Well-chosen weak solutions of the instationary Navier-Stokes system and their uniqueness

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Dedicated to Professor W. Zajaczkowski on the occasion of his 70th birthday

Abstract. We clarify the notion of well-chosen weak solutions of the instationary Navier-Stokes system recently introduced by the authors and P.-Y. Hsu in the article *Initial values for the Navier-Stokes equations in spaces with weights in time, Funkcialaj Ekvacioj* (2016). Well-chosen weak solutions have initial values in $L_\sigma^2(\Omega)$ contained also in a quasi-optimal scaling-invariant space of Besov type such that nevertheless Serrin's Uniqueness Theorem cannot be applied. However, we find universal conditions such that a weak solution given by a concrete approximation method coincides with the strong solution in a weighted function class of Serrin type.

Key words: Navier-Stokes equations, initial values, strong $L_\alpha^s(L^q)$-solutions, well-chosen weak solutions, Serrin's uniquenes theorem.

1. Introduction

The aim of this article is to clarify the relation between so-called *strong $L_\alpha^s(L^q)$-solutions* and *well-chosen weak solutions* of the instationary Navier-Stokes system

$$
\begin{aligned}
u_t - \Delta u + u \cdot \nabla u + \nabla p = f & \quad \text{in } \Omega \times (0, T) \\
\operatorname{div} u = 0 & \quad \text{in } \Omega \times (0, T) \\
u = 0 & \quad \text{on } \partial\Omega \times (0, T) \\
u(0) = u_0 & \quad \text{at } t = 0
\end{aligned}
\tag{1.1}
$$

on a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary of class $C^{2,1}$. Given $2 < s < \infty$, $3 < q < \infty$ and $0 < \alpha < 1/2$ such that

$$
\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha \in (0, 1)
\tag{1.2}
$$

we consider solutions $u \in L_\alpha^s(0, T; L^q(\Omega))$ where $L_\alpha^s(0, T; L^q(\Omega))$ is the Bochner space with weight τ^α in time and with norm

$$\|u\|_{L_\alpha^s(0,T;L^q)} = \left(\int_0^T \left(\tau^\alpha \|u(\tau)\|_q \right)^s \mathrm{d}\tau \right)^{1/s}$$

and initial values

$$u_0 \in L_\sigma^2(\Omega) \cap \mathbb{B}_{q,s}^{-1+3/q}(\Omega). \tag{1.3}$$

Here $\mathbb{B}_{q,s}^{-1+3/q}(\Omega)$ is the real interpolation space

$$\mathbb{B}_{q,s}^{-1+3/q} = \left(L_\sigma^{q'}, \mathcal{D}(A_{q'}) \right)'_{\alpha+1/s,s'} = \left(\mathcal{D}(A_{q'})', L_\sigma^q \right)_{1-\alpha-1/s,s},$$

and $A = A_q$ is the Stokes operator with domain $\mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega) \subset L_\sigma^q(\Omega)$. As equivalent norm on $\mathbb{B}_{q,s}^{-1+3/q}(\Omega)$ we can use

$$\|u_0\|_{\mathbb{B}_{q,s}^{-1+3/q}} \approx \|A^{-1} u_0\|_q + \left(\int_0^\infty \left(\tau^\alpha \|e^{-\tau A} u_0\|_q \right)^s \mathrm{d}\tau \right)^{1/s}.$$

Since the semigroup $e^{-\tau A}$ is exponentially decreasing, we may omit the term $\|A^{-1} u_0\|_q$ in the last norm above, see [9, Thm. 1.14.5], and take the integral over an arbitrary interval $(0, T)$ instead of $(0, \infty)$. Hence

$$\|u_0\|_{\mathbb{B}_{q,s}^{-1+3/q}} \approx \left(\int_0^T \left(\tau^\alpha \|e^{-\tau A} u_0\|_q \right)^s \mathrm{d}\tau \right)^{1/s}.$$

For details on these Besov spaces we refer to [2, Chapter 4]. We note that the spaces $L_\alpha^s(0, T, L^q(\Omega))$ and $\mathbb{B}_{q,s}^{-1+3/q}(\Omega)$ (with the above norm) are scaling invariant under the classical scaling $u(t, x) \mapsto u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$, $\lambda > 0$, for solutions of the 3D-Navier-Stokes system when modifying the time interval $(0, T)$ and the domain Ω correspondingly.

In addition to an initial value in $u_0 \in L_\sigma^2(\Omega) \cap \mathbb{B}_{q,s}^{-1+3/q}(\Omega)$ we consider an external force $f = \operatorname{div} F$ with a matrix-valued function

$$F \in L^2\left(0, T; L^2(\Omega) \right) \cap L_{2\alpha}^{s/2}\left(0, T; L^{q/2}(\Omega) \right). \tag{1.4}$$

Let us recall the main existence result in this setting.

Theorem 1.1 ([2, Theorem 1.2]) *Under the above assumptions on u_0 and f there exists a constant $\varepsilon_* = \varepsilon_*(q, s, \alpha, \Omega) > 0$ with the following property: If*

$$\|e^{-\tau A}u_0\|_{L^s_\alpha(0,T;L^q)} + \|F\|_{L^{s/2}_{2\alpha}(0,T;L^{q/2})} \leq \varepsilon_*, \tag{1.5}$$

then the Navier-Stokes system (1.1) has a unique strong $L^s_\alpha(L^q)$-solution u with data u_0, f on $[0,T)$, i.e., u is a weak solution in the sense of Leray-Hopf, contained in the Leray-Hopf class

$$\mathcal{LH}_T = L^\infty\big(0,T;L^2_\sigma(\Omega)\big) \cap L^2\big(0,T;H^1_0(\Omega)\big)$$

satisfying the energy inequality (EI)

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2\,\mathrm{d}\tau \leq \frac{1}{2}\|u_0\|_2^2 + \int_0^t \langle f, u\rangle\,\mathrm{d}\tau, \tag{1.6}$$

and

$$u \in L^s_\alpha\big(0,T;L^q(\Omega)\big).$$

Note that $\langle f, u\rangle = -(F, \nabla u)$. Moreover, we recall that the strong $L^s_\alpha(L^q)$-solution u even satisfies the energy equality (EE) on $[0,T]$, *i.e.*,

$$\frac{1}{2}\|u(t)\| + \int_0^t \|\nabla u\|_2^2\,\mathrm{d}\tau = \frac{1}{2}\|u_0\|_2^2 - \int_0^t (F, \nabla u)\,\mathrm{d}\tau \tag{1.7}$$

for all $t \in [0,T]$; this result is based on the integrability $u \in L^4\big(\varepsilon, T; L^4(\Omega)\big)$ [2, Lemma 3.3] and the convergence $u(t) \to u_0$ in $L^2(\Omega)$ as $t \to 0+$, see [2, (3.7)] and subsequent lines. A simple consequence of (1.7) is that u also satisfies the strong energy inequality (SEI), *i.e.*

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^t \|\nabla u\|_2^2\,\mathrm{d}\tau \leq \frac{1}{2}\|u(t_0)\|_2^2 - \int_{t_0}^t (F, \nabla u)\,\mathrm{d}\tau \tag{1.8}$$

for almost all $t_0 \in (0,T)$, including $t_0 = 0$. Of course, (1.7) implies (SEI) even for all $t_0 \in (0,T)$.

Actually, the space $\mathbb{B}_{q,s}^{-1+3/q}(\Omega)$ of initial values with (1.2) is too large compared to the optimal space of initial values $\mathbb{B}_{q,s_q}^{-1+3/q}$ with $2/s_q + 3/q = 1$, cf. [4], [1], [3], [5]. We also note that the authors of [7] proved the existence and uniqueness of global strong solutions with values in the critical space $\mathbb{B}_{3,\infty}^{0}(\Omega)$ for $\mathbb{R}^3$, $\mathbb{R}_{+}^3$ and bounded domains of $\mathbb{R}^3$. The drawback of the space $\mathbb{B}_{q,s}^{-1+3/q}(\Omega)$ with (1.2) is the fact that an analogue of the classical Serrin-Masuda Uniqueness Theorem cannot be proved. For this reason, the authors of [2] introduced the notion of so-called well-chosen weak solutions. Roughly spoken, for given data u_0, f as above a well-chosen weak solution is the limit of a sequence of approximate weak and strong $L_{\alpha}^{s}(L^q)$-solutions (u_n) of an approximate Navier-Stokes system. Then by [2, Theorem 1.4] the unique $L_{\alpha}^{s}(L^q)$-strong solution is unique within the class of all well-chosen weak solutions on some subinterval $[0, T'] \subset [0, T)$. In other words, Serrin's Uniqueness Theorem holds in this setting (on a subinterval) provided that the approximation scheme for the construction of the weak solution is known and admissible in the sense that norms relevant for both a weak L^2- and a strong $L_{\alpha}^{s}(L^q)$-theory can be controlled.

However, the definition in [2] is too much restricted to the construction of weak solutions by Yosida approximation operators and analytic semigroup theory, see Assumptions 5.1 and 5.4 as well as Remarks 5.2 and 5.3 in [2]. The purpose of this paper is to clarify and weaken the assumptions on well-chosen weak solutions and to improve or extend the restricted uniqueness theorem of [2].

For simplicity, in the sequel we always assume that $T < \infty$.

Definition 1.2   A *well-chosen weak solution* v is a weak solution of the Navier-Stokes system (1.1) with $v(0) = u_0 \in L_{\sigma}^{2}(\Omega)$ satisfying the strong energy inequality (SEI), see (1.8), and defined by a concrete, so-called admissible approximation procedure, compatible with the notion of $L_{\alpha}^{s}(L^q)$-solutions in the following sense:

(1) There are initial data $(u_{0n}) \subset L_{\sigma}^{2}(\Omega) \cap \mathbb{B}_{q,s}^{-1+3/q}(\Omega)$ converging to u_0 in $L_{\sigma}^{2}(\Omega) \cap \mathbb{B}_{q,s}^{-1+3/q}(\Omega)$ as $n \to \infty$

(2) The external force $F \in L^2\big(0, T; L^2(\Omega)\big) \cap L_{2\alpha}^{s/2}\big(0, T; L^{q/2}(\Omega)\big)$ is aproximated by a sequence $(F_n) \subset L^2\big(0, T; L^2(\Omega)\big) \cap L_{2\alpha}^{s/2}\big(0, T; L^q(\Omega)\big)$ such that $F_n \to F$ in $L^2\big(0, T; L^2(\Omega)\big) \cap L_{2\alpha}^{s/2}\big(0, T; L^{q/2}(\Omega)\big)$.

(3) The admissible approximation method yields approximate weak so-

lutions u_n on $(0, T)$ uniformly bounded in $\mathcal{LH}_T$ and containing a subsequence (u_{n_k}) such that $u_{n_k} \rightharpoonup v$ in $\mathcal{LH}_T$, *i.e.*, $u_{n_k} \rightharpoonup v$ in $L^2\big(0, T; H^1(\Omega)\big)$ and $u_{n_k} \overset{*}{\rightharpoonup} v$ in $L^\infty\big(0, T; L^2_\sigma(\Omega)\big)$ as $k \to \infty$

(4) (u_n) is uniformly bounded in $L^s_\alpha(0, T'; L^q)$ for some $T' \in (0, T]$.

Remark 1.3   (1) The crucial part of Definition 1.2 is the assumption (4) on (u_n).

(2) The strong convergence $u_{0n} \to u_0$ in $L^2(\Omega)$ in Definition 1.2 (1) can be replaced by the corresponding weak convergence. By analogy, the strong convergence $F_n \to F$ in $L^2\big(0, T; H^1(\Omega)\big)$ may be replaced by a weak one.

(3) However, the strong convergence $u_{0n} \to u_0$ in $\mathbb{B}^{-1+3/q}_{q,s}(\Omega)$ is crucial to find $0 < T' \leq T$ independent of $n \in \mathbb{N}$ such that

$$\int_0^{T'} \big(\tau^\alpha \|e^{-\tau A} u_{0n}\|_q\big)^s \, d\tau \leq \varepsilon_*.$$

Now our main theorem reads as follows.

Theorem 1.4   *Let* $2 < s < \infty$, $3 < q < \infty$, $0 < \alpha < 1/2$ *with* $2/s + 3/q = 1 - 2\alpha$ *and suppose that* $u_0 \in L^2_\sigma(\Omega) \cap \mathbb{B}^{-1+3/q}_{q,s}(\Omega)$ *and an external force* $f = \operatorname{div} F$ *with* $F \in L^2\big(0, T; L^2(\Omega)\big) \cap L^{s/2}_{2\alpha}\big(0, T; L^{q/2}(\Omega)\big)$ *are given. Furthermore, let* $u \in L^s_\alpha\big(0, T; L^q(\Omega)\big)$ *be the unique strong* $L^s_\alpha(L^q)$*-solution of* (1.1) *with data* u_0, F.

(i) *Then* u *is unique within the class of all well-chosen weak solutions of* (1.1) *in the sense of Definition* 1.2.

(ii) *Given an admissible approximation scheme yielding an approximate sequence* (u_n) *as in Definition* 1.2 *assume that each subsequence of* (u_n) *converging weakly in* $\mathcal{LH}_T$ *converges weakly to a weak solution of* (1.1). *Then the whole sequence* (u_n) *converges to* u. *Moreover, for any sequence of initial values* (u_{0n}) *and external forces* (F_n) *approximating* u_0 *and* F *in the sense of Definition* 1.2 (1), (2), *respectively, and generating approximate solutions with a subsequence weakly convergent to a weak solution of* (1.1) *in* $\mathcal{LH}_T$, *the whole sequence of approximate solutions converges weakly in* $\mathcal{LH}_T$ *to* u.

The real work is to show that a concrete approximation procedure for the construction of weak solutions is compatible with the notion of $L^s_\alpha(L^q)$-solutions as required in Definition 1.2. In the following we need for $1 <$

$q < \infty$ the Helmholtz projection $P = P_q : L^p(\Omega) \to L^q_\sigma(\Omega)$ and the Stokes operator $A = A_q : \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \subset L^q_\sigma(\Omega) \to L^q_\sigma(\Omega)$.

Theorem 1.5   *Let $2 < s < \infty$, $3 < q < \infty$, $0 < \alpha < 1/2$ and $2/s + 3/q = 1 - 2\alpha$.*

(i) *The Yosida approximation scheme and, if $3 < q \leq 4$, the Galerkin approximation scheme are admissible, i.e., they define well-chosen weak solutions in $L^s_\alpha(L^q)$. To be more precise, we consider the following approximation methods*:

<u>Case 1</u> (*Yosida approximation scheme*)   *Let $J_n = (I + (1/n)A^{1/2})^{-1}$ denote the Yosida operator, let $u_{0n} = J_n u_0$, and assume that $F_n \to F$ in $L^2(0, T; L^2(\Omega)) \cap L^{s/2}_{2\alpha}(0, T^*; L^{q/2}(\Omega))$ for some $0 < T^* \leq T$. Then the approximate solution u_n is defined as the solution of the approximate Navier-Stokes system*

$$\partial_t u_n - \Delta u_n + (J_n u_n) \cdot \nabla u_n + \nabla p_n = \operatorname{div} F_n, \quad \operatorname{div} u_n = 0,$$
$$u_n|_{\partial\Omega} = 0, \quad u_n(0) = u_{0n}. \tag{1.9}$$

<u>Case 2</u> (*Galerkin approximation scheme*)   *Let Π_n denote the L^2_σ-projection onto the space of the first n eigenfunctions of the Stokes operator A_2, and suppose that $u_{0n} \in \Pi_n L^2_\sigma(\Omega)$ as well as $F_n \in L^2(0, T; L^2(\Omega))$ satisfy the assumptions of Definition 1.2 (1), (2). Then let u_n denote the Galerkin approximation of the Navier-Stokes sytem with data u_{0n}, F_n.*

(ii) *In both cases the assumption in Theorem 1.4 (ii) is satisfied. Hence the whole sequence given by the Yosida approximation scheme or the Galerkin approximation converges to the well-chosen weak solution, irrespective of the sequences (u_{0n}) and (F_n).*

Remark 1.6   (1) The condition $u_0 \in \mathbb{B}^{-1+3/q}_{q,s}(\Omega)$ seems to be quite strong compared to the assumption $u_0 \in L^2_\sigma(\Omega)$ needed in the classical Serrin Uniqueness Theorem. However, in the classical theorem the existence of the strong solution $u \in L^{s_q}(0, T; L^q(\Omega))$ (where $2/s + 3/q = 1$) with $u(0) = u_0$ even implies that $u_0 \in \mathbb{B}^{-1+3/q}_{q,s_q}(\Omega) \subset \mathbb{B}^{-1+3/q}_{q,s}(\Omega)$.

(2) The assumption in Theorem 1.4 (ii) can be weakened to the following one: Assume that each subsequence (u_n) converging weakly in $\mathcal{LH}_T$ contains a subsubsequence converging weakly to a weak solution of (1.1). This results can easily be seen from the proofs.

(3) The assumptions for Case 1 in Theorem 1.5 may be generalized to an arbitrary family of operators $\tilde{J}_n \in \mathcal{L}\big(L^q_\sigma(\Omega); \mathcal{D}(A_q^{1/2})\big)$ commuting with the Stokes operator A_q such that the fundamental properties of the Yosida operator

$$\begin{cases} \|\tilde{J}_n\|_{\mathcal{L}(L^q_\sigma)} + \left\|\dfrac{1}{n} A_q^{1/2} \tilde{J}_n\right\|_{\mathcal{L}(L^q_\sigma)} \leq C_q < \infty, \\ \tilde{J}_n u \to u \text{ in } L^q_\sigma(\Omega) \text{ for each } u \in L^q_\sigma(\Omega) \text{ as } n \to \infty \end{cases} \tag{1.10}$$

are fulfilled. Then let $u_{0n} = \tilde{J}_n u_0$ and $F_n = Y_n F$ where $Y_n = (I + (1/n)(-\Delta_q)^{1/2})^{-1}$ is defined by the Dirichlet-Laplacian $-\Delta_q$ on $L^q(\Omega)$. In particular, since the operators $\tilde{J}_n$ are uniformly bounded in $\mathcal{L}\big(L^q_\sigma(\Omega)\big)$ and converge strongly to I in $L^q_\sigma(\Omega)$, Lebesgue's Theorem on Dominated Convergence implies that

$$\int_0^T \|e^{-\tau A_q} u_{0n}\|_q^s \, \mathrm{d}\tau = \int_0^T \|\tilde{J}_n e^{-\tau A} u_0\|_q^s \, \mathrm{d}\tau \to \int_0^T \|e^{-\tau A} u_0\|_q^s \, \mathrm{d}\tau$$

as $n \to \infty$. By analogy, we argue for the convergence $F_n \to F$ in $L^2(0, T; L^2(\Omega))$ and in $L^{s/2}_{2\alpha}\big(0, T; L^{q/2}(\Omega)\big)$.

(4) To satisfy the assumptions on u_0 for Case 2 in Theorem 1.5 let $u_0 \in L^2_\sigma(\Omega) \cap \mathbb{B}_{q,s}^{-1+3/q}(\Omega)$ be given. Then $u_{0\delta} = e^{-\delta A} u_0$ converges in $\mathbb{B}_{q,s}^{-1+3/q}(\Omega)$ to u_0 as $\delta \to 0+$. Actually,

$$\int_0^T \tau^{\alpha s} \|e^{-\tau A}\big(e^{-\delta A} u_0 - u_0\big)\|_q^s \, \mathrm{d}\tau = \int_0^T \tau^{\alpha s} \|\big(e^{-\delta A} - I\big) e^{-\tau A} u_0\|_q^s \, \mathrm{d}\tau \to 0$$

as $\delta \to 0+$ by Lebesgue's Theorem. Next, due to a semigroup estimate, see (2.10) below, and by Lebesgue's Theorem, for fixed $\delta > 0$

$$\int_0^T \tau^{\alpha s} \|e^{-\tau A} (\Pi_k - I) e^{-\delta A} u_0\|_q^s \, \mathrm{d}\tau = \int_0^T \tau^{\alpha s} \|e^{-\delta A} (\Pi_k - I) e^{-\tau A} u_0\|_q^s \, \mathrm{d}\tau$$

$$\leq c \int_0^T \tau^{\alpha s} \delta^{-(3/2)(1/2 - 1/q)} \|(\Pi_k - I) e^{-\tau A} u_0\|_2^s \, \mathrm{d}\tau \to 0$$

as $k \to \infty$. Summarizing these two results with adequate numbers $\delta = \delta_n > 0$ we find a sequence (u_{0n}) satisfying $u_{0n} \in \Pi_n L^2_\sigma(\Omega)$ and converging to u_0

in $L^2_\sigma(\Omega) \cap \mathbb{B}^{-1+3/q}_{q,s}(\Omega)$ as $n \to \infty$.

(5) The restriction $3 < q \le 4$ in the case of the Galerkin approximation method in Theorem 1.5 will become clear from the crucial estimate (2.11) below which uses the elementary inclusion $q/2 \le 2 < q \le 4$.

2. Proofs

Proof of Theorem 1.4. (i) By Definition 1.2 (3) there exists a sequence of approximate weak solutions (u_n) bounded in $\mathcal{LH}_T$ such that a subsequence (u_{n_k}) converges to a weak solution $v \in \mathcal{LH}_T$ of (1.1) satisfying (SEI).

Since (u_n) is uniformly bounded in $L^s_\alpha(0, T'; L^q)$ with T' in Definition 1.2 (4) we find a subsequence $(u_{n'_k})$ of (u_{n_k}) converging weakly in $L^s_\alpha(0, T'; L^q)$ to an element $v' \in L^s_\alpha(0, T'; L^q)$. Now, since $u_{n_k} \rightharpoonup v$ in $\mathcal{LH}_{T'}$, we may conclude that $v = v'$ on $(0, T')$; in particular, v' is a weak and even a strong $L^s_\alpha(L^q)$-solution of (1.1) on $(0, T')$. Since strong $L^s_\alpha(L^q)$-solutions are unique by [2, Theorem 1.2], $v = v' = u$ on $(0, T')$.

If $T' < T$, then we find due to (SEI) applied to v some $0 < T'' \le T'$ such that the weak solution v satisfies the energy estimate on $[T'', T)$ with initial time T''. Since $u \in L^s(T'', T; L^q(\Omega))$ with $2/s + 2/q < 1$ is a "classical" strong solution, Serrin's Uniqueness Theorem implies that $u = v$ even on $[0, T)$.

(ii) Consider any subsequence (u_{m_k}) of (u_n) converging weakly in $\mathcal{LH}_T$. By the assumption in Theorem 1.4 (ii) this limit is a weak solution v'' on $(0, T)$ and even a well-chosen solution on an interval $(0, T')$. Now the assertion on uniqueness in (i) proves that $v'' = v = u$. Then classical arguments imply that the whole sequence (u_n) converges weakly to v. Moreover, again due to uniqueness, this result will hold for any sequence (u_{0n}) and (F_n) with convergence properties as described in (ii). $\qquad\square$

Proof of Theorem 1.5. (i) <u>Case 1</u>: Given u_0, u_{0n} and F, F_n as in Definition 1.2 classical L^2-methods, see [8, Ch. V.2], prove the existence of a unique approximate solution $u_n \in \mathcal{LH}_T$ of (1.9) and the convergence of a subsequence of (u_n) to a weak solution $u \in \mathcal{LH}_T$ of (1.1). Indeed, u_n satisfies the energy equality (EE), see (1.7), and consequently the energy estimate

$$\|u_n(t)\|^2_2 + \int_0^t \|\nabla u_n\|^2_2 \, d\tau \le \|u_{0n}\|^2_2 + \int_0^t \|F_n\|^2_2 \, d\tau,$$

where the right-hand side is uniformly bounded with respect to $n \in \mathbb{N}$ and $0 < t < T$ due to the weak convergence properties in Definition 1.2. Finally, $(\partial_t u_n)$ is uniformly bounded in $L^{4/3}\big(0, T; H^1_{0,\sigma}(\Omega)'\big)$, see [8, Lemma V. 2.6.1, Theorem V. 1.6.2]. Hence, by the Aubin-Lions-Simon compactness theorem for Bochner spaces, there exists a subsequence (u_{n_k}) of (u_n) and $v \in \mathcal{LH}_T$ such that

$$u_{n_k} \rightharpoonup v \ \text{ in } \ \mathcal{LH}_T, \quad u_{n_k} \to v \ \text{ in } \ L^2\big(0, T; L^2_\sigma(\Omega)\big) \qquad (2.1)$$

as $k \to \infty$. Furthermore,

$$u_{n_k}(t) \to v(t) \ \text{ in } L^2_\sigma(\Omega) \text{ for a.a. } t \in (0, T) \qquad (2.2)$$

as $k \to \infty$; this step needs the extraction of a further subsequence, as the case may be. Now (2.1) allows us to pass to the limit in (1.9) and show that v is a weak solution of (1.1) in the sense of Leray-Hopf. In particular, v satisfies the energy inequality (EI), see (1.6), and due to (2.2) even the strong energy inequality (SEI), see (1.8).

In the second step of the proof we improve the previous results by exploiting the properties of u_0 in $\mathbb{B}^{-1+3/q}_{q,s}(\Omega)$ and of F in $L^{s/2}_{2\alpha}\big(0, T; L^{q/2}(\Omega)\big)$, see Definition 1.2. Since (u_{0n}) converges strongly to u_0 in $\mathbb{B}^{-1+3/q}_{q,s}(\Omega)$ we find some $T' \in (0, T]$ such that

$$\|u_{0n}\|_{\mathbb{B}^{-1+3/q}_{q,s,T'}} \leq \frac{\varepsilon_*}{2}$$

for all $n \in \mathbb{N}$ where $\varepsilon_* > 0$ is the absolute constant from (1.5). Furthermore, since $F_n \to F$ in $L^{s/2}_{2\alpha}\big(0, T; L^{q/2}(\Omega)\big)$ we may also assume that

$$\|F_n\|_{L^{s/2}_{2\alpha}(0,T';L^{q/2})} \leq \frac{\varepsilon_*}{2}$$

for all $n \in \mathbb{N}$. We follow the construction of strong $L^s_\alpha(L^q)$-solutions in [2], decompose the solution u_n of (1.9) into $u_n = \tilde{u}_n + E_n$ where E_n solves the linear nonhomogeneous Stokes problem with data u_{0n}, F_n, *i.e.*,

$$E_n(t) = e^{-tA} u_{0n} + \int_0^t A^{1/2} e^{-(t-\tau)} \big(A^{-1/2} P \operatorname{div}\big) F_n(\tau) \, \mathrm{d}\tau. \qquad (2.3)$$

We note that the formal operator $A^{-1/2}P\,\mathrm{div}$ can be defined rigorously by duality arguments as a bounded operator from $L^q(\Omega)$ to $L_\sigma^q(\Omega)$, $1 < q < \infty$, which goes back to [6].

As in [4], [2] $\tilde u_n = u_n - E_n$ has an integral representation based on the variation of constants formula and can be considered as solution of the fixed point problem $\tilde u_n = \mathcal{F}\tilde u_n$ in $L_\alpha^s\big(0, T'; L^q(\Omega)\big)$ where

$$\mathcal{F}_n\tilde u_n(t) = -\int_0^t A^{1/2}e^{-(t-\tau)A}\big(A^{-1/2}P\,\mathrm{div}\,\big)\big(J_n(\tilde u_n + E_n)\otimes(\tilde u_n + E_n)\big)(\tau)\,\mathrm{d}\tau;$$

note that $\mathcal{F}_n$ differs from $\mathcal{F}$ in [2, (3.2)] only by the additional term J_n. Due to fundamental properties of the Yosida operators J_n, cf. (1.10), the fixed point of $\mathcal{F}_n$ can be constructed by Banach's Fixed Point Theorem in the same way as in [2]. By the assumptions on u_n, F_n and [2, (3.5)] (see also [4, (2.45)] for the case without weights) $\tilde u_n, u_n$ satisfy the estimate

$$\|\tilde u_n\|_{L_\alpha^s(0,T';L^q)}, \|u_n\|_{L_\alpha^s(0,T';L^q)} \le C\varepsilon^* \tag{2.4}$$

with a constant $C > 0$ independent of $n \ge n_0(\varepsilon^*, T')$.

Case 2: It is well known that the Stokes operator A_2 on the bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^3$ admits an orthonormal basis of eigenfunctions $\psi_k \in \mathcal{D}(A_2) = H^2(\Omega) \cap H_{0,\sigma}^1(\Omega)$ with corresponding eigenvalues λ_k monotonically increasing to ∞ as $k \to \infty$. For $n \in \mathbb{N}$ let

$$\Pi_n : L_\sigma^2(\Omega) \to V_n := \mathrm{span}\{\psi_1, \ldots, \psi_n\} \subset L_\sigma^2(\Omega)$$

denote the orthogonal projection. Obviously, $\|\Pi_n\|_{\mathcal{L}(L_\sigma^2(\Omega))} = 1$ for all $n \in \mathbb{N}$. In the Galerkin method we are looking for a solution $u_n : [0, T) \to V_n$ of the ordinary differential $n \times n$-system

$$(\partial_t u_n, \psi_k) + (\nabla u_n, \nabla \psi_k) - (u_n \otimes u_n, \nabla \psi_k) = -(F_n, \nabla \psi_k)$$
$$u_n(0) = u_{0n} \in V_n \tag{2.5}$$

on $(0, T)$ for each $k = 1, \ldots, n$. By the L^2-assumptions on u_{0n} and F_n we know that there exists a sequence of unique solutions (u_n) to (2.5) bounded in $\mathcal{LH}_T$. Moreover, $(\partial_t u_n)$ is uniformly bounded in $L^{4/3}\big(0, T; H_{0,\sigma}^1(\Omega)'\big)$. As in the first part of the proof we find a subseqence (u_{n_k}) of (u_n) and a vector field v satisfying (2.1) and (2.2). In particular, $v \in \mathcal{LH}_T$ is a weak solution

to (1.1) satisfying (SEI).

The crucial question is whether u_n is also a strong $L_\alpha^s(0, T'; L^q)$-solution, uniformly bounded in n. To address this problem we consider arbitrary linear combinations of $(2.5)_1$ to see that for all $w \in H_{0,\sigma}^1(\Omega)$

$$(\partial_t u_n, \Pi_n w) + (\nabla u_n, \nabla \Pi_n w) - (u_n \otimes u_n, \nabla \Pi_n w) = -(F_n, \nabla \Pi_n w)$$
$$u_n(0) = u_{0n} \in V_n. \tag{2.6}$$

Since $\Pi_n = P\Pi_n$, $P^* = P$, A commutes with Π_n, and $(\nabla u_n, \nabla \Pi_n w) = (Au_n, w)$, we may omit the test function $w \in H_{0,\sigma}^1(\Omega)$ and rewrite (2.6) in the form

$$\partial_t u_n + Au_n + \Pi_n P \text{div} \, (u_n \otimes u_n) = \Pi_n P \text{div} \, F_n, \quad u_n(0) = u_{0n} \in V_n. \tag{2.7}$$

Thus $u_n(t)$ can be considered as a solution in $W^{1,4/3}(0, T)$ (with respect to time) of an abstract Cauchy problem and as a mild solution with integral representation

$$u_n(t) = e^{-tA}u_{0n} - \int_0^t A^{1/2}e^{-(t-\tau)A}\big(A^{-1/2}\Pi_n P \text{div}\,\big)(u_n \otimes u_n - F_n)(\tau)\, d\tau. \tag{2.8}$$

Although $\|\Pi_n\|_{\mathcal{L}(L_\sigma^2(\Omega))} = 1$ and $A^{-1/2}P\text{div} \in \mathcal{L}(L^q(\Omega))$ for each $1 < q < \infty$, similar estimates will not hold for Π_n on $L_\sigma^q(\Omega)$ and for the operator $A^{-1/2}\Pi_n P\text{div}$ on $L^q(\Omega)$ uniformly in $n \in \mathbb{N}$. Actually, there seems to exist no estimate of the type $\|\Pi_n\|_{\mathcal{L}(L_\sigma^q(\Omega))} \leq c(q)$ uniformly in $n \in N$ when $q \neq 2$; the reason is the non-uniform distribution of eigenvalues λ_k as $k \to \infty$ compared to a Fourier series setting. Therefore, the question occurs how to estimate L^q-norms uniformly in n when Π_n is involved.

Let us recall the embedding and semigroup estimates

$$\|v\|_q \leq c\|A_2^\gamma v\|_2 \, , \quad v \in D(A_2^\gamma), \; 2\gamma + \frac{3}{q} = \frac{3}{2}, \; 0 \leq \gamma \leq 1, \tag{2.9}$$

$$\|e^{-tA_q}v\|_q \leq ct^{-(3/2)(1/\rho - 1/q)}\|v\|_\rho \, , \quad v \in L_\sigma^\rho(\Omega), \; q \geq \rho > 1, \; t > 0, \tag{2.10}$$

with constants $c = c(\Omega, q, \rho) > 0$, see [2], [4]. Applying (2.9) with $3 < q \leq 4$, exploiting the uniform boundedness and commutator properties of Π_n on $L_\sigma^2(\Omega)$ and finally (2.10) with $2, q/2$ instead of q, ρ we get the estimate

$$\|A^{1/2}e^{-(t-\tau)A}(A^{-1/2}\Pi_n P\mathrm{div}\,)(u_n \otimes u_n - F_n)\|_q$$

$$\leq c\|A^{\gamma+1/2}e^{-(t-\tau)A}(A^{-1/2}\Pi_n P\mathrm{div}\,)(u_n \otimes u_n - F_n)\|_2$$

$$\leq c\|A^{\gamma+1/2}e^{-(t-\tau)A}(A^{-1/2}P\mathrm{div}\,)(u_n \otimes u_n - F_n)\|_2$$

$$\leq c(t-\tau)^{-1+\alpha+1/s}\|u_n \otimes u_n - F_n\|_{q/2}. \tag{2.11}$$

Now the weighted Hardy-Littlewood-Sobolev inequality, *cf.* [2, Lemma 2.1], implies that with a constant $c > 0$ independent of $n \in \mathbb{N}$ and T

$$\|u_n - e^{-tA}u_{0n}\|_{L_\alpha^s(0,T;L^q)} \leq c\big(\|u_n\|_{L_\alpha^s(0,T;L^q)}^2 + \|F_n\|_{L_{2\alpha}^{s/2}(0,T;L^{q/2})}\big).$$

Then by standard arguments we find $T' \in (0,T)$ independent of $n \in \mathbb{N}$ such that $(u_n) \subset L_\alpha^s(0,T';L^q)$ is uniformly bounded.

Now we complete the proof as in the previous case.

(ii) It is well-known that each sequence of approximate solutions, (u_n), given by the Yosida approximation scheme or the Galerkin method, respectively, allows for a uniform estimate of the time derivative $\partial_t u_n$. Hence by compactness arguments $(u_n) \subset L^2(0,T;L^2(\Omega))$ is precompact. Consequently, there exists a subsequence converging strongly in $L^2(0,T;\Omega)$ to a limit function which is a weak solution to (1.1). $\square$

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