']))\supseteq\{e_{ij}\text{'s}\}$ and $H\not\subseteq V_{R'}(R')$ then H is a simple ring and R/S is locally finite.*

Proof. If T is an arbitrary intermediate ring of R/R', then $H\not\subseteq V_T(T)$. In fact, $H\subseteq V_T(T)$ implies $H\subseteq R'\cap V_T(R')=V_{R'}(R')$, which is a contradiction. At first, we shall prove that there exists an element $r_1\in R$ such that $R_1=V_R(V_R(S[r_1]))\supseteq\{e_{ij}\text{'s}\}$, $H\not\subseteq V_{R_1}(R_1)$ and $V_R(S[r_1])\subseteq C_0$. To this end, we shall distinguish two cases, and set $S'=S[r']$:

Case I: $V_{R'}(R')\not\subseteq C_0$. As V is (inner) Galois and finite over C_0, by [6, Theorem 5], for arbitrary $x\in V_{R'}(R')(\subseteq V)$ not contained in C_0 there exists some $y\in V$ such that $V=C_0[x, y]$. If we set $S''=S'[y]=S[r', y]$, then $R''=V_R(V_R(S''))\supseteq S''[V_R(V_R(S'))]=S''[R']\supseteq H[x, y, \{e_{ij}\text{'s}\}]$. And so, $R''(\supseteq\{e_{ij}\text{'s}\})$ is a simple ring containing $V=C_0[x, y]$. Further, recalling that R'' is (S'-R''- and so) S''-R''-irreducible, we see that $\infty>[S'':S]_l\geqslant[V_{R''}(S):V_{R''}(R'')]=[V:V_{R''}(R'')]$ by [7, Lemma 1.5]. Hence, R'' is inner Galois and finite over (simple) $H=V_{R''}(V)$ and, as is noted at the opening, $H\not\subseteq V_{R''}(R'')$. Accordingly, by [6, Corollary 2], $R''=H[r_1]$ with some r_1. Evidently, there holds $V_R(S[r_1]))=V_R(H[r_1])=V_R(R'')=V\cap V_R(R'')\subseteq V\cap V_R(V)=C_0$ and $R_1=V_R(V_R(S[r_1]))\supseteq R'$. Needless to say, as is noted at the opening, $H\not\subseteq V_{R_1}(R_1)$.

Case II: $V_{R'}(R')\subseteq C_0$. Evidently, $V_R(S')=V_R(R')$ is a division subring of D, and its center coincides with the center C' of R'. Since $V'=C_0[V_R(S')]=C_0\times_{C'}V_R(R')$ and $[V':C_0]\leqslant[V:C_0]<\infty$, we see that $[V_R(R'):C']<\infty$ and V' is a simple ring. Consequently, there holds $V_{V}(V_{V}(V'))=V'$. Now, in case $V'\subsetneqq V$, there holds $C_0\subsetneqq V_V(V')=V_V(V_R(S'))\subseteq R'$. And so, by [6, Theorem 5], for an arbitrary $x\in V_V(V')\backslash C_0$ there exists some $y\in V$ such that $V=C_0[x, y]$. If we set $S''=S'[y]$, it follows $R''=V_R(V_R(S''))\supseteq S''[V_R(V_R(S'))]=S''[R']\supseteq H[x,y, \{e_{ij}\text{'s}\}]\supseteq V$. Accordingly the rest of the proof will proceed just as in the latter half of Case I. While, in case $V'=V$, $R''=R'\cdot V_R(R')=R'\times_C V_R(R')$

is a simple ring containing $V'=V$ and the center C'' of R'' coincides with C'. Noting here that R is S'-R-irreducible and $[V_R(R'):C']<\infty$, we obtain $\infty>[S':S]_l\cdot[V_R(R'):C']\geqslant[V:V_R(S')]_r\cdot[V_R(R'):C']=[V:C']=[V:C'']$. Hence, R'' is inner Galois and finite over $H=V_{R''}(V)$. Accordingly, the rest of the proof proceeds just as in the last part of Case I.

Thus, we have proved our first step assertion, and at the same time we have seen that H is a simple ring.

Now, let $\{x_1,\cdots,x_m\}$ be an arbitrary finite subset of R. As $R=S[Q]$, there exists a finite subset $\{q_1,\cdots,q_t\}$ of Q such that $S[x_1,\cdots,x_m]\subseteq S[q_1,\cdots,q_t]$. We set here $S_2=S[r_1,q_1]$ and $R_2=V_R(V_R(S_2))\,(\supseteq R_1\supseteq V)$. Then, the simple ring R_2 being S_2-R_2-irreducible, $\infty>[S_2:S]_l\geqslant[V_{R_2}(S):V_{R_2}(S_2)]_r=[V:V_{R_2}(R_2)]_r$ by [7, Lemma 1.5]. R_2 is therefore inner Galois and finite over H and $H\not\subseteq V_{R_2}(R_2)$. And so, by [6, Corollary 2], $R_2=H[r_2]$ with some r_2. Noting here that $V_R(S[r_2])=V_R(H[r_2])=V_R(R_2)\subseteq V_R(R_1)\subseteq C_0$, one will easily see that r_2 possesses the property that r_1 enjoyed. Accordingly, we can repeat the same argument for $S[r_2,q_2]$ instead for $S_2=S[r_1,q_1]$ to obtain such an element r_3 that $V_R(V_R(S[r_2,q_2]))=H[r_3]$. Continuing the same procedures step by step, we obtain eventually $r_2,\cdots,r_{t+1}\in R$ such that $V_R(V_R(S[r_k,q_k]))=H[r_{k+1}]$ $(k=1,\cdots,t)$. As $q_1,\cdots,q_t\in H[r_{t+1}]$, there exists a finite subset F of H such that $S[q_1,\cdots,q_t]\subseteq S[F,r_{t+1}]$. Recalling here that the simple ring H is outer Galois and left algebraic over S, $S[F]=S[h]$ with some $h\in H$ by Corollary 3. Hence, $S[x_1,\cdots,x_m]$ $(\subseteq S[q_1,\cdots,q_t])$ is contained in $S[h,r_{t+1}]$ that is left finite over S.

Theorem 5. *If R is Galois and left 2-algebraic over S and $[V:C_0]<\infty$ then H is simple and R/S is locally finite.*

Proof. If $S\subseteq C$ then $V=R$, whence $[V:C]=[V:C_0]<\infty$. And so, our theorem for this case is contained in Theorem 4. And next, in case D is finite there is nothing to prove. Thus, in what follows, we shall restrict our attention to the case where $S\not\subseteq C$ and D is infinite. Let Q be the set of all $q\in R$ with the property that each intermediate ring T of $R/S[q,\{e_{ij}\text{'s}\}]$ is $S[q]$-T-irredccible. We shall distinguish here two cases:

Case I: $n=1$. As each intermediate (division) ring T of R/S is S-T-irreducible, there holds evidently $R=Q$. Now, let a be an arbitrary element of $S\backslash C$. Then, $ar'\neq r'a$ for some $r'\in R$. And, it is evident that $H(\ni a)$ is not contained in the center of $R'=V_R(V_R(S[r']))$. Our assertion is therefore a direct consequence of Lemma 7.

Case II: $n>1$. By the same reason as in the proof of Lemma 4, without loss of generality, we may assume that S contains an element $a=\sum_1^n d_{ij}e_{ij}$ with $d_{1n}=1$ and $d_{in}=0$ $(i\geqslant 2)$. Then, we have seen in Case II of the proof of Lemma 4 that $R=S[Q]$. We set here $r'=\sum_2^n e_{ii-1}$, $S'=S[r']$ and $R'=V_R(V_R(S'))$. r' is contained in Q by [6, Lemma 8] and S' contains $e_{k1}=r'^{k-1}ar'^{n-1}$ $(k=1,\cdots,n)$. And so, $V_R(S')\subseteq V_R(\{e_{11},e_{21},\cdots,e_{n1}\})=D$, whence $R'\supseteq V_R(D)\supseteq\{e_{ij}\text{'s}\}$. And, noting that $ar'\neq r'a$, it will be easy to see that H is not contained in the center of R'. Hence, again by Lemma 7, H is simple and R/S is locally finite.

Corresponding to Corollary 4, Theorem 5 together with [7, Conclusion 2.2] and [7, Theorem 4.2] yields the next:

Corollary 5. *Let R be Galois and left 2-algebraic over S, $[V:C_0]<\infty$, and $[R:H]_l\leqslant\aleph_0$. And let R_1, R_2 be f-regular intermediate rings of R/S.*

(i) If ρ is an S-(ring) isomorphism of R_1 onto R_2 then ρ can be extended to an automorphism of R.

(ii) R/R_1 is $\mathfrak{G}(R_1)$-locally Galois.

(iii) There exists a 1–1 dual correspondence between closed $(*_f)$-regular subgroups of $\mathfrak{G}$ and f-regular intermediate rings of R/S, in the usual sense of Galois theory.

3. In this section, we shall give an improvement of [7, Theorem 3.1]. The realization of our improvement is essentially due to the next lemma.

Lemma 8. *Let R be locally finite over a regular subring S, and $\mathfrak{G}R_r$ dense in $\mathrm{Hom}_{S_l}(R,R)$. If R^* is a regular intermediate ring of R/S, then for each finite subset F of R^* there exists a regular subring S' of R such that $R^*\supseteq S'\supseteq S[F]$ and $[S':S]_l<\infty$.*

Proof. Let $R^*=\sum D^*e^*_{ij}$ and $V^*=V_R(R^*)=\sum U^*g^*_{pq}$, where e^*_{ij}'s, g^*_{pq}'s are matrix units of R^*, V^* respectively and $D^*=V_{R^*}(\{e^*_{ij}\text{'s}\})$, $U^*=V_{V^*}(\{g^*_{pq}\text{'s}\})$ are division rings. Then, $R^{**}=R^*[\{g^*_{pq}\text{'s}\}]=\sum R^*g^*_{pq}=\sum D^*(e^*_{ij}g^*_{pq})$, where $(e^*_{ij}g^*_{pq})$'s form evidently a system of matrix units and $V_{R^{**}}(\{(e^*_{ij}g^*_{pq})\text{'s}\})=D^*$. And so, for an arbitrary finite subset E of R^{**}, $S_E=S[\{e^*_{ij}\text{'s}\},\{g^*_{pq}\text{'s}\},E]$ is an intermediate simple ring of R^{**}/S left finite over S. As R is S_E-R-completely reducible by Lemma 2, we shall denote by $n(E)$ the length of its composition series, which is evidently bounded with the capacity of R and so finite. Now, we set $n(E_0)=\mathrm{Min}\ n(E)$, where E ranges over all the finite subsets of R^{**}. Then, $R=N^*_1\oplus\cdots\oplus N^*_{n(E_0)}$ with S_{E_0}-R-irreducible N^*_i's. Here, we assume that E is an arbitray finite subset of R^{**} containing E_0 and $R=N_1\oplus\cdots\oplus N_{n(E)}$ is a direct decomposition of R into S_E-R-irreducible submodules. Since each N_i is yet S_{E_0}-R-admissible, we have $n(E)\leqslant n(E_0)$, whence it follows $n(E)=n(E_0)$. Hence, we see that each N_i is S_{E_0}-R-irreducible as well. And so, if $R=M_1\oplus\cdots\oplus M_t$ is the direct decomposition of the S_{E_0}-R-module R into homogeneous components M_j's, each M_j is S_E-R-admissible. Noting here that E is an arbitrary finite subset of R^{**} containing E_0, we see that M_j is R^{**}-R-admissible. And then, as $V_R(R^{**})=U^*$ is a division ring, t is to be 1, that is, R is homogeneously completely reducible as an S_{E_0}-R-module. Further, by the same reason, $n(E)=n(E_0)$ implies that R is homogeneovsly completely reducible as an S_E-R-module. Accordingly, we see that $V_R(S_E)$ is simpe. Evidently, $S'_E=V_{S_E}(\{g^*_{pq}\text{'s}\})$ is a simple subring of $R^*=V_{R^{**}}(\{g^*_{pq}\text{'s}\})$ containing S and left finite over S. Recalling here that $S_E=\sum S'_E g^*_{pq}$, it will be easy to see that $V_R(S'_E)=\sum V_R(S_E)g^*_{pq}$, which proves that S'_E is a regular subring of R. Now, to be easily verified, $S'_{E_0\cup F}$ can be adopted as our S' for any finite subset F of R^*.

Lemma 9. *Let R be locally finite over a regular subring S, $\mathfrak{G}R_r$ dense in* $\mathrm{Hom}_{S_l}(R,R)$, *and let $V_R(V_R(S'))$ be simple for each regular intermediate ring S' of R/S with $[S':S]_l<\infty$. If R' is a regular intermediate ring of R/S such that $[H[R']:H]_l<\infty$, then $H[R']$ is outer Galois and locally finite over R' and $\mathfrak{G}(H[R']/R')\cong\mathfrak{G}(H/H\cap R')$ by the restriction map.*

Proof. There exists a simple intermediate ring S' of R'/S such that $H[R']=H[S']$ and $[S':S]_l<\infty$. As $H'=V_R(V_R(S'))\supseteq H[S']\supseteq S'$ and H' is outer Galois and locally finite over S' by Theorem 2, $H[S']$ is simple by [7, Theorem 1.1]. And, noting that $H\mathfrak{G}(S')=H$, Theorem 2 proves that $\mathfrak{G}(S')|H[S']$ is a Galois group of $H[S']/S'$, whence $\mathfrak{G}(S')|H[S']$ is dense in $\mathfrak{G}(H[S']/S')$ by Corollary 4. It follows therefore $H\mathfrak{G}(H[S']/S')=H$. Now, the required isomorphism will be given by the restriction map $\rho:\mathfrak{G}(H[R']/R')\ni\sigma\to\sigma|H\in\mathfrak{G}(H/H\cap R')$. Here, one should remark that $\mathfrak{G}(H[R']/R')$ is compact ([7, §1]) and $\mathfrak{G}(H[R']/R')|H$ is dense in $\mathfrak{G}(H/H\cap R')$ (Corollary 4).

Now, by the validity of Lemmas 8, 9, we can prove the following improvement of [7, Theorem 3.1].

Theorem 6. *Let R be locally finite over a regular subring S, $\mathfrak{G}R_r$ dense in* $\mathrm{Hom}_{S_l}(R,R)$, *and let $V_R(V_R(S'))$ be simple for each regular intermediate ring S' of R/S with $[S':S]_l<\infty$. If R' is a regular intermediate ring of R/S, and H' an intermediate ring of H/S such that H'/S is Galois, then $H'[R']$ is outer Galois and locally finite over R', and $\mathfrak{G}(H'[R']/R')\cong\mathfrak{G}(H'/H'\cap R')$ by the restriction map.*

Proof. As $\mathfrak{G}(H'/S)=\mathfrak{G}(H/S)|H'$ by [7, Corollary 3.9] and Corollary 4, it will suffice to prove our theorem for the case where $H'=H$. Let $R'=\sum D'e'_{ij}$, where $D'=V_{R'}(\{e'_{ij}\text{'s}\})$ is a division ring. Then, Lemma 8 enables us to set $H[R']=\cup_\nu R_\nu$, where $R_\nu=H[S_\nu]$ and S_ν runs over all the regular subrings of R such that $R'\supseteq S_\nu\supseteq S[\{e'_{ij}\text{'s}\}]$ and $[S_\nu:S]_l<\infty$. As $[R_\nu:H]<\infty$ by [7, Lemma 3.2 (iii)], R_ν is simple by Lemma 9, whence so is $H[R']=\cup_\nu R_\nu$

by [7, Lemma 1.1]. Evidently, $R'_\nu = R' \cap R_\nu$ is a simple ring. Moreover, $V_R(S_\nu) \supseteq V_R(R'_\nu) \supseteq V_R(R_\nu) = V_R(S_\nu)$ shows that R'_ν is a regular subring of R. And, one will easily verify that $R_\nu = H[R'_\nu]$, $R' = \cup_\nu R'_\nu$ and $R'_\nu \cap H = R' \cap H$. Now, in virtue of Lemma 9, R_ν is outer Galois and locally finite over R'_ν and $\mathfrak{G}(R_\nu/R'_\nu) \cong \mathfrak{G}(H/H \cap R'_\nu) = \mathfrak{G}(H/R' \cap H)$ by the restriction map. Hence, for each $\tau \in \mathfrak{G}(H/R' \cap H)$ there exists a uniquely determined extension $\tau_\nu \in \mathfrak{G}(R_\nu/R'_\nu)$ of τ. Accordingly, one will easily see that if $R_\mu \subseteq R_\nu$ then $\mathfrak{G}(R_\nu/R'_\nu)|R_\mu = \mathfrak{G}(R_\nu/R'_\nu)|H[R'_\mu] = \mathfrak{G}(R_\mu/R'_\mu)$. By the light of this fact, we can define an R'-(ring) automorphism $\hat{\tau}$ of $H[R']$ by $\hat{\tau}|R_\nu = \tau_\nu$. We set here $\hat{\mathfrak{T}} = \{\hat{\tau}\,; \tau \in \mathfrak{G}(H/R' \cap H)\}$. Evidently, $\hat{\mathfrak{T}}$ forms a group and $\hat{\mathfrak{T}}|R_\nu = \mathfrak{G}(R_\nu/R'_\nu)$. And so, there holds $J(\hat{\mathfrak{T}}, H[R']) = \cup_\nu J(\hat{\mathfrak{T}}|R_\nu, R_\nu) = \cup_\nu R'_\nu = R'$, which means that $H[R']/R'$ is outer Galois. Further, to be easily seen $\hat{\mathfrak{T}}$ is locally finite (cf. [10, (**a***)]). And then, the method used in the proof of [10, (**a***)] enables us to see that $H[R']/R'$ is locally finite. It follows therefore that $\hat{\mathfrak{T}}$ is dense in $\mathfrak{G}(H[R']/R')$ (Corollary 4), whence it will be easy to see that $H\mathfrak{G}(H[R']/R') = H$. Consequently, there holds $\mathfrak{G}(H[R']/R')|H = \mathfrak{G}(H/R' \cap H)$. Now, the rest of the proof will be easy (cf. the proof of Lemma 9).

Combining Theorem 6 with [5, Corollary 1.4], one will readily obtain

Corollary 6. *Let R and S satisfy the assumptions cited in Theorem 6. If R' is a regular intermediate ring of R/S, then $H^* = H^*[R'] \cap H$ for each intermediate ring H^* of $H/R' \cap H$ and $R^* = (R^* \cap H)[R']$ for each intermediate ring R^* of $H[R']/R'$.*

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