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Regular homeomorphisms of $\mathbb{R}^3$ and of $\mathbb{S}^3$

Khadija Ben Rejeb

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Abstract. This paper is the paper announced in [Be2, References [2]]. We show that every compact abelian group of homeomorphisms of $\mathbb{R}^3$ is either zero-dimensional or equivalent to a subgroup of the orthogonal group O(3). We prove a similar result if we replace $\mathbb{R}^3$ by $\mathbb{S}^3$, and we study regular homeomorphisms that are conjugate to their inverses.

Key words: Recurrent homeomorphisms of $\mathbb{R}^3$, compact abelian groups of homeomorphisms of $\mathbb{R}^3$, topologically equivalent, reversible.

1. Introduction

A homeomorphism h of a metric space (E, d) onto itself is called

(a) *periodic* if h^n is the identity for some integer $n \in \mathbb{N}$;
(b) *regularly almost periodic* if for every $\epsilon > 0$, there exists an integer $n > 0$ such that $d(h^{mn}(x), x) < \epsilon$ for all $x \in E$ and all $m \in \mathbb{N}$;
(c) *almost periodic* if for every $\epsilon > 0$, there exists an integer $N > 0$ such that every block of N consecutive integers contains an integer $m \in \mathbb{Z}$ such that $d(h^m(x), x) < \epsilon$, for all $x \in E$;
(d) *recurrent* if for every $\epsilon > 0$, there exists an integer $n > 0$ such that $d(h^n(x), x) < \epsilon$ for all $x \in E$;
(e) *regular* provided that the family $\{h^m,\ m \in \mathbb{Z}\}$ is equicontinuous; this means that for every $x \in E$ and for every $\epsilon > 0$, there exists $\eta > 0$, such that: $\forall\ m \in \mathbb{Z}$, $\forall\ y \in E$, $d(x, y) \leq \eta \Rightarrow d(h^m(x), h^m(y)) < \epsilon$.

The homeomorphism h is said to be *positively regular* if the family $\{h^m,\ m \in \mathbb{N}\}$ is equicontinuous. Clearly the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ and (e) hold. Moreover, if E is compact we also have $(c) \Leftrightarrow (e)$ ([GH]).

The homeomorphism h is said to be *pointwise periodic* if for every $x \in E$, there exists an integer $n \neq 0$ depending on x such that $h^n(x) = x$. It is said to be *pointwise regularly almost periodic* if for each $x \in E$, h is regularly almost periodic at x, i.e., for each $\epsilon > 0$ there exists a positive integer n

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depending on x such that $d(h^{mn}(x), x) < \epsilon$ for all $m \in \mathbb{N}$.

We denote by $Homeo(E)$ the group of all homeomorphisms of the metric space E onto itself equipped with the compact-open topology, and by id the identity map. Let $x \in E$. For $h \in Homeo(E)$, the orbit of x under h is defined by

$$O_x = \{h^n(x) \mid n \in \mathbb{Z}\};$$

and for a subgroup G of $Homeo(E)$, the orbit of x under G is $G(x) = \{g(x) \mid g \in G\}$.

Two elements f and h (resp. two subgroups G_1 and G_2) of $Homeo(E)$ are said to be topologically equivalent (equivalent) or conjugate if there exists $\varphi \in Homeo(E)$ satisfying $f = \varphi h \varphi^{-1}$ (resp. $G_1 = \varphi G_2 \varphi^{-1}$).

The following results are well known:

- For $E = \mathbb{S}^1$, $\mathbb{S}^2$, or $\mathbb{R}^2$, every almost periodic homeomorphism of E is topologically equivalent to an isometry ([F], [Br1], [Rit]).
- Every compact subgroup of $Homeo(\mathbb{S}^2)$ is equivalent to a closed subgroup of the orthogonal group $O(3)$ ([Ke1], [Ke2], [Ko]).
- Recurrent homeomorphisms of $\mathbb{R}^2$ are periodic ([OT]).
- If E is compact, then every almost periodic homeomorphism h of E is the uniform limit of a sequence $(h_n)_n$ of regularly almost periodic homeomorphisms such that for each integer n, h_n is the limit of some sequence $(h^{p_{n,k}})_k$ of iterates of h ([GH]).
- If E is a 2-dimensional manifold, then every regularly almost periodic homeomorphism of E is periodic ([MZ2]).

Note that an irrational rotation of the circle $\mathbb{S}^1$ (or the sphere $\mathbb{S}^2$) is an almost periodic homeomorphism which is neither periodic nor regularly almost periodic; indeed, from [GH], for pointwise regularly almost periodic homeomorphisms, the orbit-closures are zero-dimensional.

In dimension 3, one can ask similar questions:

(1) Is a recurrent homeomorphism of $\mathbb{R}^3$ necessarily periodic?
(2) How are compact subgroups of $Homeo(\mathbb{R}^3)$ or of $Homeo(\mathbb{S}^3)$ characterized?

In this paper we give partial answers to questions (1) and (2). More precisely, we show the following results:

1. Let h be a recurrent homeomorphism of $\mathbb{R}^3$. If h commutes with the group $\Re$ of all rotations around the z-axis, then h is periodic (Theorem 2.2).
2. Let G be a compact abelian subgroup of $Homeo(\mathbb{S}^3)$ with a fixed point. Then G is either finite or equivalent to a subgroup of $O(4)$ (Theorem 3.9).
   In particular, regular homeomorphisms of $\mathbb{R}^3$ or of $\mathbb{S}^3$ with nonempty fixed point set are completely characterized. Moreover, we show the following statement:
3. Every regular (resp. positively regular) homeomorphism of $\mathbb{R}^3$ with a bounded orbit (resp. of $\mathbb{S}^3$ with a fixed point) is either periodic or the product of two involutions (Theorem 4.7 and Theorem 4.11).

We use a unified notation $\Re$ for two groups of rotations: on $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, $\Re$ means the group of rotations $R : (z, u) \longmapsto (ze^{i\theta}, u)$ around the z-axis, and on $\mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$, $\Re$ means the group of rotations $R : (z_1, z_2) \longmapsto (z_1 e^{i\theta}, z_2)$. The notation R_θ or R means an element of $\Re$, and the notation ρ means, on $\mathbb{R}^3$ the reflection $(x, y, u) \longmapsto (x, y, -u)$, and on $\mathbb{R}^4$ the reflection $(x, y, u, v) \longmapsto (x, y, -u, v)$. We denote by $C(\Re)$ the subgroup of $Homeo(\mathbb{R}^3)$ consisting of elements h that commute with elements of $\Re$;

$$C(\Re) = \{h \in Homeo(\mathbb{R}^3) \mid hR = Rh, \ \forall\ R \in \Re\}.$$

The set of all fixed points under $h \in Homeo(E)$ is

$$Fix(h) = \{x \in E \mid h(x) = x\},$$

and for $G \subset Homeo(E)$, $Fix(G) = \{x \in E \mid g(x) = x, \ \forall\ g \in G\}$.

Let $Homeo^+(\mathbb{R}^3) = \{$orientation preserving elements of $Homeo(\mathbb{R}^3)\}$, and $Homeo^-(\mathbb{R}^3) = \{$orientation reversing elements of $Homeo(\mathbb{R}^3)\}$. If G is a subgroup of $Homeo(\mathbb{R}^3)$, we shall make use of the following notations:

$G_+ = G \cap Homeo^+(\mathbb{R}^3)$, $G_- = G \cap Homeo^-(\mathbb{R}^3)$, and for $h \in G$, $\langle h \rangle = \{h^n \mid n \in \mathbb{Z}\}$ is the subgroup of G generated by h, its closure $\overline{\langle h \rangle}$ is called the monothetic group generated by h.

2. Recurrent elements of $C(\Re)$ are periodic

In [OT], Oversteegen and Tymchatyn showed that recurrent homeomorphisms of the plane are periodic. The main result of this section is to extend this result to recurrent elements of $C(\Re)$.

Let

$$\mathbb{R}_+ = \{r \in \mathbb{R} \mid r > 0\} \ ; \ (oz) = \{(0,0,z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\},$$

and let (x, y, z), (r, z, θ) be respectively the cartesian coordinates and the cylindrical coordinates. For every $(x, y, z) \in \mathbb{R}^3 \setminus (oz)$, there exists a unique $(r, z, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times [0, 2\pi[$ such that $(x, y, z) = (r\cos\theta, r\sin\theta, z)$.

For $h \in C(\Re)$, the form of the restriction of h to $\mathbb{R}^3 \setminus (oz)$ is described in the following lemma.

Lemma 2.1 *Let $h \in Homeo(\mathbb{R}^3)$. The following statements are equivalent*

(a) $h \in C(\Re)$.
(b) *For every $(r, z, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times [0, 2\pi[$, $h(r, z, \theta) = (f(r, z), \theta + g(r, z))$; where $f \in Homeo(\mathbb{R}_+ \times \mathbb{R})$ and g is a map from $\mathbb{R}_+ \times \mathbb{R}$ to $[0, 2\pi[$.*

Proof. We have $Fix(\Re) = (oz)$. If $h \in C(\Re)$, then $h((oz)) = (oz)$ and the restriction $h_{|\mathbb{R}^3\setminus(oz)}$ is a homeomorphism of $\mathbb{R}^3 \setminus (oz)$. There exist three maps h_1, h_2, and h_3 from $\mathbb{R}_+ \times \mathbb{R} \times [0, 2\pi[$ to $\mathbb{R}_+$, $\mathbb{R}$, and $[0, 2\pi[$ respectively satisfying: for every $(r, z, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times [0, 2\pi[$,

$$h(r, z, \theta) = (h_1(r, z, \theta), h_2(r, z, \theta), h_3(r, z, \theta)).$$

Since $h \in C(\Re)$, for every $\theta \in [0, 2\pi[$, we have

$$hR_\theta(r, z, 0) = R_\theta h(r, z, 0), \ \forall \ (r, z) \in \mathbb{R}_+ \times \mathbb{R}.$$

Then

$$h(r, z, \theta) = (f(r, z), \theta + g(r, z)), \ \forall \ (r, z, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times [0, 2\pi[;$$

where $f(r, z) = (h_1(r, z, 0), h_2(r, z, 0))$ and $g(r, z) = h_3(r, z, 0)$.

We will show that f is a homeomorphism. For proving the continuity of f, let $(r, z) \in \mathbb{R}_+ \times \mathbb{R}$, and let $((r_n, z_n))$ be a sequence in $\mathbb{R}_+ \times \mathbb{R}$ converging to (r, z). Since h is continuous, we obtain that $h(r_n, z_n, 0) \longrightarrow h(r, z, 0)$, in particular, $h_1(r_n, z_n, 0) \longrightarrow h_1(r, z, 0)$ and $h_2(r_n, z_n, 0) \longrightarrow h_2(r, z, 0)$. Which means that $f(r_n, z_n) \longrightarrow f(r, z)$. Thus, f is continuous. Now, let (r, z), $(r', z') \in \mathbb{R}_+ \times \mathbb{R}$ such that $f(r, z) = f(r', z')$. Since $h_{|\mathbb{R}^3\setminus(oz)}$ is surjective there exist θ, $\theta' \in [0, 2\pi[$ such that $h(r, z, \theta) = (f(r, z), 0)$ and

$h(r', z', \theta') = (f(r', z'), 0)$, and since h is injective we obtain that $(r, z, \theta) = (r', z', \theta')$. Which implies that $(r, z) = (r', z')$. So, f is injective. Moreover, f is surjective since h is. Let f^{-1} denote the inverse map of f, we have $h^{-1}(r, z, \theta) = (f^{-1}(r, z), \theta - g(f^{-1}(r, z)))$ and the continuity of h^{-1} implies the continuity of f^{-1}. Therefore, $f \in Homeo(\mathbb{R}_+ \times \mathbb{R})$. We conclude that Item (a) implies Item (b). The converse implication is clear. □

Theorem 2.2 *Let h be a recurrent homeomorphism of $\mathbb{R}^3$ commuting with $\Re$. Then h is periodic.*

Proof. First, we show that the homeomorphism f defined in Lemma 2.1 is recurrent. Let $\epsilon > 0$. Since h is recurrent, there exists an integer $n > 0$ satisfying

$$\|h^n(r, z, \theta) - (r, z, \theta)\| < \epsilon, \ \forall \ (r, z, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times [0, 2\pi[. \quad (*)$$

From Lemma 2.1 we have $h^n(r, z, \theta) = (f^n(r, z), \theta + g_n(r, z))$; where g_n is a map from $\mathbb{R}_+ \times \mathbb{R}$ to $[0, 2\pi[$. Then the inequality $(*)$ implies that

$$\|f^n(r, z) - (r, z)\| < \epsilon, \ \forall \ (r, z) \in \mathbb{R}_+ \times \mathbb{R}.$$

Thus, f is a recurrent homeomorphism of $\mathbb{R}_+ \times \mathbb{R}$.

The restriction of h to the z-axis (oz) is a recurrent homeomorphism of (oz). Then $h^2_{|(oz)} = id$. Assume that $h_{|(oz)} = id$. By the continuity of h, the homeomorphism f can be extended as follows:

$$\tilde{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x, y) \longmapsto \begin{cases} f(x, y) & \text{if } x > 0, \\ (x, y) & \text{if } x \leq 0; \end{cases}$$

It is easy to show that $\tilde{f}$ is a recurrent homeomorphism of the plane $\mathbb{R}^2$, and by [OT] $\tilde{f}$ is periodic. Since $\tilde{f}$ coincides with the identity on a nonempty open subset of $\mathbb{R}^2$, by [N], $\tilde{f} = id$. Then $f = id$, it follows that for every $(r, z, \theta) \in \mathbb{R}^2 \setminus \{(0, 0)\} \times \{z\}$,

$$h(r, z, \theta) = (r, z, \theta + g(r, z)) \in \mathbb{R}^2 \times \{z\}.$$

Moreover, for every $z \in \mathbb{R}$, $h(0, 0, z) = (0, 0, z)$ since $h_{|(oz)} = id$. Then

$h(\mathbb{R}^2 \times \{z\}) = \mathbb{R}^2 \times \{z\}$, for every $z \in \mathbb{R}$. In other words, every horizontal plane $P_z = \mathbb{R}^2 \times \{z\}$ is invariant by h. Now, for every z, $h_{|P_z}$ is a recurrent homeomorphism of the plane P_z, and by [OT], it is periodic. Therefore, h is pointwise periodic, and by [MZ2] h is periodic. If $h_{|(oz)} \neq id$, we consider the map h^2 instead of h. So, we obtain that h^2 is periodic, and then h is periodic. □

3. Characterization of compact abelian sub-groups of $Homeo(\mathbb{R}^3)$ and of $Homeo(\mathbb{S}^3)$

3.1. Compact abelian sub-groups of $Homeo(\mathbb{R}^3)$

In this subsection we show that every compact abelian subgroup of $Homeo(\mathbb{R}^3)$ is either zero-dimensional or equivalent to a subgroup of $O(3)$ and we characterize regular homeomorphisms of $\mathbb{R}^3$ with bounded orbits. We begin by characterizing compact abelian subgroups of $Homeo^+(\mathbb{R}^3)$ in the following.

Proposition 3.1 *Let G be an abelian compact subgroup of $Homeo^+(\mathbb{R}^3)$. Then G is either zero-dimensional or equivalent to the group $\Re$.*

Proof. Let G_0 be the connected component of G containing the identity map id. If $G_0 = \{id\}$, then G is totally disconnected and from Theorem 12.3.1 of [Pal], G is zero-dimensional. If not, then by [MZ2], G_0 is equivalent to the group $\Re$. We can assume that $G_0 = \Re$.

We will show that $G = \Re$. Let $h \in G$. Since G is abelian, then $h \in C(\Re)$, and by Lemma 2.1, we have $h(r, z, \theta) = (f(r, z), \theta + g(r, z))$, for every $(r, z, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times [0, 2\pi[$, where $f \in Homeo(\mathbb{R}_+ \times \mathbb{R})$. Since G is compact, then h is a regular homeomorphism of $\mathbb{R}^3$ with bounded orbits. It follows that f is also regular and with bounded orbits (see Proof of Theorem 2.2). We know that $h((oz)) = (oz)$, then either $h_{|(oz)} = id$ or $h_{|(oz)}$ is equivalent to the reflection $z \longmapsto -z$.

Case 1. $h_{|(oz)} = id$. Let $\tilde{f}$ be the extension of f on the plane $\mathbb{R}^2$ as in the Proof of Theorem 2.2. It is easy to see that $\tilde{f}$ is a regular homeomorphism of $\mathbb{R}^2$ with bounded orbits. Then by [Br2], $\tilde{f}$ is equivalent to either a rotation or a reflection, and since $P^- = \{(x, y) \in \mathbb{R}^2 \mid x \leq 0\} \subset \text{Fix}(\tilde{f})$, then $\tilde{f} = id$ and $f = id$. Thus, in the same way as in the Proof of Theorem 2.2, every horizontal plane $P_z = \mathbb{R}^2 \times \{z\}$ is invariant by h and $h_{|P_z}(r, z, \theta) = (r, z, \theta + g(r, z))$. Which means that for every horizontal circle $C_r = \{(r, z, \theta) \in P_z \mid$

$\theta \in [0, 2\pi[\}$ in the plane P_z, the restriction $h_{|C_r}$ is a rotation around the z-axis through the angle $g(r,z)$. By the fact that $h_{|P_z}$ is regular all rotations $h_{|C_r}$, $r \in \mathbb{R}_+$ must have the same angle $g(r,z)$. Then for every $z \in \mathbb{R}$, $h_{|P_z}$ is a rotation through an angle θ_z, and since h is regular, all rotations $h_{|P_z}$ have the same angle θ_z denoted simply by θ. Thus $h = R_\theta \in \Re$. We conclude that $G \subset \Re$ and then $G = \Re$.

Case 2. $h_{|(oz)}$ is equivalent to the reflection $z \mapsto -z$. Then $h^2_{|(oz)} = id$, and by case 1, $h^2 = R^2_\theta$, for some $R_\theta \in \Re$. Which means that hR_θ^{-1} is an orientation-preserving involution in G. By Smith [Sm], either $\mathrm{Fix}(hR_\theta^{-1}) = \mathbb{R}^3$ and $h = R_\theta$ or $\mathrm{Fix}(hR_\theta^{-1})$ is a line $L \subset \mathbb{R}^3$ that will be invariant by $\Re$, i.e., necessarily equal to (oz). Then, since $\mathrm{Fix}(R_\theta) = (oz)$, in both cases, we obtain that $h_{|(oz)} = id$, which contradicts the hypothesis of Case 2. Thus, we cannot have case 2.

We conclude that G is equivalent to $\Re$. □

Theorem 3.2 *Let G be an abelian compact subgroup of $Homeo(\mathbb{R}^3)$. Then G is either zero-dimensional or equivalent to $\Re$ or to $\Re \cup \rho\Re$.*

Proof. If $G_0 = \{id\}$, then G is zero-dimensional. If not, G_0 is equivalent to $\Re$ by [MZ2] and $G_0 \subset G_+$, then by Proposition 3.1, G_+ is equivalent to $\Re$. We have $G = G_+ \cup G_-$.

If $G_- = \emptyset$, then G is equivalent to $\Re$.

If $G_- \neq \emptyset$, we can assume that $G = \Re \cup G_-$. It is easy to see that for every $h \in G_-$, $G_- = h\Re$. Let $h \in G_-$ (orientation-reversing) such that h is nonperiodic. Then $h^2 \in \Re$ (orientation-preserving). Therefore, we may write $h^2 = R^2_\theta$ for some irrational θ. Because G is abelian, hR_θ^{-1} is an involution. Since h reverses the orientation and R_θ^{-1} preserves the orientation, hR_θ^{-1} is an orientation-reversing involution.

We know that $h(r,z,\theta) = (f(r,z), \theta + g(r,z))$ (Lemma 2.1), and that the restriction $h_{|(oz)}$ is either the identity map or equivalent to the reflection $\rho_{|(oz)}$. If $h_{|(oz)} = id$, then by Smith [Sm], $\mathrm{Fix}(hR_\theta^{-1})$ will be a topological plane containing (oz) and invariant by $\Re$, but such a plane cannot exist. So $h_{|(oz)}$ is equivalent to $\rho_{|(oz)}$; which means that there exists $u \in Homeo((oz))$ satisfying $uh_{|(oz)}u^{-1} = \rho_{|(oz)}$. The homeomorphism u can be extended on $\mathbb{R}^3$ by $v(x,y,z) = (x,y,u(z))$, for every $(x,y,z) \in \mathbb{R}^3$. We have $(vhv^{-1})_{|(oz)} = \rho_{|(oz)}$. By remarking that $v \in C(\Re)$ (i.e., it satisfies $v\Re v^{-1} = \Re$) and that the group vGv^{-1} has the same properties as G, we can assume that $h_{|(oz)} = \rho_{|(oz)}$. Then, from the continuity of h, f can be extended as follows

$$\tilde{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto \begin{cases} f(x, y) & \text{if } x > 0, \\ (x, -y) & \text{if } x \leq 0; \end{cases}$$

$\tilde{f}$ is a regular homeomorphism of $\mathbb{R}^2$ with bounded orbits and coinciding with the reflection $\sigma : (x, y) \longmapsto (x, -y)$ on P^-. Then $\tilde{f}$ is equivalent to σ. In the same way as previously there exists $\gamma \in C(\Re)$ satisfying $\gamma h \gamma^{-1}(r, z, \theta) = (r, -z, \theta + g(r, z))$. We can assume that

$$h(r, z, \theta) = (r, -z, \theta + g(r, z)).$$

Therefore, for every horizontal plane $P_z = \mathbb{R}^2 \times \{z\}$, we have $h(P_z) = P_{-z}$. In particular, $h(P_0) = P_0$ and the fixed points of h lie in P_0. The plane P_0 divides the space $\mathbb{R}^3$ into two connected components $\overline{E_1} = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$ and $\overline{E_2} = \rho(\overline{E_1})$ satisfying $h(\overline{E_1}) = \overline{E_2}$. We know that $h_{|P_0}$ is equivalent to either a rotation or a reflection, and the fact that $\mathrm{Fix}(h_{|P_0})$ is invariant by $\Re$ implies that $h_{|P_0}$ is equivalent to a rotation. On the other hand, we have $h^2_{|\overline{E_1}} = R^2_\theta$, then either $h_{|P_0} = R_\theta$ or $h_{|P_0} = R_{\theta+\pi}$, in both cases put simply $h_{|P_0} = R_\theta$. Let ϕ be the homeomorphism of $\mathbb{R}^3$ defined as follows

$$\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$x \longmapsto \begin{cases} x & \text{if } x \in \overline{E_1}, \\ R_\theta \rho h^{-1}(x) & \text{if } x \in \overline{E_2} \end{cases}$$

We can easily show that the homeomorphism ϕ satisfies $h = \phi^{-1} R_\theta \rho \phi$. Which means that $G = \Re \cup \phi^{-1} R_\theta \rho \phi \Re$, equivalently, $\phi G \phi^{-1} = \phi \Re \phi^{-1} \cup R_\theta \rho \phi \Re \phi^{-1}$. In particular, we have $R_\theta \rho \in \phi G \phi^{-1}$, and since R_θ is irrational we obtain that $\overline{\langle R^2_\theta \rangle} = \Re \subset \phi G \phi^{-1}$. Then, $\Re \subset \phi \Re \phi^{-1}$ and $\Re = \phi \Re \phi^{-1}$. It follows that $\phi G \phi^{-1} = \Re \cup \rho \Re$. This completes our proof. □

Lemma 3.3 *Let h be a regular homeomorphism of the euclidean space $\mathbb{R}^n$. If h has a bounded orbit, then every orbit is bounded.*

Proof. By equicontinuity of the family $\{h^n,\ n \in \mathbb{Z}\}$, the set

$$\{x \in \mathbb{R}^n \mid O_x \text{ is bounded}\}$$

is both open and closed in $\mathbb{R}^n$ (see [EHS] for more details). □

Corollary 3.4 *Let h be a regular homeomorphism of $\mathbb{R}^3$ with a bounded orbit, then the following statements hold.*

(1) *h is either periodic or equivalent to R_θ or to ρR_θ.*
(2) *If h commutes with an irrational rotation R_{θ_0}, then either h is a rotation R_θ or h is equivalent to ρR_θ and h is the product of a rotation with an involution.*

Proof.

(1) Let $G = \overline{\langle h \rangle}$. By Lemma 3.3 the homeomorphism h is regular with bounded orbits. Then by Ascoli's Theorem G is a compact abelian subgroup of $Homeo(\mathbb{R}^3)$. In [Par], author showed that every locally compact group acting effectively on a connected 3-manifold is a Lie group. This result and Theorem 3.2 permit us to deduce that every abelian compact subgroup of $Homeo(\mathbb{R}^3)$ is either finite or equivalent to $\Re$ or to $\Re \cup \rho\Re$. Then G is either finite or equivalent to $\Re$ or to $\Re \cup \rho\Re$. Then h is either periodic or equivalent to an irrational rotation R_θ if it is orientation-preserving or to ρR_θ if it is orientation-reversing.
(2) Let $G = \overline{\langle h, R_{\theta_0} \rangle}$ be the closure of the group generated by h and R_{θ_0}. Since h commutes with R_{θ_0} and h is regular, the group $\langle h, R_{\theta_0} \rangle$ is equicontinuous. Moreover, since every orbit $O_h(x)$ is bounded, then the diameter $\delta(O_h(x))$ of $O_h(x)$ is $< +\infty$. So, for all n, $p \in \mathbb{Z}$, we have $d(h^n R^p_{\theta_0}(x), x) = d(h^n(x), R^{-p}_{\theta_0}(x)) \leq d(h^n(x), x) + d(x, R^{-p}_{\theta_0}(x)) \leq \delta(O_h(x)) + \delta(\Re(x)) < +\infty$. Therefore, every orbit $\langle h, R_{\theta_0} \rangle(x)$ is bounded and relatively compact. Then, by Ascoli's Theorem, G is a compact abelian subgroup of $Homeo(\mathbb{R}^3)$. Then, G is either finite or equivalent to $\Re$ or to $\Re \cup \rho\Re$. Since $R_{\theta_0} \in G$ is irrational, then $\Re \subset G$ and either $G = \Re$ or $G = \Re \cup \alpha\rho\alpha^{-1}\Re$ for some homeomorphism α of $\mathbb{R}^3$. Therefore, if h is orientation-preserving then $h = R_\theta \in \Re$ and if h is orientation-reversing then h is equivalent to ρR_θ and $h = \alpha\rho\alpha^{-1} R_\theta = \tau R_\theta$; where $R_\theta \in \Re$ and $\tau = \alpha\rho\alpha^{-1}$. This completes our proof. □

Remark 3.5 A periodic homeomorphism of $\mathbb{R}^3$ need not be equivalent to an orthogonal map ([Bi] and [MZ1]); however, Corollary 3.4 says that for every regular homeomorphism h of $\mathbb{R}^3$ with a bounded orbit, h is conjugate to an orthogonal map if and only if h commutes with a topological irrational rotation $\alpha R_\theta \alpha^{-1}$.

Proof. Assume that h is conjugate to an orthogonal map, then $h = \alpha R_{\theta_0} \alpha^{-1}$ or $h = \alpha R_{\theta_0} \rho \alpha^{-1}$. Then for every irrational rotation R_θ, we have $h\alpha R_\theta \alpha^{-1} = \alpha R_\theta \alpha^{-1} h$. The converse is true by Corollary 3.4.(2). □

Corollary 3.6 *Let h be a regular homeomorphism of $\mathbb{R}^3$. Then the following statements are equivalent.*

(a) *h has a bounded orbit.*
(b) *Every orbit is bounded.*
(c) *h has a fixed point.*
(d) *h has an almost periodic point.*

Proof. (a) $\Longrightarrow$ (b). Follows from Lemma 3.3.

(b) $\Longrightarrow$ (c). By Corollary 3.4.(1), h is either periodic or equivalent to R_θ or to ρR_θ. By [Sm], every periodic homeomorphism of $\mathbb{R}^3$ must have a fixed point, moreover $Fix(R_\theta) \neq \emptyset$ and $Fix(\rho R_\theta) \neq \emptyset$. So, h has a fixed point.

(c) $\Longrightarrow$ (d). Let $x \in Fix(h)$, then $h(x) = x$ and so x is almost periodic.

(d) $\Longrightarrow$ (a). If x is an almost periodic point by h, then from [GH, Theorem 4.09] the orbit O_x is relatively compact, so O_x is bounded. □

Remark 3.7 A translation $T : x \longmapsto x+a$ $(a \neq 0)$ of $\mathbb{R}^3$ is a regular homeomorphism without bounded orbits. Also, a nonregular homeomorphism of $\mathbb{R}^3$ with bounded orbits need not have a fixed point ([Br2]).

3.2. Compact abelian sub-groups of $\boldsymbol{Homeo}(\mathbb{S}^3)$

In this subsection, we characterize compact abelian subgroups of $Homeo(\mathbb{S}^3)$ with a fixed point.

Lemma 3.8 *Let E and F be two locally compact metric spaces and let H be a subset of $Homeo(E)$. If H is compact, then, for every homeomorphism $\varphi : E \longrightarrow F$, the subset $\varphi H \varphi^{-1}$ is homeomorphic to H and is compact.*

Proof. We will show that the map

$$\phi : H \longrightarrow \varphi H \varphi^{-1}$$
$$h \longmapsto \varphi h \varphi^{-1}.$$

is a homeomorphism. It is clear that ϕ is bijective. Let $h_0 \in H$. For showing the continuity of ϕ at h_0, let

$$B_K(\phi(h_0), \epsilon) = \{g \in \varphi H \varphi^{-1} \mid d_K(\phi(h_0), g) < \epsilon\}$$

be an open neighborhood of $\phi(h_0) = \varphi h_0 \varphi^{-1}$ in $\varphi H \varphi^{-1}$; where K is a compact subset of F. Then $\varphi^{-1}(K)$ is compact in E. Since E is locally compact the evaluation map $e : Homeo(E) \times E \longrightarrow E$ defined by $e(h, x) = h(x)$ is continuous. Then $H(\varphi^{-1}(K))$ is compact since H is. For ϵ, there exists $\eta > 0$ such that for all $u,\ v \in H\varphi^{-1}(K)$,

$$d(u, v) \leq \eta \Longrightarrow d(\varphi(u), \varphi(v)) \leq \epsilon. \quad (*)$$

For every $g \in B_{\varphi^{-1}(K)}(h_0, \eta) = \{g \in H \mid d_{\varphi^{-1}(K)}(h_0, g) < \eta\}$, we have

$$d(h_0\varphi^{-1}(x), g\varphi^{-1}(x)) \leq \eta, \quad \forall\ x \in K.$$

Then by $(*)$, we obtain that $d_K(\varphi h_0 \varphi^{-1}, \varphi g \varphi^{-1}) \leq \epsilon$, equivalently, $\phi(g) \in B_K(\phi(h_0), \epsilon)$. So ϕ is continuous. Since H is compact, then ϕ is a homeomorphism and $\phi(H) = \varphi H \varphi^{-1}$ is compact. □

Theorem 3.9 *Let G be a compact abelian subgroup of $Homeo(\mathbb{S}^3)$ with nonempty fixed point set. Then G is either finite or equivalent to $\Re$ or to $\Re \cup \rho\Re$.*

Proof. Let $a \in Fix(G)$, we can assume that $a = (0, 0, 0, 1)$. Let $\varphi : \mathbb{S}^3 \setminus \{a\} \longrightarrow \mathbb{R}^3$ be the stereographic projection. By Lemma 3.8, the group $G' = \varphi G_{|\mathbb{S}^3\setminus\{a\}} \varphi^{-1}$ is compact. So, G' is either finite or equivalent to $\Re$ or to $\Re \cup \rho\Re$ (see Proof of Corollary 3.4.(1)). If G' is finite, then $G_{|\mathbb{S}^3\setminus\{a\}}$ is finite and G is also finite. Now, assume that G' is equivalent to $\Re$. This means that there exists $\psi \in Homeo(\mathbb{R}^3)$ such that $G' = \psi \Re \psi^{-1}$. Therefore, for each $g \in G$, there exists $R_\theta \in \Re$ such that $g_{|\mathbb{S}^3\setminus\{a\}} = \varphi^{-1}\psi R_\theta \psi^{-1}\varphi$. If we put $\beta = \varphi^{-1}\psi\varphi$ and $R = \varphi^{-1} R_\theta \varphi$, then $g_{|\mathbb{S}^3\setminus\{a\}} = \beta R \beta^{-1}$. Then, we can easily see that the rotation R is defined by $R(z_1, z_2) = (z_1 e^{i\theta}, z_2)$; we recall that φ and its inverse φ^{-1} are defined respectively by

$$\varphi(x, y, u, v) = \frac{1}{1 - v}(x, y, u), \quad \text{and}$$

$$\varphi^{-1}(x,y,u)=\frac{1}{1+x^2+y^2+u^2}(2x,2y,2u,x^2+y^2+u^2-1).$$

By compactness of $\mathbb{S}^3$ and by the fact that β is a homeomorphism of $\mathbb{S}^3\setminus\{a\}$, β can be extended on $\mathbb{S}^3$ by $\beta(a)=a$, the extension is also denoted by β. So, we have $g=\beta R_\theta\beta^{-1}$. It follows that $G\subset\beta\Re\beta^{-1}$. In order to show that $\beta\Re\beta^{-1}\subset G$, let $g=\beta R_\theta\beta^{-1}$ be a nonperiodic element in G. Then $\overline{\beta\langle R_\theta\rangle\beta^{-1}}=\beta\Re\beta^{-1}\subset G$ since G is compact. We conclude that $G=\beta\Re\beta^{-1}$. In the same way, we can show that if G' is equivalent to $\Re\cup\rho\Re$ on $\mathbb{R}^3$ then G is equivalent to $\Re\cup\rho\Re$ on $\mathbb{S}^3$; where ρ is the reflection of $\mathbb{S}^3$ defined by $\rho(x,y,u,v)=(x,y,-u,v)$. □

Regular homeomorphisms of $\mathbb{S}^3$ having a fixed point are characterized in the following corollary, we recall that every orientation-reversing homeomorphism of $\mathbb{S}^3$ has a fixed point ([Sm]).

Corollary 3.10 *Let h be a regular homeomorphism of $\mathbb{S}^3$ with a fixed point. Then the following statements hold.*

(1) *h is either periodic or equivalent to R_θ or to $R_\theta\rho$.*
(2) *If h commutes with an irrational rotation R_{θ_0} such that $Fix(hR_{\theta_0})\neq\emptyset$, then h is equivalent to R_θ or to $R_\theta\rho$.*

Proof. The group $G=\overline{\langle h\rangle}$ is compact since h is regular and $\mathbb{S}^3$ is compact. Since $Fix(h)\neq\emptyset$, we have $Fix(G)\neq\emptyset$ and by Theorem 3.9, G is either finite or equivalent to $\Re$ or to $\Re\cup\rho\Re$. It follows that h is either periodic or equivalent to R_θ when it is orientation-preserving and to $R_\theta\rho$ when it is orientation-reversing. So, Item (1) is true. For showing Item (2), assume that $hR_{\theta_0}=R_{\theta_0}h$; where R_{θ_0} is irrational and $Fix(hR_{\theta_0})\neq\emptyset$. By Item (1), it suffices to consider the case of h is periodic. Then, there exists an integer $q>0$ such that $h^q=id$. Let $a\in Fix(hR_{\theta_0})$. Thus, $a=(hR_{\theta_0})^q(a)=h^qR^q_{\theta_0}(a)=R^q_{\theta_0}(a)$ and since R_{θ_0} is irrational, we have $R_{\theta_0}(a)=a$. It follows that $h(a)=a$. Let $G=\overline{\langle h,R_{\theta_0}\rangle}$. Clearly $a\in Fix(G)$ and G is compact, abelian (see Proof of Corollary 3.4.(2)). Therefore, G is equivalent to $\Re$ or to $\Re\cup\rho\Re$ since G contains R_{θ_0} which is irrational and G cannot be finite. It follows that h is equivalent to R_θ if it is orientation-preserving and to $R_\theta\rho$ if it is orientation-reversing. □

4. Reversibility for regular homeomorphisms

Let G be a group and let id be its identity element. An element $g \in G$ is said to be

(a) *reversible* in G if it is conjugate to its own inverse in G, that is, if there exists $h \in G$ such that $g^{-1} = hgh^{-1}$;
(b) *strongly reversible* in G if it is reversible in G by an involution, that is, if there exists $\tau \in G$ such that $\tau^2 = id$ and $g^{-1} = \tau g \tau$.

Clearly the implication (b) $\Longrightarrow$ (a) holds and every involution is strongly reversible. A subgroup G of $Homeo(E)$ is said to be *reversible* (resp. *strongly reversible*) in $Homeo(E)$ if there exists $h \in Homeo(E)$ (resp. an involution $\tau \in Homeo(E)$) such that for each element $g \in G$, we have $g^{-1} = hgh^{-1}$ (resp. $g^{-1} = \tau g \tau$).

In this section, we determine the reversible elements in compact Lie groups. On the other hand, for $M = \mathbb{R}^3$ or $\mathbb{S}^3$, we characterize regular homeomorphisms of M that are reversible in $Homeo(M)$.

Lemma 4.1 *Let G be a group, and let $g \in G$. If g is reversible by a periodic element of G, then one of the following statements holds.*

(1) $g^2 = id$.
(2) *There exists an involution $\tau \in G$ such that $g^{-1} = \tau g \tau$.*
(3) *g is reversible by a periodic element of G of period 2^n; where $n \geq 2$ and there exists an involution $\tau \in G$ such that $g = \tau g \tau$.*

Proof. Assume that there exists a periodic element $h \in G$ such that

$$g^{-1} = hgh^{-1}. \quad (*)$$

The period of h can be writen in the form $2^n q$; where $n \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{N}$ and q is odd. By equality $(*)$ we have $h^2 g h^{-2} = h[hgh^{-1}]h^{-1} = hg^{-1}h^{-1} = g$. Then we can easily show by induction that for each integer $p \in \mathbb{N}$, h^{2p} commutes with g. It follows that

$$g^{-1} = h^q g h^{-q}. \quad (**)$$

Then g is reversible by the periodic element h^q of period 2^n.

Case 1. $n = 0$. In this case we have $h^q = id$, so $g = g^{-1}$ and $g^2 = id$.

Case 2. $n = 1$. Then g is strongly reversible by the involution $\tau = h^q$; $g^{-1} = \tau g \tau$.

Case 3. $n > 1$. Let $\tau = h^{2^{n-1}q}$. We have $\tau^2 = id$. Since $n > 1$, $(2^{n-1}q)$ is even and the equality $(*)$ implies that $\tau g \tau = g$. □

Lemma 4.2 *Let G be a compact Lie group. Then the set of torsion elements $T = \{g \in G \mid g$ is periodic$\}$ is dense in G.*

Proof. Let $g \in G$. The closure $H = \overline{\langle g \rangle}$ is a compact abelian Lie group, then the quotient of H by the connected component of the identity H_0 is a finite abelian subgroup A. It follows that H is an extension of A and H_0. But H_0 is a torus, then the set of torsion elements of H_0 is dense in H_0. Therefore, the set of torsion elements of H is dense in H. Thus, T is dense in G. □

Theorem 4.3 *Let G be a compact Lie group, and let $g \in G$. Then the following statements hold.*

(1) *g is reversible in G if and only if $g^2 = id$ or g is reversible by a periodic element f of G of period 2^n; where $n \in \mathbb{N}$.*
(2) *If g is reversible in G, then either g is strongly reversible in G or g commutes with an involution $\tau \in G$.*

Proof.

(1) Let $g \in G$. Assume that g is reversible in G, then there exists $h \in G$ such that

$$g^{-1} = hgh^{-1}. \quad (*)$$

The closure $\overline{\langle h \rangle}$ is a compact Lie group since G is, and by Lemma 4.2 there exists a sequence (h_n) of periodic elements in $\overline{\langle h \rangle}$ such that h_n converges to h when $n \longrightarrow +\infty$. We have

$$\overline{\langle h \rangle} = \overline{\langle h^2 \rangle} \cup h\overline{\langle h^2 \rangle}.$$

First, assume that $h_n \in \overline{\langle h^2 \rangle}$ for each integer n. Then for each integer n, h_n is the limit of a sequence $(h^{2p_{k,n}})_k$. The equality $(*)$ implies that $h^{2p_{k,n}} g h^{-2p_{k,n}} = g$ for each integer k. Then, when $k \longrightarrow +\infty$, we obtain that $h_n g {h_n}^{-1} = g$. So, when $n \longrightarrow +\infty$, we obtain that

$hgh^{-1} = g$ and by equality $(*)$, we have $g^2 = id$. Now, assume that there exists an integer n such that $h_n \in h\overline{\langle h^2\rangle}$, then h_n is the limit of some sequence $(h^{2p_{k,n}+1})_k$ and the equality $(*)$ implies that $h^{2p_{k,n}+1}gh^{-(2p_{k,n}+1)} = g^{-1}$ for each k. When $k \longrightarrow +\infty$, we obtain that $h_n g {h_n}^{-1} = g^{-1}$. If we put $H = h_n$, we have

$$g^{-1} = HgH^{-1}; \quad (**)$$

where H is a periodic element in G. So, by Lemma 4.1, g is reversible by a periodic element of G of period 2^n; where $n \in \mathbb{N} \cup \{0\}$. The converse is clear. We conclude that Item (1) is true.

(2) Follows from Item (1) and Lemma 4.1. □

In the remainder of this section we focus on reversibility of regular homeomorphisms of M in $Homeo(M)$; where $M = \mathbb{R}^3$ or $\mathbb{S}^3$. In [Sh], reversible and strongly reversible maps have been determined in the isometry groups of spherical, Euclidean and hyperbolic space in each finite dimension. In particular, we have the following lemma.

Lemma 4.4 *The group* $\Re \cup \rho\Re$ *is strongly reversible by the reflection* $\sigma : (x, y, u) \longmapsto (-x, y, u)$ *on* $\mathbb{R}^3$ *(resp.* $\sigma : (x, y, u, v) \longmapsto (-x, y, u, v)$ *on* $\mathbb{S}^3$*).*

Lemma 4.5 *Let* h *be a positively regular homeomorphism of a locally compact metric space* E. *Then* h *is regular with relatively compact orbits if one of the following conditions holds*:

(a) *E is compact.*
(b) *h is reversible with relatively compact positive orbits.*

Proof. Let $G = \langle h\rangle$, $G^+ = \{h^n \mid n \in \mathbb{N} \cup \{0\}\}$ and $G^- = \{h^{-n} \mid n \in \mathbb{N}\}$. We have $G = G^+ \cup G^-$ and $\overline{G} = \overline{G^+} \cup \overline{G^-}$. In both cases (a) and (b), every closure positive orbit $\overline{G^+(x)}$ is compact, so by Ascoli's Theorem $\overline{G^+}$ is compact.

(a) If E is compact, then $Homeo(E)$ is a topological group and the map

$$\phi : Homeo(E) \longrightarrow Homeo(E)$$
$$g \longmapsto g^{-1}$$

is a homeomorphism. Which implies that $\phi(\overline{G^+}) = \overline{\phi(G^+)} = \overline{G^-}$ is compact. Thus $\overline{G}$ is compact. Since E is compact, every orbit is relatively compact and by Ascoli's Theorem h is regular.

(b) If h is reversible, then there exists $f \in Homeo(E)$ such that $h^{-1} = fhf^{-1}$. Then, for each $x \in E$, we have $O_x^- = \{h^{-n}(x) \mid n \in \mathbb{N}\} = \{fh^n f^{-1}(x) \mid n \in \mathbb{N}\} = f(O^+_{f^{-1}(x)})$. Since f is a homeomorphism, we have $\overline{O_x^-} = f(\overline{O^+_{f^{-1}(x)}})$ is compact since every positive orbit is relatively compact. Then for each $x \in E$, $\overline{O_x} = \overline{O_x^+} \cup \overline{O_x^-}$ is compact. Moreover, $\overline{G^-} = \overline{\{fh^n f^{-1} \mid n \in \mathbb{N}\}} = f\overline{G^+}f^{-1}$ and by Lemma 3.8, G^- is relatively compact. So $\overline{G}$ is compact and h is regular. □

In the following Theorem we characterize positively regular homeomorphisms of $\mathbb{R}^3$ with bounded positive orbits that are reversible.

Theorem 4.6 *Let h be a positively regular homeomorphism of $\mathbb{R}^3$ with a bounded positive orbit. Then the following statements hold.*

(1) *If h is nonperiodic or h commutes with an irrational rotation. Then the following are equivalent.*
  (a) *h is reversible.*
  (b) *h is strongly reversible.*
(2) *If h is reversible then h is either periodic or the product of two involutions.*

Proof.

(1) (a) $\Longrightarrow$ (b). Assume that h is reversible. Since h is positively regular with a bounded positive orbit, then by Lemma 3.3 every positive orbit is relatively compact and from Lemma 4.5, h is regular with bounded orbits. By Corollary 3.4.(1), h must be equivalent to an orthogonal map, and by Lemma 4.4, h is strongly reversible.
(b) $\Longrightarrow$ (a). Trivial.
(2) Assume that h is reversible. If h is nonperiodic then by Item (1) there exists an involution $\tau \in Homeo(\mathbb{R}^3)$ such that $h^{-1} = \tau h \tau$. Then $(h\tau)^2 = id$, and we have $h = (h\tau)\tau$; where $\tau^2 = id$. □

In the following Theorem we study reversibility for positively regular homeomorphisms of $\mathbb{S}^3$ with a fixed point.

Theorem 4.7 *Let h be a positively regular homeomorphism of $\mathbb{S}^3$ with a fixed point, then the following statements hold.*

(1) *h is strongly reversible if one of the following conditions holds.*
   (a) *h is nonperiodic.*
   (b) *h commutes with an irrational rotation R_{θ_0} and $Fix(hR_{\theta_0}) \neq \emptyset$.*

(2) *h is either periodic or the product of two involutions.*

Proof.

(1) Since $\mathbb{S}^3$ is compact, from Lemma 4.5, h is regular. In both cases (a) and (b) h must be equivalent to an orthogonal map by Corollary 3.10. So, by Lemma 4.4, h is strongly reversible.

(2) Follows from Item (1). □

Lemma 4.8 (1) *Every compact abelian subgroup of $Homeo(\mathbb{R}^3)$ (resp. of $Homeo(\mathbb{S}^3)$ with a fixed point) is either finite or strongly reversible.*

(2) *Let G be a compact subgroup of $Homeo(M)$ and let $g \in G$, then g is either periodic or the product of two involutions if one of the following conditions holds*:
   (a) *$M = \mathbb{R}^3$.*
   (b) *$M = \mathbb{S}^3$ and g has a fixed point.*

Proof.

(1) Let $M = \mathbb{R}^3$ or $\mathbb{S}^3$. If G is infinite, then it is conjugate either to $\Re$ or to $\Re \cup \rho\Re$, that is, there exists an element $\alpha \in Homeo(M)$ such that $G = \alpha\Re\alpha^{-1}$ or $G = \alpha(\Re \cup \rho\Re)\alpha^{-1}$. Let $g \in G$. Then either $g = \alpha R_\theta \alpha^{-1}$ or $g = \alpha R_\theta \rho \alpha^{-1}$ for some $R_\theta \in \Re$. By Lemma 4.4, we have $R_\theta^{-1} = \sigma R_\theta \sigma$ and $R_\theta^{-1}\rho = \sigma(R_\theta \rho)\sigma$. So, $g^{-1} = (\alpha\sigma\alpha^{-1})g(\alpha\sigma\alpha^{-1})$. We conclude that G is strongly reversible by the involution $\alpha\sigma\alpha^{-1}$.

(2) Follows from Item (1) by considering the compact abelian group $\overline{\langle g \rangle}$. □

In the following theorem we study reversibility for equicontinuous flows.

Theorem 4.9 (1) *Every positively equicontinuous flow on $\mathbb{S}^3$ with a fixed point is strongly reversible.*

(2) *Every equicontinuous flow on $\mathbb{R}^3$ with a bounded orbit is strongly reversible.*

(3) *Let* $G = \{h_t \mid t \in \mathbb{R}\}$ *be a nontrivial positively equicontinuous flow on* $\mathbb{R}^3$ *with a bounded positive orbit, then the following statements are equivalent*:
 (a) h_1 *is reversible.*
 (b) G *is strongly reversible.*
 (c) G *is equivalent to* $\mathfrak{R}$.

Proof.

(1) Let $G = \{h_t \mid t \in \mathbb{R}\}$ be a positively equicontinuous flow on $\mathbb{S}^3$. We have $G = G^+ \cup G^-$; where $G^+ = \{h_t \mid t \geq 0\}$ and $G^- = \{h_t \mid t < 0\}$. Then $\overline{G} = \overline{G^+} \cup \overline{G^-}$. Since $\mathbb{S}^3$ is compact, $\overline{G^+}$ is compact and in the same way as in the Proof of Lemma 4.5, $\overline{G^-}$ is also compact. So, $\overline{G}$ is compact, and by Lemma 4.8, $\overline{G}$ is either finite or strongly reversible. If G is finite it must be trivial (i.e. $G = \{id\}$) since G is connected. We conclude that G is strongly reversible.

(2) Let G be an equicontinuous flow on $\mathbb{R}^3$ with a bounded orbit, then $\overline{G}$ is compact abelian, and in the same way as in Item (1), G is strongly reversible.

(3) (a) $\Longrightarrow$ (c). Since G is positively equicontinuous with a bounded positive orbit, then h_1 is positively regular with a bounded positive orbit and by Lemma 4.5, h_1 is regular with bounded orbits. Then G is equicontinuous with relatively compact orbits (see Proof of [Be1, Corollary 2.6]). Therefore G is equivalent to $\mathfrak{R}$.
 (c) $\Longrightarrow$ (b). Follows from Lemma 4.4.
 (b) $\Longrightarrow$ (a). Trivial. □

Lemma 4.10 *Let* h *be a regular homeomorphism of a metric space* E *such that positive orbits are relatively compact. Then every orbit is relatively compact.*

Proof. Let $G = \{h^n \mid n \in \mathbb{Z}\}$, $G^+ = \{h^n \mid n \geq 0\}$, and $G^- = \{h^{-n} \mid n > 0\}$. We will show that every orbit $G(x)$ is relatively compact. By Ascoli's Theorem $\overline{G^+}$ is compact. We have $\overline{G^+}(x) = \overline{G^+(x)}$ is compact and $\overline{G(x)} = \overline{G^+(x)} \cup \overline{G^-(x)}$. For showing that $\overline{G^-(x)}$ is compact, let $(y_n)_n$ be every sequence in $\overline{G^-(x)}$. Then, for each integer n, y_n is the limit of some sequence $(h^{-p_{k,n}}(x))_k$. Since $\overline{G^+(x)}$ is compact, for each integer k, the sequence $(h^{p_{k,n}}(x))_n$ has a subsequence $(h^{p_{k,\varphi(n)}}(x))_n$ converging to some point

$g_k(x) \in \overline{G^+}(x)$. Now, the sequence $(g_k(x))_k$ has a subsequence $(g_{\psi(k)}(x))_k$ converging to some point $g(x) \in \overline{G^+}(x)$. By the inequality

$$d(h^{p_{\psi(k),\varphi(n)}}(x), g(x)) \leq d(h^{p_{\psi(k),\varphi(n)}}(x), g_{\psi(k)}(x)) + d(g_{\psi(k)}(x), g(x)),$$

we deduce that the sequence $(h^{p_{\psi(k),\varphi(n)}}(x))_{n,k}$ converges to $g(x)$, when $n,\ k \longrightarrow +\infty$. Let $\epsilon > 0$. For ϵ, there exists $\eta > 0$ given by the regularity of h at $g(x)$. For η, there exists $k_0 > 0$ and $n_0 > 0$ such that for each $k \geq k_0$ and for each $n \geq n_0$, $d(h^{p_{\psi(k),\varphi(n)}}(x), g(x)) < \eta$, which implies that $d(x, h^{-p_{\psi(k),\varphi(n)}}g(x)) < \epsilon$. By the fact that g commutes with h, when $k \longrightarrow +\infty$ we obtain that $d(x, g(y_{\varphi(n)})) < \epsilon$ for each $n \geq n_0$. Then $(g(y_{\varphi(n)}))_n$ converges to x and $(y_{\varphi(n)})_n$ converges to $g^{-1}(x)$. Remark that since $g(x) \in \overline{G^+(x)}$, there exists a sequence $(h^{p_k}(x))_k$ in $G^+(x)$ converging to $g(x)$. Then $g^{-1}h^{p_k}(x)$ converges to x and since h is regular, $h^{-p_k}(x)$ converges to $g^{-1}(x)$, so $g^{-1}(x) \in \overline{G^-(x)}$. We conclude that $\overline{G^-(x)}$ is compact and so $\overline{G(x)}$ is compact. □

Theorem 4.11 *Let h be an orientation-preserving homeomorphism of M; where $M = \mathbb{R}^3$ or $\mathbb{S}^3$. Consider the following statements*:

(I) *h is reversible.*
(II) *h is regular.*
(III) *h is equivalent to a rotation.*
(IV) *h is strongly reversible.*
(V) *h is embeddable in a flow, that is, there exists a flow $G = \{h_t \mid t \in \mathbb{R}\}$ on M such that $h_1 = h$.*

Then the following statements hold.

(1) *If h is nonperiodic and positively regular, then*
  (a) *If $M = \mathbb{R}^3$ and h has a bounded positive orbit, then* (I), (II), (III) *and* (IV) *are equivalent, and* (I) *implies* (V).
  (b) *If $M = \mathbb{S}^3$ and h has a fixed point, then* (I), (II), (III), (IV) *and* (V) *hold.*
(2) *If h is periodic, then* (III) *is equivalent to* (V), *and* (III) *implies* (IV).

Proof.

(1) Assume that h is nonperiodic and positively regular. For showing (a), assume that $M = \mathbb{R}^3$ and h moreover has a bounded positive orbit.

(I) $\Longrightarrow$ (II). If h is reversible then by Lemma 4.5, h is regular.

(II) $\Longrightarrow$ (III). If h is regular, then by Lemma 4.10, h has bounded orbits and by Corollary 3.4.(1), h is equivalent to a rotation since h is nonperiodic.

(III) $\Longrightarrow$ (IV). Follows from the fact that every rotation is strongly reversible.

(IV) $\Longrightarrow$ (I). Trivial.

(I) $\Longrightarrow$ (V). If condition (I) holds, then h is regular and equivalent to a rotation and by [Be1, Corollary 2.6.(1)] h is embeddable in a flow. Item (b) is true since by Lemma 4.5, h is regular and by Corollary 3.10, h is equivalent to a rotation which implies that h is strongly reversible and embeddable in a flow.

(2) (III) $\Longleftrightarrow$ (V). Follows from [Be1, Corollary 2.6.(1)] if $M = \mathbb{R}^3$, and if $M = \mathbb{S}^3$ it follows from Proof of [Be1, Corollary 2.6.(1)] and [Ric, Theorem A]. (III) $\Longrightarrow$ (IV) is clear. □

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Khadija Ben Rejeb
Isitcom Sousse
22 Rue Habib Bourguiba Hergla 4012 Sousse, Tunisia
E-mail: kbrjeb@yahoo.fr