Hokkaido Mathematical Journal Vol. 47 (2018) p. 387–426

Automorphisms of order three of the moduli space of Spin-Higgs bundles

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(Received November 2, 2015; Revised April 24, 2017)

Abstract. In this work we consider a family of Spin complex groups constructed in [1] which have outer automorphisms of order three. We define an action of $\mathrm{Out}(\mathrm{Spin}(n, \mathbb{C})) \times \mathbb{C}^*$ on the moduli space of Spin-Higgs bundles and we study the subvariety of fixed points of the induced automorphisms of order three. These fixed points can be expressed in terms of some kind of Higgs pairs associated to certain subgroups of $\mathrm{Spin}(n, \mathbb{C})$ equipped with a representation of the subgroup. We further the study for the simple case, $G = \mathrm{Spin}(8, \mathbb{C})$.

Key words: triality, Spin-Higgs bundles, moduli space, fixed points, Higgs pair.

Introduction

Let X be a compact Riemann surface of genus $g \geq 2$ and let G be a reductive complex Lie group with Lie algebra $\mathfrak{g}$. As it is stated in Definition 1.3, a G-Higgs bundle over X is a pair (E, φ) where E is a holomorphic principal G-bundle and φ is a holomorphic global section of the adjoint bundle of E, $E(\mathfrak{g})$, twisted by the canonical bundle, K. A notion of polystability can be given for Higgs bundles generalizing the notion of polystability given by Ramanathan in [17], [18] and [19] for principal bundles obtaining that the moduli space of polystable G-Higgs bundles, $\mathcal{M}(G)$, is a complex variety of dimension $2 \dim G(g-1)$.

Higgs bundles were introduced by Hitchin in [11] and are of interest in many different areas including surface group representations, gauge theory, Kähler and hyperkähler geometry, integrable systems, Langlands duality and mirror symmetry.

A way of studying the geometry of $\mathcal{M}(G)$ is by the study of subvarieties of the moduli space. Given an automorphism of $\mathcal{M}(G)$, we have a natural subvariety given by the subset of fixed points in $\mathcal{M}(G)$ for this automorphism. Then, it is natural to study automorphisms of finite order of $\mathcal{M}(G)$,

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2010 Mathematics Subject Classification : 14D20.

in the spirit of [15]. The case of involutions was developed by García-Prada and Ramanan in [7] and [10], where they showed that they are related to representations of the fundamental group of the surface in real forms of G. In [1] we studied the case of order three automorphisms for the moduli space of principal Spin-bundles. In this paper we will deal with the case of automorphisms of order three of the moduli space of $\mathrm{Spin}(n,\mathbb{C})$-Higgs bundles. In [7] the same problem is considered but with a different approach.

Our order three automorphisms come from the triality automorphism. If the group G is complex, simple and simply connected with Lie algebra $\mathfrak{g}$, the group $\mathrm{Out}(G)$ is isomorphic to the group of symmetries of the Dynkin diagram of $\mathfrak{g}$. Then the only possibility for $\mathfrak{g}$ to have outer automorphisms of order three is when $\mathfrak{g} = \mathfrak{so}(8,\mathbb{C})$, for which the group of symmetries of its Dynkin diagram is S_3. In [1], a family of Spin groups with outer automorphisms of order three are constructed. Here, we define an appropriate action of $\mathrm{Out}(G)$ on $\mathcal{M}(G)$ for these groups.

We also define a natural action of $\mathbb{C}^*$ on $\mathcal{M}(\mathrm{Spin}(n,\mathbb{C}))$ which will give rise to new automorphisms of order three of the moduli. These automorphisms come from primitive cubic roots of unity in $\mathbb{C}^*$. These two actions can be combined to give rise to an action of $\mathrm{Out}(\mathrm{Spin}(n,\mathbb{C}))\times\mathbb{C}^*$ on $\mathcal{M}(\mathrm{Spin}(n,\mathbb{C}))$. This action will provide a family of automorphisms of order three of the moduli whose subvarieties of fixed points will be studied here for the case in wich $n \cong 0 \mod 8$. Moreover, we will give a complete description in the simple case, when $n = 8$.

This paper is organized as follows. In Section 1 we present some basic notions of stability of Higgs bundles and Higgs pairs we will use throughout the article. In Sections 2 and 3 we describe the action of the group $\mathrm{Out}(G)\times\mathbb{C}^*$ in $\mathcal{M}(G)$. In Section 4 we recall the basic concepts of triality and present the groups that work later and in Section 5 we study the subvariety of fixed points for the action of the automorphisms of order three in $\mathcal{M}(\mathrm{Spin}(n,\mathbb{C}))$ coming from the preceding action, for $n \cong 0 \mod 8$. Finally, in Section 6 we further study the case $\mathrm{Spin}(8,\mathbb{C})$ and in Section 7 we study in detail the moduli spaces of Higgs pairs which play a role in the description of fixed points.

1. The moduli space of G-Higgs bundles

From now on, X will be a compact Riemann surface of genus $g \geq 2$.

Let G be a complex reductive Lie group and $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G. Here, we give the notion of (G,ρ)-Higgs pair. A G-Higgs bundle will be a particular case of (G,ρ)-Higgs pair in which we take the adjoint representation of G. This will be relevant for us because the fixed points which we will describe will usually be Higgs pairs but not Higgs bundles. Then, we will explain the notions of stability and polystability for (G,ρ)-Higgs pairs and a simplified notion of stability for G-Higgs bundles.

Definition 1.1 Let E be a principal G-bundle and P be a parabolic subgroup of G. Let $\chi : P \to \mathbb{C}^*$ a character of P. Let $\sigma : E_P \to E$ be a reduction of the structure group of E to P. We define the degree of E with respect to σ and χ by

$$\deg E(\sigma,\chi) = \deg \chi_* E_P.$$

Let E be a principal G-bundle over X and $\rho : G \to \mathrm{GL}(V)$ a complex representation of V. Associated to E and V, we define the vector bundle

$$E(V) = E \times_\rho V.$$

Definition 1.2 ((G,ρ)-Higgs pair) Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional complex representation of G. A (G,ρ)-Higgs is a pair (E,φ) where E is a principal G-bundle and $\varphi \in H^0(X, E(V) \otimes K)$ and K denotes the canonical bundle over X.

Definition 1.3 A G-Higgs bundle is a (G,Ad)-Higgs pair, where $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation of G and $\mathfrak{g}$ denotes the Lie algebra of G.

If (E,φ) is a G-Higgs bundle, the section φ is called the Higgs field of (E,φ).

Let P be a parabolic subgroup of G, χ be a character of P and s_χ be its dual by the Killing form. Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G. We denote by $V_\chi^{t,-}$ the subspace of V spanned by the eigenvectors of s_χ whose eigenvalues are less or equal to 1. Then, we define

$$\begin{aligned} V_\chi^- &= \{v \in V \ : \ \exists t_0 \text{ such that } \rho(e^{ts_\chi})v \in V_\chi^{t,-} \ \forall t > t_0\} \\ V_\chi^0 &= \{v \in V \ : \ \rho(e^{ts_\chi})v = v \ \forall t\}. \end{aligned} \tag{1}$$

Let P_{s_χ} and L_{s_χ} be the Lie subgroups of G whose Lie algebras are

$$\mathfrak{p}_{s_\chi} = \{x \in \mathfrak{g} \ : \ \mathrm{Ad}(e^{ts_\chi})(x) \text{ is bounded as } t \to \infty\}$$
$$\mathfrak{l}_{s_\chi} = \{x \in \mathfrak{g} \ : \ [x, s_\chi] = 0\},$$

respectively. These subgroups are studied in detail in [9, Lemma 2.5]. The group P_{s_χ} is a parabolic subgroup of G and L_{s_χ} is a Levi subgroup of P_{s_χ}. It can be proved that V_χ^- is invariant under the action of P_{s_χ} and V_χ^0 is invariant under the action of L_{s_χ}. If $V = \mathfrak{g}$ and ρ is the adjoint representation, then $V_\chi^- = \mathfrak{p}_{s_\chi}$ and $V_\chi^0 = \mathfrak{l}_{s_\chi}$.

Let (E, φ) be a (G, ρ)-Higgs pair for the complex representation ρ of G. A reduction of structure group, $\sigma : E_P \to E$, of E to P will be called *admissible* if

$$\varphi \in H^0(X, E_P(V_\chi^-) \otimes K). \tag{2}$$

Now, we give the notion of stability for (G, ρ)-Higgs pairs.

Definition 1.4 (Semistable (G, ρ)-Higgs pair) Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G. Let (E, φ) be a (G, ρ)-Higgs pair. The pair (E, φ) is called semistable if for any parabolic subgroup P of G, any antidominant character χ of P and any reduction of the structure group of E to P, $\sigma : E_P \to E$, such that $\varphi \in H^0(X, E_P(V_\chi^-) \otimes K)$, we have

$$\deg E(\sigma, \chi) \geq 0.$$

Definition 1.5 (Stable (G, ρ)-Higgs pair) Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G and (E, φ) a (G, ρ)-Higgs pair. The pair (E, φ) is called stable if it is semistable and for any nontrivial P, χ and $\sigma : E_P \to E$ as above such that $\varphi \in H^0(X, E_P(V_\chi^-) \otimes K)$,

$$\deg E(\sigma, \chi) > 0.$$

Definition 1.6 (Polystable (G, ρ)-Higgs pair) Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G and (E, φ) a (G, ρ)-Higgs pair. The pair (E, φ) is called polystable if it is semistable and for each P, $\sigma : E_P \to E$ and χ as in the definition of semistable (G, ρ)-Higgs pair such that $\deg E(\sigma, \chi) = 0$, there exists a holomorphic reduction of the structure group of E_P to a

Levi subgroup L of P, $\sigma_L : E_L \to E_P$. Moreover, in this case, we require $\varphi \in H^0(X, E(V_\chi^0) \otimes K)$.

The general theory of (G,ρ)-Higgs pairs can be found in [8] and in [24], where a construction of the moduli space of (G,ρ)-Higgs pairs is given.

It is easy to prove that the condition given in Definition 1.4 for a (G,ρ)-Higgs pair to be semistable is necessary to be checked only for maximal parabolic subgroups of G (of course, this also works for the particular case of G-Higgs bundles).

The moduli space of G-Higgs bundles, denoted by $\mathcal{M}(G)$, is the projective variety parametrizing all possible isomorphism classes of polystable G-Higgs bundles on X. An explicit construction of the moduli space of G-Higgs bundles can be found in [21] or, with more algebraic techniques, in [24].

Definition 1.7 (Simple G-Higgs bundle) A G-Higgs bundle (E,φ) is called simple if $\mathrm{Aut}(E,\varphi) = Z(G)$, where $Z(G)$ denotes the center of G.

The following result, which relates smoothness to simplicity, is proved in [9, Proposition 3.18].

Proposition 1.1 *Let (E,φ) be a stable and simple G-Higgs bundle. Then, (E,φ) is a smooth point in $\mathcal{M}(G)$.*

In a smooth point, the expected dimension of the moduli space of G-Higgs bundles is $2\dim G(g-1)$.

We will now sum-up some useful notions about maps between moduli spaces of G-Higgs bundles.

Suppose that G is semisimple. Let G_0 be a maximal compact subgroup of G. Let $\mathfrak{g}_0$ be the Lie algebra of G_0 and k the Killing form of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, k is a nondegenerate symmetric bilinear form on $\mathfrak{g}$ which stablishes an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$. We define the following hermitian product on $\mathfrak{g}$:

$$h(Y,Z) = k(Y, \sigma(Z)),$$

where σ denotes the Cartan involution which induces G_0. Since k is invariant under the adjoint action of G on $\mathfrak{g}$, we have that the restriction of the adjoint representation of G, $\mathrm{Ad}_0 : G_0 \to \mathrm{U}(\mathfrak{g})$, is a unitary representation. The restriction of k to G_0 is also nondegenerate.

Let H be a semisimple Lie subgroup of G with maximal compact subgroup H_0. Let $\mathfrak{h} = \mathrm{Lie}H$ and $\mathfrak{h}_0 = \mathrm{Lie}H_0$. The Killing form of G, k, restricts to the Killing form of H (denote it by k, too) and it is a nondegenerate symmetric bilinear form, since H is semisimple. It is also nondegenerate on $\mathfrak{h}_0$. Suppose that we have a subspace $\mathfrak{m}$ of $\mathfrak{g}$ in which h is a nondegenerate Hermitian form and the restriction of the adjoint action of G to H respects $\mathfrak{m}$, $\mathrm{Ad}^H : H \to \mathrm{GL}(\mathfrak{m})$. Then, we have that the diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{\mathrm{Ad}_0} & \mathrm{U}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ H_0 & \xrightarrow[\mathrm{Ad}_0^H]{} & \mathrm{U}(\mathfrak{m}) \end{array}$$

commutes.

Let (E_H, φ_H) be an (H, Ad^H)-Higgs pair. We have a natural inclusion of vector bundles $E_H(\mathfrak{m}) \hookrightarrow E_G(\mathfrak{g})$ which induces an inclusion of groups $H^0(X, E_H(\mathfrak{m}) \otimes K) \hookrightarrow H^0(X, E_G(\mathfrak{g}) \otimes K)$. We define φ_G to be the image of the element φ_H by this inclusion. Then, to each (H, Ad^H)-Higgs pair (E_H, φ_H) we may associate a G-Higgs bundle (E_G, φ_G). We will see the following result.

Proposition 1.2 *Let H be a semisimple Lie subgroup of the complex Lie group G. Let (E_H, φ_H) be a polystable (H, Ad^H)-Higgs pair. Then, the G-Higgs bundle (E_G, φ_G) is polystable.*

In order to prove Proposition 1.2, we briefly introduce the notation and techniques used by Hitchin in [11], who introduced a natural gauge equation for Higgs bundles. Details can be found in [8, Section 2].

Let $\rho : G \to \mathrm{GL}(V)$ be a Hermitian complex representation of G such that its restriction to the maximal compact subgroup G_0 of G is unitary. Let $\rho_0 : G_0 \to \mathrm{U}(V)$ be this restriction. Let K be the canonical line bundle over the curve X and let h be any Hermitian metric on K. Let $F \in \Omega^2(X, \mathbb{C})$ be the curvature of the Chern connection of h. Let E be a principal G-bundle over X which admits a reduction of structure group, $\sigma : E_{G_0} \to E$, to G_0. The vector bundle $E(V)$ induced by the representation ρ admits a Hermitian structure, induced by the canonical isomorphism $E(V) \cong E_{G_0} \times_{G_0} V$. Given any Higgs field $\varphi \in H^0(X, E(V) \otimes K)$, we may identify $i\varphi \otimes \varphi^*$ with a skew symmetric section of $\mathrm{End}(E(V) \otimes K)$ (here, the dual of φ is taken with respect to the Hermitian forms of $E(V)$ and K). We define the holomorphic global section

$$\mu_{E_{G_0}}(\varphi) = \rho_0^*\Big(-\frac{i}{2}\varphi \otimes \varphi^*\Big), \tag{3}$$

where the map $\rho_0^* : E_{G_0}(\mathfrak{u}(V))^* \to E_{G_0}(\mathfrak{g}_0)^*$ is induced by the dual of the infinitesimal action of $\mathfrak{g}_0$ on V. Now, since $\mathfrak{g}_0 \cong \mathfrak{g}_0^*$ via its non degenerate bilinear form, we may identify $\mu_{E_{G_0}}$ with an element of $H^0(X, E_{G_0}(\mathfrak{g}_0))$. In [9, Theorem 2.24] the following result is proved.

Theorem 1.1 *Let h be a Hermitian metric on the canonical bundle K and let F be the curvature of the corresponding Chern connection. Let $\rho : G \to \mathrm{GL}(V)$ be a Hermitian complex representation of G such that its restriction to the maximal compact subgroup G_0 of G is unitary. Let (E, φ) be a polystable (G, ρ)-Higgs pair. Then there exists a reduction of structure group $\sigma : E_{G_0} \to E$ of E to G_0 such that*

$$\Lambda(F_{E_{G_0}} + F) + \mu_{E_{G_0}}(\varphi) = 0, \tag{4}$$

where $F_{E_{G_0}} \in \Omega^2(X, E_{G_0}(\mathfrak{g}_0))$ is the curvature of the Chern connection on E with respect to its Hermitian metric, $\Lambda : \Omega^2(X) \to \Omega^0(X)$ is the adjoint of wedging with the volume form on X and $\mu_{E_{G_0}}(\varphi)$ is defined in (3).

Conversely, if (E, φ) is a (G, ρ)-Higgs pair which admits a reduction of structure group to G_0 which is a solution of (4)*, then (E, φ) is polystable.*

Proof of Proposition 1.2. Suppose that the (H, Ad^H)-Higgs pair (E_H, φ_H) is polystable. From Theorem 1.1 applied to (H, Ad^H), there exists a reduction of structure group $\sigma_H : E_{H,H_0} \to E_H$ to the maximal compact subgroup H_0 of H such that this reduction is a solution of (4). The composition of σ_H with the map $E_H \to E_G$ induces a reduction of structure group of E_G to G_0,

$$\sigma_G : E_{G,G_0} = E_{H,H_0} \times_{H_0} G_0 \to E_G, \quad \sigma_G([e, g]) = [e, g].$$

The image of the Chern connection of $E_{H,H_0}(\mathfrak{h}_0)$ by the inclusion map $\Omega^1(X, E_{H,H_0}(\mathfrak{h}_0)) \hookrightarrow \Omega^1(X, E_{G,G_0}(\mathfrak{g}_0))$ is the Chern connection of $E_{G,G_0}(\mathfrak{g}_0)$. Then $F_{E_{H,H_0}} = F_{E_{G,G_0}}$. The same argument shows that

$\varphi_H \otimes \varphi_H^* = \varphi_G \otimes \varphi_G^*$, so $\mu_{E_{H,H_0}}(\varphi_H) = \mu_{E_{G,G_0}}(\varphi_G)$. From this,

$$\Lambda(F_{E_{G,G_0}} + F) + \mu_{E_{G,G_0}}(\varphi_G) = \Lambda(F_{E_{H,H_0}} + F) + \mu_{E_{H,H_0}}(\varphi_H) = 0.$$

From Theorem 1.1, the G-Higgs bundle (E, φ) is polystable. $\square$

If we take $\mathfrak{m} = \mathfrak{h}$, then Ad^H is the adjoint representation of H in its Lie algebra, so we have, as a particular case, the following.

Proposition 1.3 *Let (E_H, φ_H) be a polystable H-Higgs bundle. Then the associated G-Higgs bundle (E_G, φ_G) is polystable.*

This says that the map $(E_H, \varphi_H) \mapsto (E_G, \varphi_G)$ induces a map $\mathcal{M}(H) \to \mathcal{M}(G)$ from the moduli space of H-Higgs bundles to the moduli space of G-Higgs bundles.

2. The action of Out (G) on $\mathcal{M}(G)$

Let G be a complex reductive Lie group and let $\mathcal{M}(G)$ be the moduli space of polystable G-Higgs bundles over X. We use the action of the group $\mathrm{Aut}(G)$ of automorphisms of the Lie group G on the set of principal G-bundles over X given in [1], in order to define the following for G-Higgs bundles.

Definition 2.1 Let (E, φ) be a G-Higgs bundle and $A \in \mathrm{Aut}(G)$. We define $A(E, \varphi)$ to be

$$A(E, \varphi) = (A(E), dA(\varphi)),$$

where $A(E)$ is equal to E as a complex manifold but equipped with the action of G on the right

$$A(E) \times G \xrightarrow{\diamond} A(E), \quad e \diamond g = eA^{-1}(g),$$

and $dA : H^0(X, E(\mathfrak{g}) \otimes K) \to H^0(X, E(\mathfrak{g}) \otimes K)$ is the map induced by the map $E(\mathfrak{g}) \to E(\mathfrak{g})$ which consists on taking the differential of A on each fibre.

This action is well-defined and, in fact, it defines an action of $\mathrm{Out}(G)$ on the set of isomorphism classes of G-Higgs bundles in the following way: if $\sigma \in \mathrm{Out}(G)$ and $A \in \mathrm{Aut}(G)$ is an automorphism of G representing σ, then $\sigma(E, \varphi) = (A(E), dA(\varphi))$.

Remark In fact, it is easy to see that, if $g \in G$ and $A = i_g$ is the inner automorphism given by g, the map $f : (E, \varphi) \to (A(E), dA(\varphi))$ defined by $f(e) = eg^{-1}$ is an isomorphism of principal G-bundles (see [13]).

Our goal now is to prove that $\mathrm{Out}(G)$ acts on the moduli space of G-Higgs bundles, $\mathcal{M}(G)$.

We shall prove the following result.

Proposition 2.1 *If* (E, φ) *is a polystable (resp. stable, semistable) G-Higgs bundle and* $A \in \mathrm{Aut}(G)$*, then* $A(E, \varphi)$ *is polystable (resp. stable, semistable).*

To do that, we will need some lemmas.

Lemma 2.1 *If P is a parabolic subgroup of G and* $A \in \mathrm{Aut}(G)$*, then $A(P)$ is a parabolic subgroup of G. The decomposition $P = L \cdot U$ is a Levi decomposition of P if and only if $A(P) = A(L) \cdot A(U)$ is a Levi decomposition of $A(P)$.*

Proof. The first assertion is trivial from the definition of parabolic subgroup. For the second part we shall take into account that the unipotent radical of $A(P)$, $\mathrm{Rad}A(P)$ coincides with $A(\mathrm{Rad}P)$ and the form of the Levi decomposition. □

Lemma 2.2 *Let (E, φ) be a G-Higgs bundle, P be a parabolic subgroup of G, L be a Levi subgroup of P,* $A \in \mathrm{Aut}(G)$*, $\sigma : (E_P, \varphi_P) \to (E, \varphi)$ be a reduction of the structure group of (E, φ) to P and $\sigma_L : (E_L, \varphi_L) \to (E_P, \varphi_P)$ be a reduction of the structure group to L. Then $A(\sigma) : A(E_P, \varphi_P) \to A(E, \varphi)$ defined by $A(\sigma)(e) = \sigma(e)$ is a reduction of the structure group of (E, φ) to $A(P)$. Moreover, if σ is an admissible reduction (see (2)), then $A(\sigma)$ is admissible.*

Proof. The first part is immediate. For the second, let $\overline{\chi} : A(P) \to \mathbb{C}^*$ be a character of $A(P)$ that is trivial in the connected component of the identity. Then $\overline{\chi}$ is of the form $\chi \circ A^{-1}$ for a character $\chi : P \to \mathbb{C}^*$ that is trivial in the connected component of the identity. The line bundle $(\chi \circ A^{-1})_* A(E_P)$ can be described as the set of classes of the form $[e, \alpha]$ where $e \in E_P$, $\alpha \in \mathbb{C}^*$ and the class of (e, α) is given by all the elements of the form

$$\big(eA^{-1}(g), \chi\big(A^{-1}(g)\big)^{-1}\alpha\big), \quad g \in A(P),$$

that is, by all the elements of the form

$$(eh, \chi(h)^{-1}\alpha).$$

Then it is clear that $(\chi \circ A^{-1})_* A(E_P)$ is isomorphic to $\chi_* E_P$. If $\chi_* E_P$ has degree 0, then $(\chi \circ A^{-1})_* A(E_P)$ has degree 0. □

Proposition 2.2 *Let (E, φ) be a semistable (resp. stable) G-Higgs bundle and $A \in \mathrm{Aut}(G)$. Then $A(E, \varphi)$ is semistable (resp. stable) as a G-Higgs bundle.*

Proof. Let P be a parabolic subgroup of G for which (E, φ) admits a reduction of structure group, $\sigma : E_P \to E$ with $\varphi \in H^0(X, E_P(\mathfrak{p}) \otimes K)$, and a character χ of P. Then, $\deg E(\sigma, \chi) \leq 0$ (resp. < 0). From the proof of Lemma 2.2 it follows that

$$\deg A(E)(A(\sigma), \chi) = \deg E(\sigma, \chi) \leq 0 \quad \text{(resp. } < 0). \qquad \square$$

Proposition 2.3 *If (E, φ) is a polystable G-Higgs bundle and $A \in \mathrm{Aut}(G)$, then $A(E, \varphi)$ is a polystable G-Higgs bundle.*

Proof. Since (E, φ) is semistable, $A(E, \varphi)$ is semistable by Proposition 2.2. Let P be a parabolic subgroup of G for which (E, φ) admits a reduction of structure group, $\sigma : E_P \to E$ with $\varphi \in H^0(X, E_P(\mathfrak{p}) \otimes K)$, and a character χ of P for which $\deg E(\sigma, \chi) = 0$. Let L be a Levi subgroup of P. Since (E, φ) is polystable, there exists a reduction of the structure group of E_P to E_L, σ_L, and $\varphi \in H^0(X, E_L(\mathfrak{l}) \otimes K)$. Then $A(E)$ admits a reduction of structure group, $A(\sigma)$, which satisfies

$$\deg A(E)(A(\sigma), \chi) = \deg E(\sigma, \chi) = 0,$$

so we deduce deduce that $A(E)_{A(P)} = A(E_P)$ admits a reduction of the structure group to $A(L)$, $A(\sigma_L)$, with $\varphi \in H^0(X, E_L(\mathfrak{l}) \otimes K) = H^0(X, A(E_L)(dA(\mathfrak{l})) \otimes K)$. □

Proof of Proposition 2.1. It is a consequence of Propositions 2.2 and 2.3. □

The preceding results allow us to define the following action of $\mathrm{Out}(G)$ on $\mathcal{M}(G)$.

Definition 2.2 Let (E, φ) be a polystable G-Higgs bundle and $\sigma \in \mathrm{Out}(G)$. We define

$$\sigma(E, \varphi) = A(E, \varphi),$$

where $A \in \mathrm{Aut}(G)$ represents σ.

This definition retrieves the action of $\mathrm{Out}(G)$ on the moduli space of polystable principal G-bundles given in [1] only by considering the closed immersion of the moduli space of principal G-bundles in $\mathcal{M}(G)$ given by $E \mapsto (E, 0)$.

3. The action of Out $(G) \times \mathbb{C}^*$ on $\mathcal{M}(G)$

We consider the following action of the group $\mathbb{C}^*$ on the set of G-Higgs bundles as follows.

Definition 3.1 If (E, φ) is a G-Higgs bundle and $\lambda \in \mathbb{C}^*$, we define

$$\lambda(E, \varphi) = (E, \lambda\varphi).$$

It is immediate that reductions of structure group of (E, φ) are in bijective correspondence with reductions of structure group of $(E, \lambda\varphi)$ for all $\lambda \in \mathbb{C}^*$ in such a way that the notions of stability, semistability and polystability are also equivalent. This, applied to parabolic subgroups, gives the following result.

Proposition 3.1 *Let (E, φ) be a semistable (resp. stable, polystable) G-Higgs bundle and $\lambda \in \mathbb{C}^*$. Then $\lambda(E, \varphi)$ is a semistable (resp. stable, polystable) G-Higgs bundle.*

The preceding result ensure that the action of $\mathbb{C}^*$ is well defined in $\mathcal{M}(G)$. Moreover, if λ is a primitive cubic root of unity, then its action on $\mathcal{M}(G)$ induces an automorphism of order three. We will be interested in this kind of automorphisms.

We can now define the following action of $\mathrm{Out}(G) \times \mathbb{C}^*$ on the moduli space of G-Higgs bundles, which is nothing but the combination of the two preceding actions of $\mathrm{Out}(G)$ and $\mathbb{C}^*$.

Definition 3.2 If (E, φ) is a polystable G-Higgs bundle and $(A, \lambda) \in \mathrm{Out}(G) \times \mathbb{C}^*$, then we define

$$(A, \lambda)(E, \varphi) = (A(E), \lambda dA(\varphi)).$$

This action is well defined by Propositions 2.1 and 3.1.

We will be interested in elements of the form (τ, λ) where λ is a primitive cubic root of unity and τ is an outer automorphism of order three of G. These elements induce automorphisms of order three on $\mathcal{M}(\mathrm{Spin}(n, \mathbb{C}))$.

4. Spin-groups and the triality automorphism

The group $G = \mathrm{Spin}(n, \mathbb{C})$, with n even and $n \geq 4$, is the simply connected simple complex group with Dynkin diagram $D_{n/2}$. It is a double covering of the group $\mathrm{SO}(n, \mathbb{C})$ and then it can be seen as an extension of $\mathrm{SO}(n, \mathbb{C})$ by $\mathbb{Z}_2$.

There is a natural action of the group $\mathrm{Aut}(G)$ on the centre of G, $Z = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence, we get a homomorphism of $\mathrm{Aut}(G)$ into the group $S(Z^2)$ of permutations of the set $Z^2 = Z \setminus \{1\}$ of elements of order 2. The subgroup $\mathrm{Int}(G)$ of inner automorphisms of G acts trivially on Z, so we get a homomorphism of $\mathrm{Out}(G)$ into $S(Z^2) \cong S_3$, which is actually an isomorphism when $n = 8$.

For the simple case, $G = \mathrm{Spin}(8, \mathbb{C})$, there are three mutually non isomorphic irreducible representations of G of dimension 8 each. Exactly one nontrivial element of Z^2 for each representation acts trivially. We have also a bijection between the set of outer involutions of G and Z^2. Thus we have bijections between the set of elements of order 2 in Z, the set of involutions in $\mathrm{Out}(G)$ and the set of isomorphism classes of irreducible 8-dimensional representations of G. Indeed, for each $z \in Z^2$, there exists a unique 8 dimensional irreducible representation ρ_z of G and a unique outer automorphism of order 2, σ_z, such that $\rho_z(z) = 1$ and $\sigma_z(z) = z$.

The representation ρ_z (with associated 8-dimensional complex vector space V_z) descends to a representation

$$\mathrm{Spin}(8, \mathbb{C}) / \langle 1, z \rangle \cong \mathrm{SO}(8, \mathbb{C}) \to \mathrm{GL}(V_z),$$

so it induces an invariant nondegenerate quadratic form q_z on V_z and so a short exact sequence of groups

$$1 \to \langle 1, z \rangle \cong \mathbb{Z}_2 \to \mathrm{Spin}(q_z) \to \mathrm{SO}(q_z) \to 1.$$

Let τ be an element of order three in $\mathrm{Out}(G)$. This is called triality automorphism. There are two such automorphisms, τ and $\tau^{-1} = \tau^2$. Let z_1, z_2, z_3 be the three elements of Z^2. We may assume that they satisfy $\tau(z_1) = z_2$, $\tau(z_2) = z_3$, $\tau(z_3) = z_1$. The election of this order induces an election of an element $z \in Z^2$ of order 2 (say $z = z_1$) and, then, a representation space V and a quadratic form q on V as above. The automorphism τ induces then an outer automorphism of $G/Z \cong \mathrm{PSO}(q)$ of order 3. There must exist subspaces of rank 2, V_1, V_2, V_3 and V_4 such that τ interchanges V_1, V_2 and V_3 and leaves V_4 invariant. This says that triality induces an automorphism of the subgroup of $\mathrm{GL}(2,\mathbb{C})^4$ consisting of matrices $\{(A, B, C, D)\}$ all of whose determinants are the same. There is a natural homomorphism of this group into $\mathrm{SO}(8,\mathbb{C})$, so into $\mathrm{PSO}(8,\mathbb{C})$, given by

$$(A, B, C, D) \mapsto (A \otimes B^t) \oplus (C \otimes D^t),$$

where by B^t and D^t we mean the transposed matrices of B and D, respectively.

The kernel of this map is the group of diagonal scalars and the image of $(-1, -1, -1, 1)$ is an element of order 2, a, in the conjugacy class. Actually, this group is a subgroup of $\mathrm{Spin}(8,\mathbb{C})$ and is the centralizer of a. Triality acts on this subgroup as we have explained above.

Now, we will study fixed points in $\mathrm{Spin}(8,\mathbb{C})$ of triality automorphism. We shall first establish some general notions which work for general semisimple Lie groups. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. We have the following exact sequence of groups

$$1 \longrightarrow \mathrm{Int}(\mathfrak{g}) \longrightarrow \mathrm{Aut}(\mathfrak{g}) \longrightarrow \mathrm{Out}(\mathfrak{g}) \longrightarrow 1.$$

It is well-known that, if G is the simply connected complex Lie group with Lie algebra $\mathfrak{g}$, then there is a natural isomorphism of short exact sequences of the form

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Int}(G) & \longrightarrow & \mathrm{Aut}(G) & \longrightarrow & \mathrm{Out}(G) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Int}(\mathfrak{g}) & \longrightarrow & \mathrm{Aut}(\mathfrak{g}) & \longrightarrow & \mathrm{Out}(\mathfrak{g}) & \longrightarrow & 1. \end{array}$$

This says that we can speak, indistinctly, of automorphisms of G and automorphisms of $\mathfrak{g}$.

The following equivalence relation on $\mathrm{Aut}(\mathfrak{g})$ will also be relevant for us.

Definition 4.1 If $\alpha, \beta \in \mathrm{Aut}(\mathfrak{g})$, we say that $\alpha \sim_i \beta$ if there exists $\theta \in \mathrm{Int}(\mathfrak{g})$ such that $\alpha = \theta \circ \beta \circ \theta^{-1}$.

The relation defined above does not define outer automorphisms, but it is easily seen that, for $\alpha, \beta \in \mathrm{Aut}(\mathfrak{g})$, if $\alpha \sim_i \beta$, then α and β define the same element in $\mathrm{Out}(\mathfrak{g})$ (for details, see [1]). This says that the obvious map

$$\mathrm{Aut}(\mathfrak{g})/\sim_i \to \mathrm{Out}(\mathfrak{g}) \tag{5}$$

is well defined.

We consider now, for $j \geq 0$, $\mathrm{Aut}_j(\mathfrak{g}) = \{\alpha \in \mathrm{Aut}(\mathfrak{g}) \ : \ \alpha \text{ is of order } j\}$. The definitions of $\mathrm{Out}_j(\mathfrak{g})$ and $(\mathrm{Aut}(\mathfrak{g})/\sim_i)_j$ are analogous. It is clear that $\mathrm{Aut}_j(\mathfrak{g})/\sim_i$ is sent onto $\mathrm{Out}_j(\mathfrak{g}) \cup \{1\}$ via the natural map if $j \in \{2, 3\}$, that is,

$$\mathrm{Aut}_j(\mathfrak{g})/\sim_i \to \mathrm{Out}_j(\mathfrak{g}) \cup \{1\}, \quad j = 2, 3.$$

We will consider this map for $j = 3$, that is,

$$\mathrm{Aut}_3(\mathfrak{g})/\sim_i \to \mathrm{Out}_3(\mathfrak{g}) \cup \{1\}. \tag{6}$$

In our case, it will be $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ and $G = \mathrm{Spin}(n, \mathbb{C})$, the simply connected complex Lie group with Lie algebra $\mathfrak{g}$, with $n \equiv 0 \mod 8$. In the discussion after [1, Proposition 3.3] it is proved that, in this case, $\mathrm{Out}(\mathfrak{g}) \cong S_3$. So, there are two outer automorphisms of order three: the triality automorphism, τ, and its inverse.

There are as many automorphisms of order three of $\mathfrak{g}$ with the same subgroup of fixed points as lifts of τ by the map (6). In order to count them, in [1, Proposition 3.5] it is proved the following result:

Proposition 4.1 *The number of elements in the pre-image of* τ *by the map* (6) *is*

$$\begin{array}{ll} 2k & \textit{if} \quad n = 4k^2 + 4k, \\ 1 & \textit{otherwise.} \end{array}$$

Proposition 4.1 says that the number of elements in the pre-image of τ by the map (6) is equal to $\sqrt{n+1}-1$ if $n+1$ is a perfect square (which occurs if and only if $n = 4k^2 + 4k$ for some $k \in \mathbb{N}$) and 1 otherwise (always if $n \equiv 0 \mod 8$).

In the case in which $n+1$ is not a perfect square, Proposition 4.1 says that there is only one possibility for the subalgebra of fixed points of an outer automorphism of order three of $\mathfrak{g}$. In [20, Theorem 4] it is proved that the dimension of this subalgebra is 14 and it must be $\mathrm{Fix}(\tau) \cong \mathfrak{g}_2$. In terms of the group, this says that $\mathrm{Fix}(\tau) \cong G_2$, where τ is seen as an automorphism of G. If $n+1$ is a perfect square, we obtain that one of the $2k$ lifts of τ by the map (6) has $\mathfrak{g}_2$ as subalgebra of fixed points. So we have the following result:

Proposition 4.2 *There is an outer automorphism of order three of the group $G = \mathrm{Spin}(n, \mathbb{C})$ with subgroup of fixed points isomorphic to G_2. If $n+1$ is not a perfect square, then G_2 is the subgroup of fixed points of every outer automorphism of order three of G.*

In [25, Theorem 5.5], Wolf and Gray proved that in the case in which $n = 8$ (that is, for $G = \mathrm{Spin}(8, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{so}(8, \mathbb{C})$), the two lifts of the map (6) have as subalgebras of fixed points $\mathfrak{g}_2$ and $\mathfrak{a}_2$ (with simply connected subgroups G_2 and $\mathrm{PSL}(3, \mathbb{C})$).

5. Spin $(n, \mathbb{C})$-Higgs bundles and triality

From now on, the group G will be $\mathrm{Spin}(n, \mathbb{C})$, where $n \equiv 0 \mod 8$.

In the case of the group $\mathrm{SO}(n, \mathbb{C})$, a maximal parabolic subgroup is the stabilizer of an isotropic subspace. Then, it is easy to see that our notion of stability turns to the following in the orthogonal case (for details, see [9, Proposition 4.16 and Theorem 4.17]).

Lemma 5.1 *A special orthogonal Higgs bundle* (E, φ) *is stable* (*resp. semistable*) *if and only if for every isotropic subbundle E' of E with $\varphi(E') \subseteq E' \otimes K$ we have* $\deg E' < 0$ (*resp.* $\deg E' \leq 0$).

A special orthogonal Higgs bundle (E, φ) is polystable if and only if it can be written as the orthogonal direct sum of stable orthogonal Higgs bundles.

We now recall some facts about stability of principal $\mathrm{SO}(n, \mathbb{C})$ and $\mathrm{Spin}(n, \mathbb{C})$-bundles. A detailed explanation can be found in [1, Section 4].

Let us express by $\pi : \mathrm{Spin}(n, \mathbb{C}) \to \mathrm{SO}(n, \mathbb{C})$ the natural double cover. Since both groups have the same Lie algebra and there is a bijection between Borel subgroups and Borel subalgebras of the group, Borel subgroups of $\mathrm{Spin}(n, \mathbb{C})$ correspond exactly to Borel subgroups of $\mathrm{SO}(n, \mathbb{C})$ via π. Moreover, $\ker \pi$ is contained in every Borel subgroup of $\mathrm{Spin}(n, \mathbb{C})$, so the same is true for parabolic subgroups. From this, it is not difficult to verify from the notion of stability given for general reductive groups that a principal $\mathrm{Spin}(n, \mathbb{C})$-bundle E is stable (resp. semistable, polystable) if and only if the corresponding $\mathrm{SO}(n, \mathbb{C})$-bundle is so.

In the rest of the section, we will describe the subspace of fixed points in $\mathcal{M}(\mathrm{Spin}(n, \mathbb{C}))$ for the action of elements of order three in $\mathrm{Out}(G) \times \mathbb{C}^*$, following the spirit of [1] and generalizing some results given there to Higgs bundles.

First of all, we study the fixed points in $\mathcal{M}(\mathrm{Spin}(8, \mathbb{C}))$ for the action of elements of order three of $\mathbb{C}^*$. Here, we will only study stable fixed points. We will need an auxiliary lemma. It generalizes results by Ramanan in [16] for principal bundles to Higgs bundles.

Lemma 5.2 *Let (E, φ) be a stable $\mathrm{Spin}(8, \mathbb{C})$-Higgs bundle and let $(E_{\mathrm{SO}}, \varphi)$ be the associated $\mathrm{SO}(8, \mathbb{C})$-Higgs bundle via the map $\pi : \mathrm{Spin}(8, \mathbb{C}) \to \mathrm{SO}(8, \mathbb{C})$. If (E, φ) admits a non trivial automorphism of order two, then $(E_{\mathrm{SO}}, \varphi)$ can be written as an orthogonal sum*

$$(E_{\mathrm{SO}}, \varphi) = ((V_1 \otimes V_2^*) \perp (V_3 \otimes V_4^*), \varphi_1 \otimes \varphi_2^* \perp \varphi_3 \otimes \varphi_4^*),$$

where (V_1, φ_1), (V_2, φ_2), (V_3, φ_3), (V_4, φ_4) are stable vector Higgs bundles of rank 2 and with the same determinant bundle.

Proof. Let a be a nontrivial automorphism of (E, φ) of order two. The automorphism a acts, in some fibre, as the product by an element of $\mathrm{Spin}(8, \mathbb{C})$, so a can be seen naturally as an element of $\mathrm{Spin}(8, \mathbb{C})$. Denote by $Z(a)$ the centralizer of a in $\mathrm{Spin}(8, \mathbb{C})$. Then the structure group of E can be reduced to $Z(a)$. To see it, consider

$$F = \{\xi \in E \ : \ a(\xi) = \xi a^{-1}\}.$$

The subvariety F of E is clearly invariant by the action of $Z(a)$ and the action of $Z(a)$ on F is simply transitive. Take $e_1, e_2 \in F$ in the same fibre.

There exists a unique $g \in \mathrm{Spin}(8,\mathbb{C})$ such that $e_2 = e_1 g$. Then,

$$a(e_2) = a(e_1 g) = a(e_1)g = e_1 a^{-1} g,$$

But, since $e_2 \in F$, $a(e_2) = e_2 a^{-1} = e_1 g a^{-1}$, so $ga^{-1} = a^{-1}g$ and $g \in Z(a)$.

This proves that E admits a reduction of the structure group to $Z(a)$. Moreover, φ clearly takes values in the Lie algebra of $Z(a)$ because, since a is an automorphism of (E,φ), we have that $\mathrm{Ad}(a)(\varphi) = \varphi$ and then,

$$\exp(\varphi) = \exp(\mathrm{Ad}(a)(\varphi)) = i_a(\exp(\varphi)),$$

where i_a denotes the inner automorphism of $\mathrm{Spin}(8,\mathbb{C})$ given by a.

The automorphism a gives a decomposition of the vector bundle E_{SO} into the orthogonal direct sum of two vector bundles (in fact, orthogonal bundles), V^+ and V^-, that are the eigenspaces of a with eigenvalues 1 and -1, respectively. The automorphism a acts in V^- as a change of sign, and this automorphism lifts to an automorphism of Spin-bundles in two ways. If $\dim V^- = m$ and $s = e_1, \dots, e_m$ is an orthogonal basis of V^-, then the action of the element of the Spin group $e_1 \cdots e_m$ on an element $x \in V$ is the following

$$sxs^{-1} = e_1 \cdots e_m x e_m \cdots e_1.$$

Using that $e_i e_j = -e_j e_i$ we have that $sxs^{-1} = -x$ if and only if m is even and, in this case, the elements of the Spin group $\pm e_1 \cdots e_m$ lift the automorphism -1 in V^-. Then we have proved that V^+ and V^- are of even dimension. Say $m = 2r$.

All this proves that $\pi(Z(a)) = Z(\overline{a}) = \mathrm{S}(\mathrm{O}(2,\mathbb{C}) \times \mathrm{O}(6,\mathbb{C}))$ or $\pi(Z(a)) = Z(\overline{a}) = \mathrm{S}(\mathrm{O}(4,\mathbb{C}) \times \mathrm{O}(4,\mathbb{C}))$. An $\mathrm{SO}(2,\mathbb{C})$-bundle cannot appear in this decomposition, because it would be hyperbolic and hence not stable. Then it must be

$$\pi(Z(a)) = Z(\overline{a}) = \mathrm{S}(\mathrm{O}(4,\mathbb{C}) \times \mathrm{O}(4,\mathbb{C})).$$

We now compute explicitly the inverse image of $Z(\overline{a})$ in order to give an explicit description of $Z(a)$ (observe that, in this case, $\pi^{-1}\pi(Z(a)) = Z(a)$ because, if $g \in Z(a)$, then $-g \in Z(a)$ and $\{g, -g\} = \pi^{-1}(g)$).

The Clifford group $\Gamma(4)$ is the quotient of the subgroup $\mathrm{GL}(2,\mathbb{C}) \times$

$\mathrm{GL}(2,\mathbb{C})$ consisting of matrices (A,B) such that $\det A = \det B$ by the multiplicative group $\{(\lambda,\lambda^{-1}) : \lambda \in \mathbb{C}^*\}$. From this, one deduces that $Z(a)$ is the quotient of the subgroup of $\mathrm{GL}(2,\mathbb{C})^4$ of quadruples of matrices all with the same determinant by the group of diagonal scalars.

Since (E,φ) admits a reduction of its structure group to $Z(\overline{a})$ and $\mathrm{Spin}(8,\mathbb{C})$ restricts to $\mathrm{GL}(2,\mathbb{C})^4$ as gives the following representation

$$(A,B,C,D) \mapsto A \otimes B^t \perp C \otimes D^t,$$

it is clear that E decomposes as an orthogonal direct sum $E = (V_1 \otimes V_2^*) \perp (V_3 \otimes V_4^*)$, where all the V_i are vector bundles of rank two with the same determinant and we can write $\varphi = (\varphi_1,\varphi_2)$ with $\varphi_1 \in H^0(X,\mathrm{End}(V_1 \otimes V_2^*) \otimes K)$ and $\varphi_2 \in H^0(X,\mathrm{End}(V_3 \otimes V_4^*) \otimes K)$. Moreover, there exist sections $\varphi_{11} \in H^0(X,\mathrm{End}(V_1) \otimes K)$, $\varphi_{12} \in H^0(\mathrm{End}(V_2) \otimes K)$, $\varphi_{21} \in H^0(X,\mathrm{End}(V_3) \otimes K)$ and $\varphi_{22} \in H^0(\mathrm{End}(V_4) \otimes K)$ such that $\varphi_1 = \varphi_{11} \otimes \varphi_{12}^*$ and $\varphi_2 = \varphi_{21} \otimes \varphi_{22}^*$. Observe that $V_1 \otimes V_2^*$ and $V_3 \otimes V_4^*$ are both stable orthogonal bundles. This concludes that (V_i,φ_i) is stable for all i as vector Higgs bundles, because $V_1 \otimes V_2^*$ and $V_3 \otimes V_4^*$ are reductions of the structure group of orthogonal groups to $\mathrm{S}(\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}))$. □

We now define the notion of *cyclotomic Higgs bundle*, which will be useful for us.

Definition 5.1 Let (E,φ) a $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle. The Higgs bundle (E,φ) is called cyclotomic of order r $(r \leq n)$ if E decomposes as a direct sum of vector subbundles

$$E = E_1 \oplus \cdots \oplus E_r$$

such that the image of the restriction of the Higgs field φ to E_i falls in $E_{i+1} \otimes K$ for $i = 1, 2, \ldots, n-1$ and $\varphi(E_r) \subseteq E_1 \otimes K$, that is, φ is of the form

$$\varphi = \begin{pmatrix} 0 & \varphi_1 & & & \\ & 0 & \varphi_2 & & \\ & & \ddots & & \\ & & & 0 & \varphi_{r-1} \\ \varphi_r & 0 & & 0 & 0 \end{pmatrix}$$

in terms of the decomposition $E = E_1 \oplus \cdots \oplus E_r$.

We are now in position to describe the fixed points for the action of the elements of order three of $\mathbb{C}^*$. Let $\mathcal{M}^\lambda(\mathrm{Spin}(8,\mathbb{C}))$ be the set of fixed points in $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ for the action of λ for $\lambda \in \mathbb{C}^*$ with $\lambda^3 = 1$, $\mathcal{M}_s(\mathrm{Spin}(8,\mathbb{C}))$ be the moduli space of stable $\mathrm{Spin}(8,\mathbb{C})$)-Higgs bundles, $\mathcal{M}_*(\mathrm{Spin}(8,\mathbb{C}))$ be the open subset of stable and simple $\mathrm{Spin}(8,\mathbb{C})$)-Higgs bundles, $\mathcal{M}_*^\lambda(\mathrm{Spin}(8,\mathbb{C}))$ be the subset of simple and stable fixed points in $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ and $\mathcal{M}_s^\lambda(\mathrm{Spin}(8,\mathbb{C}))$ be the subset of stable fixed points for the action of λ.

Theorem 5.1 *Let $\lambda \in \mathbb{C}^*$ with $\lambda^3 = 1$ and $\lambda \neq 1$. Let (E,φ) be a stable and simple* $\mathrm{Spin}(8,\mathbb{C})$*-Higgs bundle. Then $(E,\varphi) \in \mathcal{M}_*^\lambda(\mathrm{Spin}(8,\mathbb{C}))$ if and only if the vector Higgs bundle of rank 8 associated to (E,φ) via the homomorphism* $\mathrm{Spin}(8,\mathbb{C}) \to \mathrm{SO}(8,\mathbb{C}) \hookrightarrow \mathrm{GL}(8,\mathbb{C})$ *is a cyclotomic Higgs bundle of order three.*

Let (E,φ) be a stable $\mathrm{Spin}(8,\mathbb{C})$*-Higgs bundle. Then $(E,\varphi) \in \mathcal{M}_s^\lambda(\mathrm{Spin}(8,\mathbb{C}))$ if and only if E_{SO} can be written as an orthogonal sum of the form*

$$E_{\mathrm{SO}} = (V_1 \otimes V_2^*) \perp (V_3 \otimes V_4^*),$$

where (V_1,φ_1), (V_2,φ_2), (V_3,φ_3) and (V_4,φ_4) are stable vector Higgs bundles of rank 2 with the same determinant bundle and such that they admit a trivialization, with respect of which the Higgs field φ_i admits the form

$$\varphi_i = \begin{pmatrix} 0 & \alpha_i \\ \beta_i & 0 \end{pmatrix}$$

for each $i = 1,2,3,4$. In terms of that decomposition, the Higgs field φ has the form

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & \varphi_{1,4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_{3,2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \varphi_{2,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \varphi_{4,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varphi_{5,8} \\ 0 & 0 & 0 & 0 & 0 & 0 & \varphi_{6,7} & 0 \\ 0 & 0 & 0 & 0 & 0 & \varphi_{7,6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \varphi_{8,5} & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Let (E,φ) be a stable $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundle and suppose that (E,φ) is a fixed point for the action of λ.

Suppose, as a first step, that (E,φ) is simple. Since (E,φ) is fixed by the action of λ, there exists an automorphism of E, $f_0 : E \to E$, such that, if $\overline{f_0}$ is the corresponding automorphism of the adjoint bundle of E, $E(\mathfrak{so}(8,\mathbb{C}))$, and $1 : K \to K$ is the identity, we have that

$$(\overline{f_0} \otimes 1) \circ \varphi = \lambda\varphi. \tag{7}$$

We consider the vector Higgs bundle of rank 8 associated to (E,φ) via the homomorphism $\mathrm{Spin}(8,\mathbb{C}) \to \mathrm{SO}(8,\mathbb{C}) \hookrightarrow \mathrm{GL}(8,\mathbb{C})$, which we denote by $(E_{\mathrm{GL}},\varphi_{\mathrm{GL}})$. The Higgs field φ is, then, a homomorphism $\varphi : E_{\mathrm{GL}} \to E_{\mathrm{GL}} \otimes K$. The isomorphism f_0 satisfies that the diagram

$$\begin{array}{ccc} E_{\mathrm{GL}} & \xrightarrow{\varphi} & E_{\mathrm{GL}} \otimes K \\ {\scriptstyle \widetilde{f_0}}\downarrow & & \downarrow{\scriptstyle \widetilde{f_0}\otimes 1} \\ E_{\mathrm{GL}} & \xrightarrow[\lambda\varphi]{} & E_{\mathrm{GL}} \otimes K, \end{array} \tag{8}$$

where $\widetilde{f_0} : E_{\mathrm{GL}} \to E_{\mathrm{GL}}$ is the automorphism of E_{GL} induced by f_0, is commutative.

Consider the isomorphism of (E,φ) given by $f = f_0^3 : (E,\varphi) \to (E,\varphi)$. Since (E,φ) is simple, we must have $f \in Z(\mathrm{Spin}(8,\mathbb{C}))$. Let $(E_{\mathrm{SO}},\varphi_{\mathrm{SO}})$ be the $\mathrm{SO}(8,\mathbb{C})$-Higgs bundle associated to (E,φ). Then if we see f as an automorphism of $(E_{\mathrm{SO}},\varphi_{\mathrm{SO}})$, we have that $f \in Z(\mathrm{SO}(8,\mathbb{C}))$, so $f = 1$ or $f = -1$. Suppose that $f = 1$ (the other case is similar). Then f induces a decomposition of E_{GL} of the form $E_{\mathrm{GL}} = E_1 \oplus E_2 \oplus E_3$, where E_i is the eigenbundle of E_{GL} corresponding to the eigenvalue λ^i of f, for $i = 1,2,3$. Moreover, if $u \in E_i$, thanks to (8) we have that

$$f(\varphi(u)) = \lambda\varphi(\lambda^i u) = \lambda^{i+1}\varphi(u),$$

so $\varphi(u) \in E_{i+1}$ if $i = 1,2$ and $\varphi(u) \in E_1$ if $i = 3$. This implies that φ is of the form

$$\begin{pmatrix} 0 & \varphi_1 & 0 \\ 0 & 0 & \varphi_2 \\ \varphi_3 & 0 & 0 \end{pmatrix}$$

in terms of the decomposition $E_{\mathrm{GL}} = E_1 \oplus E_2 \oplus E_3$. Therefore, $(E_{\mathrm{GL}}, \varphi_{\mathrm{GL}})$ is a cyclotomic bundle of order three.

For the converse, suppose that (E, φ) is a cyclotomic bundle with decomposition $E \cong E_1 \oplus E_2 \oplus E_3$. It is easy to see that the isomorphism $f : E \to E$ given by $f(e_1, e_2, e_3) = (\lambda e_1, e_2, \lambda^2 e_3)$ gives an isomorphism $(E, \varphi) \cong (E, \lambda\varphi)$.

Suppose now that (E, φ) is not simple. The bundle (E, φ) is fixed by the action of λ, so, as before, there exists an automorphism of E, $f : E \to E$, such that $(\overline{f} \otimes 1) \circ \varphi = \lambda\varphi$. If f is given by multiplication by an element of $Z(\mathrm{Spin}(8, \mathbb{C}))$, then we proceed as in the preceding case. So suppose that it is not the case. Since every automorphism of (E, φ) has order two, applying Lemma 5.2 to f^3, which is an automorphism of (E, φ), it induces a decomposition of $(E_{\mathrm{GL}}, \varphi_{\mathrm{GL}})$ of the form

$$E = (V_1 \otimes V_2^*) \perp (V_3 \otimes V_4^*), \quad \varphi_{\mathrm{GL}} = \varphi_1 \otimes \varphi_2^* \perp \varphi_3 \otimes \varphi_4^*.$$

where (V_1, φ_1), (V_2, φ_2), (V_3, φ_3), (V_4, φ_4) are stable vector Higgs bundles of rank 2. The bundles $V_1 \otimes V_2^*$ and $V_3 \otimes V_4^*$ are stable $\mathrm{SO}(4, \mathbb{C})$-Higgs bundles. In terms of this orthogonal direct sum decomposition, the automorphism f^3, seen as an automorphism of vector bundles, must be of the form

$$f^3 = \begin{pmatrix} aI & 0 \\ 0 & \frac{1}{a} I \end{pmatrix}$$

for $a \in \mathbb{C}^*$, because each (V_i, φ_i) is stable as a vector Higgs bundle and hence simple.

Now, we will put our attention in $V_1 \otimes V_2^*$. For the other summand the analysis is analogous.

Since (V_1, φ_1) is stable as a vector Higgs bundle, we have that f is of the form $f = \alpha \mathrm{Id}$. For the same reason, the restriction to V_2 of f is of the form $f = \beta \mathrm{Id}$ for some $\beta \in \mathbb{C}^*$. Then, the restriction of f to $V_1 \otimes V_2^*$ is multiplication by $\alpha\beta$.

Suppose now that the Higgs field for V_1 has the form

$$\varphi_{V_1} = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{pmatrix}$$

and, for V_2^*

$$\varphi_{V_2^*} = \begin{pmatrix} \varphi_1' & \varphi_2' \\ \varphi_3' & \varphi_4' \end{pmatrix}.$$

Then imposing the condition $f\varphi_{V_1} = \lambda\varphi_{V_1} f$, that is,

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{pmatrix} = \lambda \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

we must have $\varphi_1 = 0$ and $\varphi_4 = 0$. Doing the same with $\varphi_{V_2^*}$, we have that $\varphi_1' = 0$, $\varphi_4' = 0$. Then computing $\varphi_{V_1} \otimes \varphi_{V_2^*}$, we have that it has to be of the expected form

$$\begin{pmatrix} & & & \varphi_{1,4} \\ & & \varphi_{2,3} & \\ & \varphi_{3,2} & & \\ \varphi_{4,1} & & & \end{pmatrix}.$$

Doing the same with V_3 and V_4^* we obtain that the Higgs field for this summand is of the form

$$\begin{pmatrix} & & & \varphi_{5,8} \\ & & \varphi_{6,7} & \\ & \varphi_{7,6} & & \\ \varphi_{8,5} & & & \end{pmatrix},$$

and the result holds. □

We can now treat the general case. Let (E, φ) be a $\mathrm{Spin}(n, \mathbb{C})$-Higgs bundle fixed by the action of (τ, λ), that is, if $A \in \mathrm{Aut}(\mathrm{Spin}(n, \mathbb{C}))$ represents τ, then $(E, \varphi) \cong (A(E), \lambda dA(\varphi))$. We may assume that A has order three. To see this, observe that there are two automorphisms of order three of $\mathrm{Spin}(n, \mathbb{C})$ not related by inner automorphisms, that are the triality automorphism and its inverse and each of them belongs to an element of order three of $\mathrm{Out}(\mathrm{Spin}(n, \mathbb{C}))$. We consider the subgroup $\mathrm{Fix}(A)$ of fixed points of A. The differential of τ is an automorphism of order three of $\mathfrak{so}(n, \mathbb{C})$. We consider the corresponding decomposition of $\mathfrak{so}(n, \mathbb{C})$ into eigenspaces for $d\tau$

$$\mathfrak{so}(n, \mathbb{C}) = \mathfrak{h}_1 \oplus \mathfrak{h}_\mu \oplus \mathfrak{h}_{\mu^2}, \tag{9}$$

where μ is a primitive third root of unity. The subspace $\mathfrak{h}_1$ is the subalgebra of fixed points of $d\tau$, so $\mathfrak{h}_1 \cong \mathfrak{g}_2$ if $n+1$ is not a perfect square. It is easily seen that the subalgebra $\mathfrak{h}_1$ is represented in $\mathfrak{h}_1$, $\mathfrak{h}_\mu$ and $\mathfrak{h}_{\mu^2}$ by the restriction of the adjoint action of $\mathfrak{so}(n,\mathbb{C})$ to $\mathfrak{h}_1$. In other words, the restriction of the adjoint representation of $\mathfrak{so}(n,\mathbb{C})$ to $\mathfrak{h}_1$ gives rise to the following representations of $\mathfrak{h}_1$:

$$\rho_1 : \mathfrak{h}_1 \to \mathfrak{gl}(\mathfrak{h}_1)$$
$$\rho_\mu : \mathfrak{h}_1 \to \mathfrak{gl}(\mathfrak{h}_\mu)$$
$$\rho_{\mu^2} : \mathfrak{h}_1 \to \mathfrak{gl}(\mathfrak{h}_{\mu^2}).$$

Let H be the connected subgroup of $\mathrm{Spin}(n,\mathbb{C})$ with Lie algebra $\mathfrak{h}_1$. We have the corresponding representations of H (which we also denote ρ_1, ρ_μ and ρ_{μ^2} for simplicity).

Let $\mathcal{M}_\lambda(H)$ be the moduli space of polystable H-Higgs pairs of the form (E,φ) where E is a principal H-bundle and $\varphi \in H^0(X,(E(\mathfrak{h}_\lambda)\otimes K)$. With this notation, $\mathcal{M}_1(H)$ coincides with the moduli space of H-Higgs bundles, $\mathcal{M}(H)$.

The following statement is the main result of this section.

Theorem 5.2 *Let (τ,λ) be an element of* $\mathrm{Out}(\mathrm{Spin}(n,\mathbb{C}))\times\mathbb{C}^*$ *of order three with $\tau \neq 1$. Let $\mathcal{M}^{(\tau,\lambda)}(\mathrm{Spin}(n,\mathbb{C}))$ be the subset of fixed points in $\mathcal{M}(\mathrm{Spin}(n,\mathbb{C}))$ for the action induced by (τ,λ) and let $\mathcal{M}_*^\tau(\mathrm{Spin}(n,\mathbb{C}))$ be the subset of stable and simple fixed points in $\mathcal{M}(\mathrm{Spin}(n,\mathbb{C}))$ for (τ,λ). Then*

$$\mathcal{M}_*^{(\tau,\lambda)}(\mathrm{Spin}(n,\mathbb{C})) \subseteq \widetilde{\mathcal{M}_{\lambda^2}(\mathrm{Fix}(\tau))} \subseteq \mathcal{M}^{(\tau,\lambda)}(\mathrm{Spin}(n,\mathbb{C})),$$

Proof. Let A be a lifting of τ for the equivalence relation $\sim_i$. Take $(E,\varphi) \in \mathcal{M}_*^{(\tau,\lambda)}(\mathrm{Spin}(n,\mathbb{C}))$. We will see that $(E,\varphi) \in \widetilde{\mathcal{M}_{\lambda^2}(\mathrm{Fix}(A))}$.

There exists an isomorphism $f : E \to A(E)$ such that $f \circ \varphi = \varphi$. Then the corresponding homomorphisms $A(f) : A(E) \to A^2(E)$ and $A^2(f) : A^2(E) \to A^3(E) = E$ are isomorphisms. If we compose them, we obtain an endomorphism

$$A^2(f) \circ A(f) \circ f : E \to E$$

of (E,φ) and, since (E,φ) is simple, there exists $z \in Z(\mathrm{Spin}(n,\mathbb{C}))$ such that

$$A^2(f) \circ A(f) \circ f = z. \tag{10}$$

The arguments exposed in the proof of [1, Theorem 6.1] show that $z = 1$ and that f admits fixed points. We then may define

$$E_H = \{e \in E \ : \ f(e) = e\} \subseteq E.$$

The subvariety E_H of E is clearly invariant under the action of $\mathrm{Fix}(A)$, so E_H is a reduction of structure group of E to $\mathrm{Fix}(A)$ via the inclusion map $E_H \hookrightarrow E$.

We will now study what happens with the Higgs field. It is clear that $(\overline{f} \otimes 1) \circ \varphi = \varphi$, where $\overline{f}$ is the automorphism of $E(\mathfrak{so}(n, \mathbb{C}))$ induced by f. Therefore, $\lambda dA(\varphi) = \varphi$, so φ takes values in $\mathfrak{h}_{\lambda^2}$. Then φ admits a reduction

$$\varphi_{\lambda^2} \in H^0(X, E_{\mathrm{Fix}(A)}(\mathfrak{h}_{\lambda^2}) \otimes K)$$

so that (E, φ) is the image by the forgetful map $\mathcal{M}_{\lambda^2}(\mathrm{Fix}(A)) \to \mathcal{M}(\mathrm{Spin}(n, \mathbb{C}))$ of the element $(E_{\mathrm{Fix}(A)}, \varphi_{\mathrm{Fix}(A)}) \in \mathcal{M}(\mathrm{Fix}(A))$. This proves that (E_H, φ) is a reduction of the structure group of (E, φ) to $H = \mathrm{Fix}\, A$. Observe that if we had fixed λ^2 in the beginning we had obtained that φ reduces to $\mathfrak{h}_\lambda$.

For the second contention of the theorem, suppose that $(E, \varphi) \in \widetilde{\mathcal{M}_{\lambda^2}(\mathrm{Fix}(A))}$. Then E admits a reduction of the structure group to $\mathrm{Fix}(A)$, $E_{\mathrm{Fix}(A)}$, and (E, φ) is isomorphic to $(A(E), \lambda dA(\varphi))$ via an isomorphism $f : E \to A(E)$ such that $E_{\mathrm{Fix}(A)}$ can be seen as the subvariety of E given by the fixed points of f. In fact, it is easily seen that $(E, \varphi) \cong (E_{\mathrm{Fix}(A)} \times_{\mathrm{Fix}(A)} \mathrm{Spin}(n, \mathbb{C}), \varphi)$ and $f : E \to A(E)$ is then given by $f([e, g]) = [e, A^{-1}(g)]$, where the brackets denote the fiber product. □

In the case in which the rank n verifies that $n+1$ is not a perfect square, the theorem says that

$$\mathcal{M}_s^\tau(\mathrm{Spin}(n, \mathbb{C})) \subseteq \widetilde{\mathcal{M}(G_2)} \subseteq \mathcal{M}^\tau(\mathrm{Spin}(n, \mathbb{C}))$$

because G_2 is the only possibility for the group $\mathrm{Fix}(A)$ where A is a lifting of the triality automorphism. There is an infinite amount of possible ranks verifying the condition required for the rank n. For example, take $n = 2^m$ for $m \geq 4$. It is easily seen that $2^m + 1$ is a perfect square if and only if $m = 3$, so this family gives rise to an infinite family of groups whose ranks verify the preceding condition.

When we do not restrict the rank to those for which $n+1$ is not a perfect square, we have more possibilities for the group $\mathrm{Fix}(A)$. The case in which $m = 3$ (that is, when $G = \mathrm{Spin}(8, \mathbb{C})$) is special because we have two lifts of the triality automorphism (as it was seen in Proposition (4.1)) and we can give a more precise description of the $\mathrm{Spin}(8, \mathbb{C})$-bundles which will help us deepen results. We will deal with this case in the next section.

6. The case of Spin $(8, \mathbb{C})$

From Theorem 5.2 and [25, Theorem 5.5], we have that

$$\mathcal{M}^\tau_*(\mathrm{Spin}(8,\mathbb{C})) \subseteq \widetilde{\mathcal{M}(G_2)} \cup \mathcal{M}(\widetilde{\mathrm{PSL}(3,\mathbb{C})}) \subseteq \mathcal{M}^\tau(\mathrm{Spin}(8,\mathbb{C})).$$

In this section we will give a complete characterization of the subvariety of fixed points of the triality automorphism when the structure group is $\mathrm{Spin}(8, \mathbb{C})$.

We will use the following auxiliary result, which is proved in [1, Proposition 7.1]:

Proposition 6.1 *Let* $\tau \in \mathrm{Out}(\mathrm{Spin}(8,\mathbb{C}))$ *be a non-trivial element of order three and* E *be a principal* $\mathrm{Spin}(8,\mathbb{C})$*-bundle with* $E \cong \tau(E)$ *via an isomorphism* $f_0 : E \to \tau(E)$ *such that* $f = \tau^2(f_0) \circ \tau(f_0) \circ f_0 : E \to E$ *is an automorphism of* E *not coming from the centre of* $\mathrm{Spin}(8,\mathbb{C})$*. Then, there exists an element* $a \in \mathrm{Spin}(8,\mathbb{C})$ *with* $a \notin Z(\mathrm{Spin}(8,\mathbb{C}))$ *such that* E *admits a reduction of the structure group to the centralizer of* a *in* $\mathrm{Spin}(8,\mathbb{C})$*,* $Z(a)$*.*

We can now prove the main result of the section:

Theorem 6.1 *Let* (τ, λ) *be an element of order three of* $\mathrm{Out}(\mathrm{Spin}(8,\mathbb{C})) \times \mathbb{C}^*$ *with* $\tau \neq 1$ *(that is,* τ *is a non-trivial outer automorphism of order three of* $\mathrm{Spin}(8,\mathbb{C})$ *and* λ *is a cubic root of unity). Let* $\mathcal{M}^{(\tau,\lambda)}(\mathrm{Spin}(8,\mathbb{C}))$ *be the subset of fixed points on* $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ *for the action induced by* (τ, λ)*. Then*

$$\mathcal{M}^{(\tau,\lambda)}(\mathrm{Spin}(8,\mathbb{C})) = \widetilde{\mathcal{M}_{\lambda^2}(G_2)} \cup \mathcal{M}_{\lambda^2}(\widetilde{\mathrm{PSL}(3,\mathbb{C})}).$$

Proof. Take A a lifting of τ by $\sim_i$. Let $\mu \in \mathbb{C}^*$ be a primitive cubic root of unity. The automorphism A induces a decomposition of $\mathfrak{g} = \mathfrak{so}(8,\mathbb{C})$ into eigenspaces, $\mathfrak{so}(8,\mathbb{C}) = \mathfrak{h}_1 \oplus \mathfrak{h}_\mu \oplus \mathfrak{h}_{\mu^2}$, as in (9).

Let $(E,\varphi) \in \mathcal{M}^{(\tau,\lambda)}(\mathrm{Spin}(8,\mathbb{C}))$. Suppose, as a first step that (E,φ) is stable. Since (E,φ) is fixed by (τ,λ), there exists an automorphism of E, $f_0 : E \to A(E)$ such that

$$(\overline{f_0} \otimes 1) \circ \varphi = \lambda dA(\varphi). \tag{11}$$

If $f = f_0 \circ A(f_0) \circ A^2(f_0)$ is an automorphism of (E,φ) given by an element of the centre of $\mathrm{Spin}(8,\mathbb{C})$, then we are in the situation of the preceding proposition. Suppose this does not happen. Then fix $x \in X$ and $e_0 \in E_x$ and, for them, consider the inclusion of groups $i : \mathrm{Aut}\, E \to \mathrm{Spin}(8,\mathbb{C})$. By Proposition 6.1, the principal $\mathrm{Spin}(8,\mathbb{C})$-bundle E admits a reduction of the structure group to $Z(i(f))$, the centralizer in $\mathrm{Spin}(8,\mathbb{C})$ of the element $i(f)$ and, from the proof of that proposition, we have that this reduction is given by

$$E_0 = \{e \in E \ : \ f(e) = ei(f)\}.$$

Since we have that $(f \otimes 1) \circ \varphi = \varphi$ and it is easy to see that, when restricted to the fibre of x, $(f \otimes 1) \circ \varphi = \mathrm{Ad}_{i(f)}(\varphi)$, we must have $\mathrm{Ad}_{i(f)}(\varphi) = \varphi$, that is, φ takes values in the Lie algebra of $Z(i(f))$.

We know that the reduction of (E,φ) to $Z(i(f))$ gives a decomposition of (E,φ) of the form $E = (V_1 \otimes V_2^*) \perp (V_3 \otimes V_4^*)$ for certain stable vector bundles of rank 2. The bundles $V_1 \otimes V_2^*$ and $V_3 \otimes V_4^*$ are stable principal $\mathrm{SO}(4,\mathbb{C})$-bundles and the direct sum is orthogonal. The triality automorphism acts on the subgroup $\mathrm{GL}(2,\mathbb{C})^4$ of $\mathrm{Spin}(8,\mathbb{C})$ by fixing one of the components and interchanging the other three. This means that a stable fixed point for the action of A, E is of the form $(W \otimes V) \perp (V \otimes V)$, that is, induces a reduction of the structure group of E to $\mathrm{Fix}(A)$. And, since $(\overline{f} \otimes 1) \circ \varphi = \varphi$, we have $\lambda dA(\varphi) = \varphi$ and, then, φ admits a reduction $\varphi_{\lambda^2} \in H^0(X, E_{\mathrm{Fix}(A)}(\mathfrak{h}_{\lambda^2}) \otimes K)$. This completes the case in which (E,φ) is stable.

The polystable case reduces to the stable case. To see this observe that from the Jordan-H der reduction we have that a polystable $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundle reduces to a stable H-Higgs bundle where H is the centralizer of a torus of $\mathrm{Spin}(8,\mathbb{C})$. It is easy to see that the centralizer of a maximal torus of $\mathrm{SO}(8, \mathbb{C})$ is of the form $\mathrm{S}(\mathrm{O}(2, \mathbb{C})^4)$. This proves that the centralizer of a torus of $\mathrm{SO}(8, \mathbb{C})$ is always a subgroup of $\mathrm{S}(\mathrm{O}(4, \mathbb{C}) \times \mathrm{O}(4, \mathbb{C}))$ and we are in the preceding situation. □

7. G_2 and PSL$(3, \mathbb{C})$-Higgs pairs

In this section we describe certain (G, ρ)-Higgs pairs which play a role in the description of fixed points in $\mathcal{M}(\mathrm{Spin}(8, \mathbb{C}))$ made in Theorem 6.1. Here, we will deal with the groups G_2 and $\mathrm{PSL}(3, \mathbb{C})$ and the representations that we will consider come from the triality automorphism, τ.

We consider the decomposition of $\mathfrak{g} = \mathfrak{so}(8, \mathbb{C})$, $\mathfrak{so}(8, \mathbb{C}) = \mathfrak{h}_1 \oplus \mathfrak{h}_\mu \oplus \mathfrak{h}_{\mu^2}$, into eigenspaces of $d\tau$, where μ is a primitive third root of unity, and the representations ρ_1, ρ_μ and ρ_{μ^2}, defined in (9).

Let H be the connected subgroup of $\mathrm{Spin}(8, \mathbb{C})$ with Lie algebra $\mathfrak{h}_1$. We know that it must be $H = G_2$ or $H = \mathrm{PSL}(3, \mathbb{C})$. Suppose first that $H = G_2$. Then, $\mathfrak{h}_\mu \cong \mathfrak{h}_{\mu^2} \cong \mathbb{C}^7$ and we have $\rho_\mu = V_7$ and $\rho_{\mu^2} = V_7$, where V_7 is the fundamental 7-dimensional representation of G_2 (see [4]).

If now $H = \mathrm{PSL}(3, \mathbb{C})$, we have $\dim \mathfrak{h}_\mu = \dim \mathfrak{h}_{\mu^2} = 10$ and then $\rho_\mu = \mathrm{Sym}^3 \mathbb{C}^3$ and $\rho_{\mu^2} = \mathrm{Sym}^3 (\mathbb{C}^3)^*$, where $\mathbb{C}^3$ is the fundamental 3-dimensional representation of $\mathrm{SL}(3, \mathbb{C})$. This representation does not descend to a representation of $\mathrm{PSL}(3, \mathbb{C})$, but the third symmetric power of it and its dual does.

The aim of the section is to describe the stability conditions for (G_2, V_7)-Higgs pairs and $(\mathrm{PSL}(3, \mathbb{C}), \mathrm{Sym}^3 \mathbb{C}^3)$ and $(\mathrm{PSL}(3, \mathbb{C}), \mathrm{Sym}^3 (\mathbb{C}^3)^*)$-Higgs pairs.

We first describe (G_2, V_7)-Higgs pairs.

The group G_2 is the group of automorphisms of $V = \mathbb{C}^7$ which preserve a non-degenerate 3-form. Then, a (G_2, V_7)-Higgs pair is a pair consisting of a rank 7 complex vector bundle, E, over X together with a holomorphic global non-degenerate 3-form $\omega \in H^0(X, \bigwedge^3 E^*)$ and a holomorphic global section $\varphi \in H^0(X, E \otimes K)$.

A subbundle F of E is said to be isotropic if $\omega(F, F, F) = 0$.

We will need to stablish some facts about parabolic subgroups of G_2 (a complete description can be found in [2]). We denote by $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}_2$. If $\{\alpha, \beta\} \subseteq \mathfrak{t}^*$ is a fundamental system of roots of $\mathfrak{g}_2$, the roots of G_2 are

$$\{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta,\\ -\alpha, -\beta, -\alpha-\beta, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta\}.$$

There are three parabolic subgroups in G_2 (those corresponding to the subsets of $\{\alpha, \beta\}$ given by $\{\alpha\}$, denoted by $\mathfrak{p}_\alpha$, $\{\beta\}$, denoted by $\mathfrak{p}_\beta$ and $\{\alpha, \beta\}$, denoted by $\mathfrak{p}_{\alpha,\beta}$, which is the intersection of the first two). The parabolic subalgebra $\mathfrak{p}_\alpha$ is the subalgebra of endomorphisms leaving certain 1-dimensional isotropic subspace of $\mathbb{C}^7$ invariant and the parabolic subalgebra $\mathfrak{p}_\beta$ is the subalgebra that leaves a 2-dimensional isotropic subspace containing the preceding 1-dimensional subspace.

The subalgebra $\mathfrak{p}_\alpha$ is given by endomorphisms of V which leaves invariant a filtration of the form $L \subseteq W \subseteq V$, where $\operatorname{rk} L = 1$ and $\operatorname{rk} W = 6$. The same works for $\mathfrak{p}_\beta$ but with $\operatorname{rk} L = 2$ and $\operatorname{rk} W = 5$.

Proposition 7.1 *A (G_2, V_7)-Higgs pair is semistable (resp. stable) if and only if for each rank one and rank two isotropic subbundle E' of E for which*

$$\varphi \in H^0(X, E/E' \otimes K)$$

we have $\deg E' \leq 0$ (*resp.* $\deg E' < 0$).

Proof. Let χ be an antidominant character of the parabolic subalgebra $\mathfrak{p}_\alpha$ and s_χ the induced element in $i\mathfrak{g}_2$ (following [8]). If $\chi = 2n\alpha/\kappa(\alpha, \alpha)$, where κ is the Killing form and $n \leq 0$, then $s_\chi = n\alpha/2$. So if $L \subseteq W \subseteq V$ is a filtration with $\operatorname{rk} L = 1$ preserved by α, is_χ has the form

$$is_\chi = \begin{pmatrix} \lambda & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & -\lambda \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$, $\lambda > 0$, in a basis induced by the filtration. The condition of semistability for this type of parabolic requires that $\varphi \in H^0(X, E(V_\chi^-))$, where V_χ^- was defined in (1). Let e_1 and e_2 the eigenvectors of V of eigenvalues λ and $-\lambda$ respectively for s_χ and $\{u_1, \ldots, u_5\}$ a basis of the kernel of s_χ. Then, e_1 is an isotropic vector (see [2]). If

$$v = a_1e_1 + a_2e_2 + \sum_{j=1}^{5} b_j u_j$$

is a generic element of V, then $v \in V_\chi^-$ if and only if

$$e^{\lambda t}a_1e_1 + e^{-\lambda t}a_2e_2 + \sum_{j=1}^{5} b_j u_j$$

satisfies $a_1 = 0$. Finally, the numerical stability condition for this kind of parabolic is that for the 1-dimensional isotropic subbundle $E' = \langle e_1 \rangle$ for which

$$\varphi \in H^0(X, E/E' \otimes K),$$

we have that $\deg E' \leq 0$.

Similarly, if χ is an antidominant character of $\mathfrak{p}_\beta$, then in some basis $\{e_1, e_1', u_1, u_2, u_3, e_2, e_2'\}$, is_χ diagonalizes in the form

$$is_\chi = \begin{pmatrix} \lambda & & & & & & \\ & \lambda & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & -\lambda & \\ & & & & & & -\lambda \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$ and $\lambda > 0$.

Here, we have that a generic element v of V is of the form

$$v = a_1e_1 + a_1'e_1' + a_2e_2 + a_2'e_2' + \sum_{j=1}^{3} b_j u_j,$$

so

$$v \in V_\chi^- \Leftrightarrow e^{\lambda t}a_1e_1 + e^{\lambda t}a_1'e_1' + e^{-\lambda t}a_2e_2 + e^{-\lambda t}a_2'e_2' + \sum_{j=1}^{3} b_j u_j$$

and should satisfy $a_1 = a'_1 = 0$. Then the condition of semistability states in this case that for any 2-dimensional isotropic subbundle, E', of E with $\varphi \in H^0(X, E/E' \otimes K)$, we have that $\deg E' \leq 0$. □

From now on, we will deal with $(\mathrm{PSL}(3,\mathbb{C}), \mathrm{Sym}^3 \mathbb{C}^3)$-Higgs pairs and $(\mathrm{PSL}(3,\mathbb{C}), \mathrm{Sym}^3(\mathbb{C}^3)^*)$-Higgs pairs. Here, we will consider principal $\mathrm{PSL}(3,\mathbb{C})$-bundles and we will develop in detail the results for those which lift to principal $\mathrm{SL}(3,\mathbb{C})$-bundles. All what we will do can be done for $\mathrm{PSL}(3,\mathbb{C})$-bundles that does not lift to a $\mathrm{SL}(3,\mathbb{C})$-bundle and the results and proofs are similar. We will only state the results for these cases.

Consider the fundamental 3-dimensional representation of $\mathrm{SL}(3,\mathbb{C})$. Then, the representations of $\mathrm{SL}(3,\mathbb{C})$, $\mathrm{Sym}^3 \mathbb{C}^3$ and $\rho_\mu \cong \mathrm{Sym}^3(\mathbb{C}^3)^*$, descend to give representations of $\mathrm{PSL}(3,\mathbb{C})$.

Let (E, φ) be a $(\mathrm{SL}(3,\mathbb{C}), \rho_\mu)$-Higgs pair. Following the notation of [8], if

$$\mathcal{E} \equiv 0 \subseteq E_1 \subseteq \cdots \subseteq E_k = E \tag{12}$$

for $k = 2, 3$ is a generic filtration of subbundles of E, we define

$$\Lambda(\mathcal{E}) = \left\{ \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k \ : \ \lambda_i \leq \lambda_{i+1} \ \forall i < k \text{ and } \sum_{i=1}^{k} \lambda_i = 0 \right\}. \tag{13}$$

Reductions of structure group of E to parabolic subgroups of $\mathrm{SL}(3,\mathbb{C})$ are in bijective correspondence with the set of filtrations defined in (12) (for details, see [9, Lemma 2.12]). Given any filtration $\mathcal{E}$ as in (12) and any $\lambda \in \Lambda(\mathcal{E})$, if P denotes the parabolic subgroup of $\mathrm{SL}(3,\mathbb{C})$ induced by $\mathcal{E}$ and E_P denotes the reduction of structure group of E to P which corresponds to $\mathcal{E}$, λ induces a character of P, χ, such that its dual by the Killing form, s_χ, diagonalizes with the eigenvalues given by λ. We may consider the subspace V_χ^- of $\mathbb{C}^3$ defined in (1). We then define

$$N(\mathcal{E}, \lambda) = \left\{ \varphi \in H^0(X, \mathrm{Sym}^3 E \otimes K) \ : \ \varphi \in H^0(E_P(V_\chi^-) \otimes K) \right\} \tag{14}$$

and

$$\deg(\mathcal{E}, \lambda) = \lambda_k \deg E + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg E_i. \tag{15}$$

We have the following (we denote by $\otimes_S$ the symmetric tensor).

Proposition 7.2 *A* $(\mathrm{SL}(3,\mathbb{C}), \operatorname{Sym}^3 \mathbb{C}^3)$*-Higgs pair is semistable if and only if the following conditions hold*:

- *For every rank one subbundle E' of E with*
$$\varphi \in H^0\big((\operatorname{Sym}^2 E \otimes_S E') \otimes K\big),$$
we have $\deg E' \leq 0$.
- *For every rank two subbundle E' of E with*
$$\varphi \in H^0\big((E \otimes_S \operatorname{Sym}^2 E') \otimes K\big),$$
we have $\deg E' \leq 0$.

Proof. Suppose that (E, φ) is semistable. Let (P, χ) be a pair with P a parabolic subgroup of $\mathrm{SL}(3,\mathbb{C})$ and χ an antidominant character of P. Then in a certain basis, s_χ will diagonalize in the form

$$is_\chi = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -\lambda-\mu \end{pmatrix},$$

where we can suppose $\lambda \geq \mu \geq -\lambda - \mu$.

If $\{e_1, e_2, e_3\}$ is a basis in which s_χ diagonalizes in the previous form, then

$$\{e_1e_1e_1, e_1e_1e_2, e_1e_1e_3, e_1e_2e_2, e_1e_2e_3, e_1e_3e_3, e_2e_2e_2, e_2e_2e_3, e_2e_3e_3, e_3e_3e_3\},$$

where the product is the symmetric product, is a basis of $\operatorname{Sym}^3 \mathbb{C}^3 = V$. If

$$v = \sum_{i \leq j \leq k} a_{ijk} e_i e_j e_k$$

is a generic element of V, then

$$\begin{aligned}\rho\big(e^{tis_\chi}\big)v = {} & \alpha_{111} e^{3\lambda t} e_1e_1e_1 + \alpha_{112} e^{(2\lambda+\mu)t} e_1e_1e_2 + \alpha_{113} e^{(\lambda-\mu)t} e_1e_1e_3 \\ & + \alpha_{122} e^{(\lambda+2\mu)t} e_1e_2e_2 + \alpha_{123} e_1e_2e_3 + \alpha_{133} e^{(-2\mu-\lambda)t} e_1e_3e_3\end{aligned}$$

$$+ \alpha_{222} e^{3\mu t} e_2 e_2 e_2 + \alpha_{223} e^{(-\lambda+\mu)t} e_2 e_2 e_3 + \alpha_{233} e^{(-2\lambda-\mu)t} e_2 e_2 e_3 + \alpha_{333} e^{-3(\lambda+\mu)t} e_3 e_3 e_3. \quad (16)$$

For the case $\lambda = \mu > 0 > -\lambda - \mu$, associated to (P, χ) we have a filtration of the form $\mathcal{E} \equiv 0 \subsetneq E' \subsetneq E$ and vector of weights $(-\lambda - \mu, \lambda, \lambda) \in \Lambda(\mathcal{E})$. In this case (16) shows that the condition on φ must be

$$\varphi \in H^0\big((E \otimes_S \operatorname{Sym}^2 E') \otimes K\big).$$

For the case $\lambda \geq 0 \geq \mu = -\lambda - \mu$, $\lambda > \mu$, associated to (P, χ) we have a filtration of the form $\mathcal{E} \equiv 0 \subsetneq E' \subsetneq E$ and vector of weights $(\mu, \mu, \lambda) \in \Lambda(\mathcal{E})$. In this case, an observation over (16) shows that the condition on φ must be

$$\varphi \in H^0\big((E \otimes_S \operatorname{Sym}^2 E' + \operatorname{Sym}^3 E') \otimes K\big).$$

As we will now see, the rest of the cases are contained in these two cases.

For the case $\lambda > \mu \geq 0 \geq -\lambda - \mu$, $\mu > -\lambda - \mu$, associated to (P, χ) we have a filtration of the form $\mathcal{E} \equiv 0 \subsetneq E_1 \subsetneq E_2 \subsetneq E$ and vector of weights $(-\lambda - \mu, \mu, \lambda) \in \Lambda(\mathcal{E})$. In this case (16) shows that the condition on φ must be

$$\varphi \in H^0((E \otimes_S E_2 \otimes_S E_1) \otimes K).$$

Finally, in the case in which $\lambda \geq 0 \geq \mu > -\lambda - \mu$, $\lambda > \mu$ (observe that, in this case, it will be $\lambda + 2\mu > 0$, $2\lambda + \mu > 0$ and $\lambda + \mu > 0$), we have a filtration of the form $\mathcal{E} \equiv 0 \subsetneq E_1 \subsetneq E_2 \subsetneq E$ and vector of weights $(-\lambda - \mu, \mu, \lambda) \in \Lambda(\mathcal{E})$. In this case, an observation over (16) shows that the condition on φ must be

$$\varphi \in H^0((E \otimes_S E_2 \otimes_S E_1 + \operatorname{Sym}^3 E_2) \otimes K).$$

For the converse, consider a reduction of the bundle to a parabolic subgroup of the third type (it is the only case we must consider). Associated to the pair (P, χ) we have a filtration $\mathcal{E}$ and certain

$$L = (-\lambda - \mu, \mu, \lambda) \in \Lambda(\mathcal{E}).$$

We may suppose that

$$\varphi \in H^0((E \otimes_S E_2 \otimes_S E_1) \otimes K).$$

Then if

$$L_1 = (-2, 1, 1), \quad L_2 = (-1, -1, 2),$$

we have that

$$L = \frac{\lambda + 2\mu}{3} L_1 + \frac{\lambda - \mu}{3} L_2.$$

With our hypothesis, the coefficients of the preceding linear combination are positive numbers and we have that, from the definitions given in (14) and (15),

$$\deg(\mathcal{E}, L) = \frac{\lambda + 2\mu}{3} \deg(\mathcal{E}, L_1) + \frac{\lambda - \mu}{3} \deg(\mathcal{E}, L_2)$$

and

$$N(\mathcal{E}, L) \subseteq N(\mathcal{E}, L_1) \cap N(\mathcal{E}, L_2),$$

hence the result holds. □

We now consider the representation $\mathrm{Sym}^3(\mathbb{C}^3)^*$ of $\mathrm{PSL}(3, \mathbb{C})$. A $(\mathrm{SL}(3, \mathbb{C}), \mathrm{Sym}^3(\mathbb{C}^3)^*)$-Higgs pair can be seen as a pair (E, φ) with V a rank 3 vector bundle over X and $\varphi \in H^0(X, \mathrm{Sym}^3 E^* \otimes K)$.

As before, if $\mathcal{E} \equiv 0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_k = E$, for $k = 2, 3$, is a generic filtration of subbundles of E, we define $\Lambda(\mathcal{E})$ as in (13) and $N(\mathcal{E}, \lambda)$ as in (14). We have the following result.

Proposition 7.3 *A* $(\mathrm{SL}(3, \mathbb{C}), \mathrm{Sym}^3(\mathbb{C}^3)^*)$*-Higgs pair is semistable if and only if the following conditions hold*:

- *For every rank one subbundle* E' *of* E *with*

  $$\varphi \in H^0\big((\mathrm{Sym}^2 E^* \otimes_S E'^*) \otimes K\big),$$

  we have $\deg E' \leq 0$.

- *For every rank two subbundle E' of E with*

$$\varphi \in H^0\big((E^* \otimes_S \operatorname{Sym}^2 E'^*) \otimes K\big),$$

we have $\deg E' \leq 0$.

The proof is the same as in the preceding case.

Let (E,φ) be a $\mathrm{GL}(3,\mathbb{C})$-Higgs bundle with $\deg E = d$, where $d = 1$ or $d = 2$. Then, the same proof of Proposition 7.2 works to show the following.

Proposition 7.4 *A* $(\mathrm{GL}(3,\mathbb{C}), \operatorname{Sym}^3 \mathbb{C}^3)$*-Higgs pair* (E,φ) *is semistable if and only if the following conditions hold*:

- *For every rank one subbundle E' of E with*

$$\varphi \in H^0\big((\operatorname{Sym}^2 E \otimes_S E') \otimes K\big),$$

we have $\deg E' \leq d/3$.
- *For every rank two subbundle E' of E with*

$$\varphi \in H^0\big((E \otimes_S \operatorname{Sym}^2 E') \otimes K\big),$$

we have $\deg E'/2 \leq d/3$.

We have a similar result for the dual representation, $\operatorname{Sym}^3(\mathbb{C}^3)^*$.

Proposition 7.5 *A* $(\mathrm{GL}(3,\mathbb{C}), \operatorname{Sym}^3(\mathbb{C}^3)^*)$*-Higgs pair* (E,φ) *is semistable if and only if the following conditions hold*:

- *For every rank one subbundle E' of E with*

$$\varphi \in H^0\big((\operatorname{Sym}^2 E^* \otimes_S E'^*) \otimes K\big),$$

we have $\deg E' \leq d/3$.
- *For every rank two subbundle E' of E with*

$$\varphi \in H^0\big((E^* \otimes_S \operatorname{Sym}^2 E'^*) \otimes K\big),$$

we have $\deg E'/2 \leq d/3$.

Finally, we can give the result for $\mathrm{PSL}(3,\mathbb{C})$-Higgs bundles.

Proposition 7.6 *Let* (E_0,φ) *be a* $(\mathrm{PSL}(3,\mathbb{C}), \operatorname{Sym}^3 \mathbb{C}^3)$*-Higgs pair which* *lifts to a* $(\mathrm{GL}(3,\mathbb{C}), \mathrm{Sym}^3 \mathbb{C}^3)$*-Higgs pair* (E,φ) *with* $\deg E = d$, $d = 0$, 1 *or* 2. *The* $(\mathrm{PSL}(3,\mathbb{C}), \mathrm{Sym}^3 \mathbb{C}^3)$*-Higgs pair* (E_0,φ) *is semistable if and only if the following conditions hold*:

- *For every rank one subbundle* E' *of* E *with*

$$\varphi \in H^0\big((\mathrm{Sym}^2 E \otimes_S E') \otimes K\big),$$

we have $\deg E' \leq d/3$.
- *For every rank two subbundle* E' *of* E *with*

$$\varphi \in H^0\big((E \otimes_S \mathrm{Sym}^2 E') \otimes K\big),$$

we have $\deg E'/2 \leq d/3$.

The analogous result holds for the dual representation, $\mathrm{Sym}^3(\mathbb{C}^3)^*$.

Proposition 7.7 *Let* (E_0,φ) *be a* $(\mathrm{PSL}(3,\mathbb{C}), \mathrm{Sym}^3(\mathbb{C}^3)^*)$*-Higgs pair which lifts to a* $(\mathrm{GL}(3,\mathbb{C}), \mathrm{Sym}^3(\mathbb{C}^3)^*)$*-Higgs pair* (E,φ) *with* $\deg E = d$, $d = 0$, 1 *or* 2. *The* $(\mathrm{PSL}(3,\mathbb{C}), \mathrm{Sym}^3(\mathbb{C}^3)^*)$*-Higgs pair* (E_0,φ) *is semistable if and only if the following conditions hold*:

- *For every rank one subbundle* E' *of* E *with*

$$\varphi \in H^0\big((\mathrm{Sym}^2 E^* \otimes_S E'^*) \otimes K\big),$$

we have $\deg E' \leq d/3$.
- *For every rank two subbundle* E' *of* E *with*

$$\varphi \in H^0\big((E^* \otimes_S \mathrm{Sym}^2 E'^*) \otimes K\big),$$

we have $\deg E'/2 \leq d/3$.

We have described the stability conditions for (G_2, V_7)-Higgs pairs and for $(\mathrm{PSL}(3,\mathbb{C}), \mathrm{Sym}^3 \mathbb{C}^3)$ and $(\mathrm{PSL}(3,\mathbb{C}), \mathrm{Sym}^3(\mathbb{C}^3)^*)$-Higgs pairs. We will now describe the maps $\mathcal{M}_\lambda(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ and $\mathcal{M}_\lambda(\mathrm{PSL}(3,\mathbb{C})) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ of Proposition 1.2, where λ is a primitive cubic root of unity.

In [3], the dimension of the moduli space of Higgs pairs is studied. Applying their results to our cases, we obtain

$$\dim \mathcal{M}_\lambda(G_2) = (\dim G_2 + 7)(g-1) = 21(g-1)$$

and

$$\dim \mathcal{M}_\lambda(\mathrm{PSL}(3,\mathbb{C})) = (\dim \mathrm{PSL}(3,\mathbb{C}) + 10)(g-1) = 18(g-1),$$

while $\dim \mathcal{M}(\mathrm{Spin}(8,\mathbb{C})) = 56(g-1)$.

Let $i_\lambda : \mathfrak{h}_\lambda \hookrightarrow \mathfrak{so}(8,\mathbb{C})$ be the natural inclusion. Let $i : \mathrm{PSL}(3,\mathbb{C}) \to \mathrm{Spin}(8,\mathbb{C})$ be the inclusion. Let $(E,\varphi) \in \mathcal{M}_\lambda(\mathrm{PSL}(3,\mathbb{C}))$ be a polystable $\mathrm{PSL}(3,\mathbb{C})$-Higgs pair. The corresponding $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundle is $(E_{\mathrm{Spin}}, \varphi_{\mathrm{Spin}})$ where

$$E_{\mathrm{Spin}} = E \times_i \mathrm{Spin}(8,\mathbb{C})$$

and

$$\varphi_{\mathrm{Spin}} = i_\lambda(\varphi).$$

Recall that, as i_λ is a Lie algebra homomorphism, it induces a homomorphism of vector bundles $i_\lambda : E(\mathfrak{h}_\lambda) \to E(\mathfrak{so}(8,\mathbb{C}))$, which gives a map

$$H^0(X, E(\mathfrak{h}_\lambda) \otimes K) \to H^0(X, E(\mathfrak{so}(8,\mathbb{C})) \otimes K)$$

denoted also by i_λ, so we denote by $i_\lambda(\varphi)$ the image of φ by this map. We can associate to E_{Spin} a holomorphic vector bundle in the following way. Let $j : \mathrm{SO}(8,\mathbb{C}) \hookrightarrow \mathrm{SL}(8,\mathbb{C})$ be the natural inclusion. Consider the homomorphism of groups

$$j \circ \pi : \mathrm{Spin}(8,\mathbb{C}) \to \mathrm{SL}(8,\mathbb{C}).$$

Then we define

$$E_{\mathrm{SL}} = E_{\mathrm{Spin}} \times_{j\circ\pi} \mathrm{SL}(8,\mathbb{C})$$

and, associated to this principal bundle, the vector bundle

$$E_0 = E_{\mathrm{SL}} \times_{\mathrm{SL}(8,\mathbb{C})} \mathbb{C}^8.$$

Let $\{\varphi_{ij}\}$ be a family of transition functions of E. Then

$$\widetilde{\varphi}_{ij} = j \circ \pi \circ i \circ \varphi_{ij}.$$

Now,

$$d(j \circ \pi \circ i) = dj \circ di : \mathfrak{sl}(3, \mathbb{C}) \to \mathfrak{sl}(8, \mathbb{C}).$$

If we consider the Killing form k on $\mathfrak{sl}(3, \mathbb{C})$, then for each $x \in \mathfrak{sl}(3, \mathbb{C})$, $\mathrm{ad}_x \in \mathfrak{so}(\mathfrak{sl}(3, \mathbb{C}))$ and we have that

$$di : \mathfrak{sl}(3, \mathbb{C}) \to \mathfrak{so}(\mathfrak{sl}(3, \mathbb{C})) = \mathfrak{so}(8, \mathbb{C}), \quad di(x) = ad_x.$$

Then $dj \circ di$ coincides with the adjoint representation $\mathfrak{sl}(3, \mathbb{C}) \to \mathfrak{so}(\mathfrak{sl}(3, \mathbb{C}))$. Therefore,

$$j \circ \pi \circ i = \mathrm{Ad},$$

so $\widetilde{\varphi}_{ij} = \mathrm{Ad} \circ \varphi_{ij}$ and, then, E_0 is isomorphic to the adjoint bundle of E. This also proves that the adjoint bundle of E, E_0, is equipped with a special orthogonal structure via the Killing form of $\mathfrak{sl}(3, \mathbb{C})$, κ. Then the image of the $\mathrm{PSL}(3, \mathbb{C})$-Higgs bundle (E, φ) in $\mathcal{M}(\mathrm{Spin}(8, \mathbb{C}))$ is a $\mathrm{Spin}(8, \mathbb{C})$-Higgs bundle whose associated orthogonal Higgs bundle is $((E_0, \kappa), \varphi_{\mathrm{Spin}})$.

Now, we will see how the Higgs field φ is modified to obtain φ_{Spin}.

If $\lambda = 1$, then it is clear that

$$\varphi_{\mathrm{Spin}} = \mathrm{ad}_{\varphi}.$$

This is, of course, an element of $\mathfrak{sl}(8, \mathbb{C})$ and it satisfies that, for every $F, G \in E_0(\mathfrak{so}(8, \mathbb{C}))$,

$$\kappa(\mathrm{ad}_{\varphi}(F), G) + \kappa(F, \mathrm{ad}_{\varphi}(G)) = 0$$

thanks to the properties of ad-invariance of the Killing form.

Suppose, then, that λ is a primitive root of unity. As $\mathrm{Sym}^3 \mathbb{C}^3 \hookrightarrow \mathfrak{sl}(8, \mathbb{C})$, the Higgs field φ induce in a natural way an endomorphism of the adjoint bundle E_0. We denote this endomorphism also by φ. We define the homomorphism of vector bundles

$$\varphi_0 : E_0 \to E_0 \otimes K$$

in the following way: if $F \in E_0$,

$$\varphi_0(F) = [F, \varphi],$$

where $[,]$ denotes the Lie algebra structure in the fibres of E_0. This homomorphism satisfies

$$\kappa(\varphi_0(F), G) + \kappa(F, \varphi_0(G)) = 0$$

for every $F, G \in E_0$, thanks to the properties of κ, so it defines a holomorphic global section of $E_0(\mathfrak{so}(8,\mathbb{C}))$. Finally, we have that

$$\varphi_{\mathrm{Spin}} = \varphi_0.$$

With similar arguments as in the preceding case, if $i' : G_2 \to \mathrm{Spin}(8,\mathbb{C})$ is an inclusion, then $j \circ \pi \circ i' : G_2 \to \mathrm{SL}(8,\mathbb{C})$ is a faithful 8-dimensional representation of G_2, so it is the direct sum of the fundamental 7-dimensional representation of G_2 and the abelian 1-dimensional representation, $G_2 \to \mathrm{SL}(7,\mathbb{C}) \oplus \mathbb{C}$. This map admits a factorization through $\mathrm{SO}(7,\mathbb{C}) \oplus \mathbb{C}$.

Let (E, φ) be a polystable G_2-Higgs bundle and let $\{\varphi_{ij}\}$ be a family of transition functions of E. Then similarly to the case of $\mathrm{SL}(3,\mathbb{C})$, the transition functions of the vector bundle associated to E are $j \circ \pi \circ i' \circ \varphi_{ij}$. This proves that the $\mathrm{SO}(8,\mathbb{C})$-Higgs bundle associated to the image in $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ of (E, φ) is

$$((E_0 \oplus \mathscr{O}, Q \oplus 1), \varphi_{\mathrm{Spin}}), \tag{17}$$

where (E_0, Q) is the orthogonal bundle associated to E via the homomorphism of groups $G_2 \to \mathrm{SO}(7,\mathbb{C})$ stated before and

$$\varphi_{\mathrm{Spin}} = i_\lambda(\varphi).$$

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