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Reeb components of leafwise complex foliations and their symmetries II

Tomohiro Horiuchi

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Abstract. We study the group of leafwise holomorphic smooth automorphisms of 5-dimensional Reeb components with leafwise complex structure which are obtained by a certain Hopf construction. In particular, in the case where the boundary holonomy is infinitely tangent to the identity, we completely determine the structure of the group of leafwise holomorphic automorphisms of such foliations.

Key words: Reeb component, Hopf surface, diffeomorphisms.

1. Introduction

In this article, we continue the study on the symmetries of Reeb components with leafwise complex structure, which we started in our previous paper [HM]. In particular, we proceed to study the symmetries of 5-dimensional Reeb components. Recall that a $(p+1)$-dimensional Reeb component is a compact manifold $R = D^p \times S^1$ with a (smooth) foliation of codimension one, whose leaves are ∂R and the graphs of $f + c$ $(c \in \mathbb{R})$, where $f : \operatorname{int} D^p \to \mathbb{R}$ is a smooth function such that $\lim_{z \to \partial D^p} f(z) = +\infty$. Here we identify R with $D^p \times \mathbb{R}/\mathbb{Z}$.

In [HM], we studied the group of all leafwise holomorphic smooth automorphisms of 3-dimensional Reeb components with leafwise complex structure and determined the structure of the group as follows.

Theorem ([HM]) *Let R be a 3-dimensional leafwise complex Reeb component obtained by the Hopf construction and $\varphi \in \mathit{Diff}^{\infty}([0,\infty))$ be the holonomy tangent to the identity to the infinite order at the boundary elliptic curve H. In particular, the modulus of H is $(-1/2\pi i)\log\lambda$, where λ is a complex number such that $|\lambda| > 1$. Then the group $\mathit{Aut}\, R$ of leafwise holomorphic automorphisms of R is isomorphic to the semi-direct product*

$$(\mathcal{Z}_{\varphi,\lambda} \rtimes Z_{\varphi}) \rtimes \mathit{Aut}_0\, H$$

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where $\mathcal{Z}_{\varphi,\lambda}$ is the space of solutions to a certain functional equation (see Section 3 for the detail), Z_φ is the centralizer of φ in $Diff^\infty([0,\infty))$ and $Aut_0\, H$ is the identity component of the automorphism group of H.

In the case of complex leaf dimension 2, the boundary leaf of a Reeb component is a primary Hopf surface. Kodaira [Ko1] classified the primary Hopf surfaces into five types and gave a normal form in each case (see Section 2). In this paper, we compute the group of leafwise holomorphic automorphisms for each type of the boundary Hopf surface relying on the normal form and obtain the following.

Main Theorem *Let R be a 5-dimensional leafwise complex Reeb component obtained by the Hopf construction and $\varphi \in Diff^\infty([0,\infty))$ be the holonomy tangent to the identity to the infinite order at the boundary Hopf surface H. Then $Aut\, R$ admits a following sequence of extensions.*

$$1 \to Aut\,(R,H) \to Aut\, R \to Aut\, H \to 1,$$
$$0 \to \mathcal{K} \to Aut\,(R,H) \to Z_\varphi \to 1$$

where $Aut\, H$ is the automorphism group of H, $Aut\,(R,H)$ is the kernel of the restriction map from $Aut\, R$ to $Aut\, H$ and $\mathcal{K}$ is an infinite dimensional vector space depending on the normal form of H (Theorem 4.9 and 4.10).

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2. Preliminaries

2.1. Leafwise complex structure

Let M be a $(2n+q)$-dimensional smooth manifold, $\mathcal{F}$ be a smooth foliation of codimension q on M, and $p = 2n$ be the dimension of leaves. We refer general basics for foliation theory to [CC].

Definition 2.1 (Leafwise complex structure, cf. [MV]) $(M,\mathcal{F})$ is said to be equipped with a *leafwise complex structure* if $(M,\mathcal{F})$ is given by a foliation atlas $\{(U_\alpha,\phi_\alpha)\}$ such that $\phi_\alpha(U_\alpha)$ is an open set of $\mathbb{C}^n \times \mathbb{R}^q$ and that in each coordinate change $\phi_\beta \circ \phi_\alpha^{-1}(z,x) = (w(z,x),\, y(x))$ $(z,w \in \mathbb{C}^n\,;\, x,y \in \mathbb{R}^q)$, w is holomorphic with respect to z.

This definition is equivalent to that the foliation has complex structures varying smoothly in transverse directions. It is eventually equivalent to that the tangent bundle $\tau\mathcal{F}$ to the foliation $\mathcal{F}$ is equipped with a smooth integrable almost complex structure J. We call $(M, \mathcal{F}, J)$ a *leafwise complex foliation.*

2.2. Hopf surfaces

Let W be the domain $\mathbb{C}^2 \backslash \{O\}$. A compact complex surface is called a *Hopf surface* if its universal covering is biholomorphic to W. Especially, a Hopf surface whose fundamental group is infinite cyclic is called a *primary Hopf surface.* Kodaira classified primary Hopf surfaces in [Ko1], [Ko2].

Theorem 2.2 (Kodaira [Ko1], [Ko2]) 1) *Any primary Hopf surface is a quotient space $W/G^{\mathbb{Z}}$ of W with respect to an infinite cyclic group $G^{\mathbb{Z}} = \langle G \rangle$ generated by a complex analytic automorphism $G : W \to W$ of the form $G(z_1, z_2) = (\lambda z_1 + \tau z_2^p,\ \mu z_2)$, where p is a positive integer and $\lambda,\ \mu,\ \tau$ are complex numbers satisfying $|\lambda| \geq |\mu| > 1$ and $(\lambda - \mu^p)\tau = 0$.*

2) *A compact complex surface S is biholomorphic to a primary Hopf surface if and only if it is diffeomorphic to $S^3 \times S^1$.*

2.3. Reeb components by Hopf construction

Construction 2.3 (Hopf construction) Let $\tilde{R}$ be $\mathbb{C}^2 \times [0, \infty) \backslash \{(O, 0)\}$ and $\tilde{\mathcal{F}} = \{\mathbb{C}^2 \times \{x\}\,;\, x > 0\} \sqcup \{W \times \{0\}\}$ be the foliation on $\tilde{R}$ with the standard leafwise complex structure J_{std}. Let T be a diffeomorphism of $\tilde{R}$ given by

$$T(z_1, z_2, x) = (G \times \varphi)(z_1, z_2, x) = (\lambda z_1 + \tau z_2^p,\ \mu z_2,\ \varphi(x))$$

where p is a positive integer, $\lambda,\ \mu,\ \tau$ are complex numbers as in Theorem 2.2 and $\varphi \in \mathit{Diff}^{\infty}([0, \infty))$ is a diffeomorphism of the half line satisfying $\varphi(x) - x > 0$ for $x > 0$, namely the origin is an expanding unique fixed point. Then the quotient $R = \tilde{R}/T^{\mathbb{Z}}$ has a foliation $\mathcal{F}$ with leafwise complex structure induced by $\tilde{\mathcal{F}}$. The boundary $H = W/G^{\mathbb{Z}}$ is a primary Hopf surface, and the holonomy along H coincides with φ. We call $(R, \mathcal{F}, J)$ the *Reeb component with leafwise complex structure* or the *LC Reeb component.*

3. Functional equations on flat functions

In this section, we review the result on the functional equations which we proved in [HM] in order to determine the automorphism groups of an

LC Reeb component.

Let $\varphi \in \mathit{Diff}^{\infty}([0,\infty))$ be a diffeomorphism of the half line which is tangent to the identity to the infinite order at $x = 0$ and satisfies $\varphi(x) - x > 0$ for $x > 0$. Also we fix a complex number λ with $|\lambda| > 1$. Let us consider the following (system of) functional equations on β, β_1 and $\beta_2 \in C^{\infty}([0,\infty);\mathbb{C})$ concerning φ and λ. If λ is a real number, we can consider the same equations for $\beta_2 \in C^{\infty}([0,\infty);\mathbb{R})$.

Equation (I) : $\beta(\varphi(x)) = \lambda\beta(x)$.
Equation (II) : $\beta_1(\varphi(x)) = \lambda\beta_1(x) + \beta_2(x), \quad \beta_2(\varphi(x)) = \lambda\beta_2(x)$.

First consider these equations on $(0,\infty)$. Then, Equation (I) has a lot of solutions. In fact, if we fix any solution $\beta^* \in C^{\infty}((0,\infty);\mathbb{C})$ which never vanishes, *i.e.* $\beta^*(x) \neq 0$ for $x > 0$, then each solution corresponds to a smooth function on $S^1 = (0,\infty)/\varphi^{\mathbb{Z}}$ by taking $\beta \mapsto \beta/\beta^*$. This gives a bijective correspondence as vector spaces between the space $\mathcal{Z} = \mathcal{Z}_{\varphi,\lambda}$ of solutions to (I) on $(0,\infty)$ and $C^{\infty}(S^1;\mathbb{C})$.

Also take the space $\mathcal{S} = \mathcal{S}_{\varphi,\lambda}$ of solutions to Equation (II) on $(0,\infty)$. If we assign β_2 to a solution $(\beta_1,\beta_2) \in \mathcal{S}$, we obtain the projection $P_2 : \mathcal{S} \to \mathcal{Z}$. Here the kernel of P_2 is nothing but $\mathcal{Z}$. We also see that P_2 is surjective because for any $\beta_2 \in \mathcal{Z}$

$$\beta_1(x) = \frac{1}{\lambda \log \lambda}\beta_2(x)\log\beta^*(x)$$

gives a solution $(\beta_1,\beta_2) \in \mathcal{S}$, where for $\log\beta^*(x)$ any smooth branch can be taken. Therefore, as a vector space, $\mathcal{S}$ has a structure such that

$$0 \to \mathcal{Z} \to \mathcal{S} \to \mathcal{Z} \to 0$$

is a short exact sequence.

We proved the flatness for $\beta \in \mathcal{Z}$ and $(\beta_1,\beta_2) \in \mathcal{S}$ in [HM]. Key points of the proof are the infinite tangency of φ and the formula of Faá di Bruno.

Theorem 3.1 ([HM]) 1) *Any solution $\beta \in \mathcal{Z}$ extends to $[0,\infty)$ so as to be a smooth function which is flat at $x = 0$.*

2) *The same applies to any solution $(\beta_1,\beta_2) \in \mathcal{S}$.*

Remark 3.2 Let us consider the following system of functional equations on β_1 and $\beta_2 \in C^{\infty}([0,\infty);\mathbb{C})$ concerning φ, λ and a non-zero constant c.

Equation (IIc) : $\beta_1(\varphi(x)) = \lambda\beta_1(x) + c\beta_2(x), \quad \beta_2(\varphi(x)) = \lambda\beta_2(x)$.

Let us take the space $\mathcal{S}(c) = \mathcal{S}_{\varphi,\lambda}(c)$ of solutions to Equation (IIc) on $(0, \infty)$. Then, by taking $(\beta_1, \beta_2) \mapsto (\beta_1, c\beta_2)$, $\mathcal{S}(c)$ is in one-to-one correspondence with $\mathcal{S}$. In particular, any solution $(\beta_1, \beta_2) \in \mathcal{S}(c)$ extends to $[0, \infty)$ so as to be smooth and flat at $x = 0$.

4. Symmetries of 5-dimensional Reeb component

In this section we compute the group of automorphisms of a Reeb component of dimension 5 which is given by a Hopf construction.

Definition 4.1 Let $(M, \mathcal{F}, J)$ be a smooth foliated manifold with leafwise complex structure. A diffeomorphism $f : M \to M$ is said to be a *leafwise holomorphic smooth automorphism* if it preserves $\mathcal{F}$ and gives rise to biholomorphic maps between leaves.

We denote by $Aut\,(M, \mathcal{F}, J)$, or $Aut\,M$ for short, the group of leafwise holomorphic smooth automorphisms of $(M, \mathcal{F}, J)$.

Let $(R, \mathcal{F}, J)$ be a 5-dimensional LC Reeb component with holonomy φ tangent to the identity to the infinite order at the origin and satisfies $\varphi(x) - x > 0$ for $x > 0$. Any element $f \in Aut\,R$ has a lift $\tilde{f} \in Aut\,(\tilde{R}, \tilde{\mathcal{F}}, J_{\mathrm{std}})$ $(= Aut\,\tilde{R})$ which takes the form

$$\tilde{f}(z_1, z_2, x) = (\xi_1(z_1, z_2, x),\, \xi_2(z_1, z_2, x),\, \eta(x))$$

in $\mathbb{C}^2 \times \mathbb{R}_{\geq 0}$-coordinate. A lift $\tilde{f}$ should commute with the covering transformation T, because, $T \circ \tilde{f} = \tilde{f} \circ T^k$ for some $k \in \mathbb{Z}$ but it is easy to see that $k = 1$ when it is restricted to the boundary. Therefore, an element in $Aut\,\tilde{R}$ is a lift of some element in $Aut\,R$ if and only if it commutes with T. Let $Aut\,(\tilde{R}; T)$ denote the centralizer of T in $Aut\,\tilde{R}$, namely, the group of all such lifts.

Proposition 4.2 *$Aut\,R$ is naturally isomorphic to $Aut(\tilde{R}; T)/T^{\mathbb{Z}}$.*

4.1. Automorphism groups of Hopf surfaces

Let $Aut\,H$ be the group of holomorphic automorphisms of a primary Hopf surface H. For any element $h \in Aut\,H$, there is a lift $\tilde{h} \in Aut\,\tilde{H}$ of h such that it commutes with G because $\tilde{H} = W$ is the universal covering of

H. Let $Aut\,(\tilde{H};G)$ denote the centralizer of G in $Aut\,\tilde{H}$. Then, $Aut\,H$ is naturally isomorphic to $Aut\,(\tilde{H};G)/G^{\mathbb{Z}}$. Moreover, by Hartogs' theorem, $\tilde{h}$ is extended to an element of the group $Aut\,(\mathbb{C}^2,\{O\})$ of all automorphisms of $\mathbb{C}^2$ fixing the origin. The automorphism $\tilde{h}\times id_{\mathbb{R}_{\geq 0}}$ of $\tilde{R}$ clearly commutes with T and defines an element in $Aut\,R$. Consequently, we obtain the following.

Proposition 4.3 *The restriction map $r_H : Aut\,R \to Aut\,H$ is surjective.*

Let $Aut\,(R,H)$ denote the kernel of r_H. By this proposition, the study of the structure of $Aut\,R$ breaks into two parts, that of $Aut\,(R,H)$ and $Aut\,H$.

Namba [Na] determined the centralizer of G in $Aut\,(\mathbb{C}^2,\{O\})$ and the automorphism group of H in the case where G is linear, namely in the cases 1)–4) below. The following classification 1)–5) easily follows from Kodaira's result (Theorem 2.2).

Theorem 4.4 *An element $F \in Aut\,\tilde{H}$ belongs to $Aut\,(\tilde{H};G)$ if and only if it is described in the following form.*

1) *If $\tau = 0$ and $\lambda = \mu$,*

$$F(z_1,z_2) = (az_1 + bz_2,\, cz_1 + dz_2), \quad a,b,c,d \in \mathbb{C},\ ad - bc \neq 0.$$

2) *If $\tau = 0$ and $\lambda = \mu^p$ for some integer $p \geq 2$,*

$$F(z_1,z_2) = (az_1 + bz_2^p,\, dz_2), \quad a,b,d \in \mathbb{C},\ ad \neq 0.$$

3) *If $\tau = 0$ and $\lambda \neq \mu^m$ for any $m \in \mathbb{N}$,*

$$F(z_1,z_2) = (az_1,\, dz_2), \quad a,d \in \mathbb{C},\ ad \neq 0.$$

4) *If $\tau \neq 0$ and $\lambda = \mu$,*

$$F(z_1,z_2) = (az_1 + bz_2,\, az_2), \quad a,b \in \mathbb{C},\ a \neq 0.$$

5) *If $\tau \neq 0$ and $\lambda = \mu^p$ for some integer $p \geq 2$,*

$$F(z_1,z_2) = (a^p z_1 + bz_2^p,\, az_2), \quad a,b \in \mathbb{C},\ a \neq 0.$$

Remark 4.5 The case 5) is proven in the same way as the case 4). It is also obtained by computing $Aut\,(\tilde{R};T)$ like Case 5 in Section 4.2.2.

Corollary 4.6 *The group $Aut\,(\tilde{H};G)$ is isomorphic to the following.*

1) *If $\tau = 0$ and $\lambda = \mu$, $Aut\,(\tilde{H};G) \cong GL(2;\mathbb{C})$.*
2) *If $\tau = 0$ and $\lambda = \mu^p$ for some integer $p \geq 2$, $Aut(\tilde{H};G) \cong \mathbb{C} \rtimes (\mathbb{C}^* \times \mathbb{C}^*)$.*
3) *If $\tau = 0$ and $\lambda \neq \mu^m$ for any $m \in \mathbb{N}$, $Aut\,(\tilde{H};G) \cong \mathbb{C}^* \times \mathbb{C}^* \subset GL(2;\mathbb{C})$.*
4) *If $\tau \neq 0$ and $\lambda = \mu$, $Aut\,(\tilde{H};G) \cong \mathbb{C} \rtimes \mathbb{C}^* \subset GL(2;\mathbb{C})$.*
5) *If $\tau \neq 0$ and $\lambda = \mu^p$ for some integer $p \geq 2$, $Aut\,(\tilde{H};G) \cong \mathbb{C} \rtimes \mathbb{C}^*$.*

4.2. Structure of $Aut\,(R, H)$

Here we determine the kernel $Aut\,(R,H)$ of the restriction map r_H. Let $Aut\,(\tilde{R},\tilde{H};T)$ be the subgroup of $Aut\,(\tilde{R};T)$ which consists of all elements which act trivially on the boundary $\tilde{H}$. Any element $f \in Aut\,(R,H)$ has a unique lift to an element in $Aut\,(\tilde{R},\tilde{H};T)$. Namely,

Proposition 4.7 *$Aut\,(R,H)$ is isomorphic to $Aut\,(\tilde{R},\tilde{H};T)$.*

So from now on we compute $Aut\,(\tilde{R},\tilde{H};T)$ instead of $Aut\,(R,H)$. Let us present an element $g \in Aut\,\tilde{R}$ in the form

$$g(z_1, z_2, x) = (\xi_1(z_1, z_2, x),\ \xi_2(z_1, z_2, x),\ \eta(x))$$

where $\xi_j : \tilde{R} \to \mathbb{C}$ is holomorphic in z_α's ($\alpha = 1, 2$) and smooth in x for $j = 1, 2$, and $\eta \in \mathit{Diff}^{\infty}([0,\infty))$. As noted in Proposition 4.2, $Aut\,(\tilde{R};T)$ exactly consists of elements that commute with T. Hence, g belongs to $Aut\,(\tilde{R};T)$ if and only if it satisfies the following conditions.

Condition (L):

$$\xi_1(\lambda z_1 + \tau z_2^p,\ \mu z_2,\ \varphi(x)) = \lambda \xi_1(z_1, z_2, x) + \tau \xi_2(z_1, z_2, x)^p \tag{L1}$$

$$\xi_2(\lambda z_1 + \tau z_2^p,\ \mu z_2,\ \varphi(x)) = \mu \xi_2(z_1, z_2, x) \tag{L2}$$

Condition (T):

$\eta \circ \varphi = \varphi \circ \eta$, namely, η belongs to the centralizer Z_φ of φ in $\mathit{Diff}^{\infty}([0,\infty))$.

Furthermore, g belongs to $Aut\,(\tilde{R},\tilde{H};T)$ if and only if the above conditions are satisfied with $\xi_1(z_1, z_2, 0) = z_1$ and $\xi_2(z_1, z_2, 0) = z_2$. For known facts on the structure of Z_φ, see [HM] and the references therein. By assigning η to an element $g \in Aut\,(\tilde{R}, \tilde{H}; T)$, we obtain a projection $P : Aut\,(\tilde{R}, \tilde{H}; T) \to Z_\varphi$. Then, we obtain a short exact sequence

$$0 \to \mathcal{K} \to Aut\,(\tilde{R}, \tilde{H}; T) \xrightarrow{P} Z_\varphi \to 1$$

where $\mathcal{K}$ is the kernel of P.

If $x = 0$, for $j = 1, 2$, $\xi_j(\cdot, 0)$ is a holomorphic function on $\tilde{H} = \mathbb{C}^2 \backslash \{O\}$. Since the origin is a removable singularity, $\xi_j(\cdot, 0)$ is extended to the holomorphic function on $\mathbb{C}^2$ with $\xi_j(O, 0) = 0$. We expand ξ_j in the power series of z_1 and z_2 at the origin:

$$\xi_j(z_1, z_2, x) = \sum_{k+l \geq 0} a^j_{kl}(x) z_1^k z_2^l \quad (j = 1, 2) \tag{4.1}$$

where $a^j_{kl} \in C^\infty([0, \infty); \mathbb{C})$ $(k,\ l = 0, 1, 2, \dots)$, and $a^j_{00}(0) = 0$.

We can redescribe Condition (L) by using a^j_{kl}'s. If g belongs to $Aut\,(\tilde{R}, \tilde{H}; T)$, then we have

$$\begin{aligned} a^1_{10}(0) = a^2_{01}(0) = 1, \quad a^1_{01}(0) = a^2_{10}(0) = 0, \\ a^j_{kl}(0) = 0 \quad (j = 1, 2, k + l \geq 2). \end{aligned} \tag{4.2}$$

In the rest of this section, we devote to determining ξ_1 and ξ_2 satisfying Condition (L).

4.2.1 Diagonal case ($\tau = 0$)

Assume $\tau = 0$. We consider three cases separately.

Case 1. $\tau = 0$ and $\lambda = \mu$.

In this case, Condition (L) is written as

$$\text{(Lj)} \quad \xi_j(\lambda z_1, \lambda z_2, \varphi(x)) = \lambda \xi_j(z_1, z_2, x) \quad (j = 1, 2)$$

By using the series expansion (4.1) of ξ_j, we rewirite (Lj) as

$$\sum_{k+l \geq 0} \lambda^{k+l} a^j_{kl}(\varphi(x)) z_1^k z_2^l = \sum_{k+l \geq 0} \lambda a^j_{kl}(x) z_1^k z_2^l \quad (j = 1, 2).$$

Comparing coefficients in both sides, we obtain

$$\lambda^{k+l-1} a^j_{kl}(\varphi(x)) = a^j_{kl}(x) \quad (j = 1, 2). \tag{4.3}$$

Proposition 4.8 *Let φ be a C^0-diffeomorphism of the half line which satisfies $\varphi(x) - x > 0$ for every $x > 0$, a be a continuous $\mathbb{C}$-valued function on the half line, and ν be a complex number with $|\nu| > 1$.*

1) *If $a(\varphi(x)) = a(x)$ for all $x \in [0, \infty)$, $a(x) = a(0)$ is constant.*
2) *If $\nu a(\varphi(x)) = a(x)$ for all $x \in [0, \infty)$, a is identically equal to 0.*

Proof. 1) For any $x \geq 0$, $a(x) = \lim_{n\to\infty} a(\varphi^{-n}(x)) = a(0)$.
2) For any $x \geq 0$, $|a(x)| = \lim_{n\to\infty}(1/|\nu|^n)|a(\varphi^{-n}(x))| = 0$. □

By (4.2), (4.3) and Proposition 4.8, we have

$$a^1_{10} = 1 \quad \text{and} \quad a^1_{kl} = 0 \ \text{ for } (k, l) \neq (1, 0);$$
$$a^2_{01} = 1 \quad \text{and} \quad a^2_{kl} = 0 \ \text{ for } (k, l) \neq (0, 1).$$

Therefore, g belongs to $Aut\,(\tilde{R}, \tilde{H}; T)$ iff it takes the following form.

$$g(z_1, z_2, x) = (z_1 + a^1_{00}(x),\ z_2 + a^2_{00}(x),\ \eta(x))$$

where a^1_{00}, $a^2_{00} \in \mathcal{Z}_{\varphi,\lambda}$ (for the definition see Section 3) and $\eta \in Z_\varphi$ (for the definition see Condition (T)).

Case 2. $\tau = 0$ and $\lambda = \mu^p$ for some integer $p \geq 2$.

In this case, Condition (L) is written as

$$\text{(L1)} \quad \xi_1(\mu^p z_1, \mu z_2, \varphi(x)) = \mu^p \xi_1(z_1, z_2, x),$$
$$\text{(L2)} \quad \xi_2(\mu^p z_1, \mu z_2, \varphi(x)) = \mu \xi_2(z_1, z_2, x).$$

In the same way as in Case 1, the above equations are reduced to

$$\sum_{k+l\geq 0} \mu^{pk+l} a^1_{kl}(\varphi(x)) z_1^k z_2^l = \sum_{k+l\geq 0} \mu^p a^1_{kl}(x) z_1^k z_2^l,$$
$$\sum_{k+l\geq 0} \mu^{pk+l} a^2_{kl}(\varphi(x)) z_1^k z_2^l = \sum_{k+l\geq 0} \mu a^2_{kl}(x) z_1^k z_2^l.$$

By comparing coefficients, we have

$$\mu^{p(k-1)+l} a^1_{kl}(\varphi(x)) = a^1_{kl}(x), \tag{4.4}$$

$$\mu^{pk+l-1} a^2_{kl}(\varphi(x)) = a^2_{kl}(x). \tag{4.5}$$

By (4.2), (4.4), (4.5) and Proposition 4.8, we have

$$a^1_{10} = 1 \quad \text{and} \quad a^1_{kl} = 0 \ \text{ for } (k,l) \neq (1,0),\ (0,j)\ (0 \leq j \leq p-1);$$
$$a^2_{01} = 1 \quad \text{and} \quad a^2_{kl} = 0 \ \text{ for } (k,l) \neq (0,1).$$

Therefore, g belongs to $Aut\,(\tilde{R}, \tilde{H}; T)$ iff it takes the following form.

$$g(z_1, z_2, x) = \left(z_1 + \sum_{j=0}^{p-1} a^1_{0j}(x) z_2^j,\ z_2 + a^2_{00}(x),\ \eta(x) \right)$$

where $a^1_{0j} \in \mathcal{Z}_{\varphi, \mu^{p-j}}$ $(j = 0, 1, \ldots, p-1)$, $a^2_{00} \in \mathcal{Z}_{\varphi,\mu}$ and $\eta \in Z_\varphi$.

Case 3. $\tau = 0$ and $\lambda \neq \mu^m$ for any $m \in \mathbb{N}$.

In this case, Condition (L) is written as

$$\text{(L1)} \quad \xi_1(\lambda z_1, \mu z_2, \varphi(x)) = \lambda \xi_1(z_1, z_2, x),$$
$$\text{(L2)} \quad \xi_2(\lambda z_1, \mu z_2, \varphi(x)) = \mu \xi_2(z_1, z_2, x).$$

In the same way as in Case 1, the above equations are reduced to

$$\sum_{k+l \geq 0} \lambda^k \mu^l a^1_{kl}(\varphi(x)) z_1^k z_2^l = \sum_{k+l \geq 0} \lambda a^1_{kl}(x) z_1^k z_2^l,$$
$$\sum_{k+l \geq 0} \lambda^k \mu^l a^2_{kl}(\varphi(x)) z_1^k z_2^l = \sum_{k+l \geq 0} \mu a^2_{kl}(x) z_1^k z_2^l.$$

By comparing coefficients, we have

$$\lambda^{k-1} \mu^l a^1_{kl}(\varphi(x)) = a^1_{kl}(x), \tag{4.6}$$

$$\lambda^k \mu^{l-1} a^2_{kl}(\varphi(x)) = a^2_{kl}(x). \tag{4.7}$$

By (4.2), (4.6), (4.7) and Proposition 4.8, we have

$$a^1_{kl} = 0 \ (k+l \geq 2,\, k \geq 1), \quad a^2_{kl} = 0 \ (k+l \geq 1,\, l \geq 1).$$

Let us consider the functional equations (L1) and (L2) concerning a^1_{0l} and a^2_{k0}. Since λ and μ satisfy the inequality $|\lambda| \geq |\mu| > 1$, they satisfy $|\lambda^k| \geq |\mu|$ for $k \geq 1$. Then, the equation (L2) on a^2_{k0} is written as follows.

$$\frac{\lambda^k}{\mu} a^2_{k0}(\varphi(x)) = a^2_{k0}(x) \ (k \geq 1)$$

If $|\lambda| > |\mu|$, then a^2_{k0} is identically equal to 0 by Proposition 4.8 2) because $|\lambda^k/\mu|$ is greater than 1 for $k \geq 1$. If $|\lambda| = |\mu|$ and $k = 1$, the equation $|a^2_{10}(\varphi(x))| = |a^2_{10}(x)|$ holds for $x \geq 0$. Then, we have $|a^2_{10}(x)| = \lim_{n\to\infty} |a^2_{10}(\varphi^{-n}(x))| = |a^2_{10}(0)| = 0$. If $|\lambda| = |\mu|$ and $k \geq 2$, then a^2_{k0} is also equal to 0 in the same way as in the case $|\lambda| > |\mu|$.

Next, we fix the positive integer $p = [\log|\lambda|/\log|\mu|]$, where $[\,\cdot\,]$ is the greatest integer function. Then, λ and μ satisfy $|\lambda| \geq |\mu^l|$ for $0 \leq l \leq p$ and $|\lambda| < |\mu^l|$ for $l \geq p+1$. The equation (L1) on a^1_{0l} is written as follows.

$$\frac{\mu^l}{\lambda} a^1_{0l}(\varphi(x)) = a^1_{0l}(x)$$

If $l \geq p+1$, then a^1_{0l} is identically equal to 0 by Proposition 4.8 2) because $|\mu^l/\lambda|$ is greater than 1. If $|\lambda| = |\mu^p|$, the equation $|a^1_{0p}(\varphi(x))| = |a^1_{0p}(x)|$ holds for $x \geq 0$, then we have $|a^1_{0p}(x)| = \lim_{n\to\infty} |a^1_{0p}(\varphi^{-n}(x))| = |a^1_{0p}(0)| = 0$.

Therefore, g belongs to $Aut\,(\tilde{R}, \tilde{H}; T)$ iff it takes the following form.

$$g(z_1, z_2, x) = \left(z_1 + \sum_{j=0}^{p} a^1_{0j}(x) z_2^j,\; z_2 + a^2_{00}(x),\; \eta(x) \right)$$

where $a^1_{0j} \in \mathcal{Z}_{\varphi,\lambda\mu^{-j}}$ $(j = 0, 1, \ldots, p)$, $a^2_{00} \in \mathcal{Z}_{\varphi,\mu}$ and $\eta \in Z_\varphi$.

Consequently, we obtain the following theorem.

Theorem 4.9 1) *If $\tau = 0$ and $\lambda = \mu$, then $Aut(R, H)$ is isomorphic to a semi-direct product*

$$(\mathcal{Z}_{\varphi,\lambda} \times \mathcal{Z}_{\varphi,\lambda}) \rtimes Z_\varphi$$

where $\eta \in Z_\varphi$ *acts on* $(\beta_1, \beta_2) \in \mathcal{Z}_{\varphi,\lambda} \times Z_{\varphi,\lambda}$ *by* $(\beta_1(x), \beta_2(x)) \mapsto (\beta_1(\eta(x)), \beta_2(\eta(x)))$.

2) *If* $\tau = 0$ *and* $\lambda \neq \mu$, *then the kernel* $\mathcal{K}$ *of the surjective homomorphism* $Aut\,(R, H) \to Z_\varphi$ *has the following structure*:

i) *If* $\lambda = \mu^p$ *for some integer* $p \geq 2$, $\mathcal{K}$ *admits an extension*

$$0 \to \mathcal{Z}_{\varphi,\mu^p} \times \mathcal{Z}_{\varphi,\mu^{p-1}} \times \cdots \times \mathcal{Z}_{\varphi,\mu} \to \mathcal{K} \to \mathcal{Z}_{\varphi,\mu} \to 0.$$

ii) *If* $\lambda \neq \mu^m$ *for any integer* $m \geq 1$, $\mathcal{K}$ *admits an extension*

$$0 \to \mathcal{Z}_{\varphi,\lambda} \times \mathcal{Z}_{\varphi,\lambda\mu^{-1}} \times \cdots \times \mathcal{Z}_{\varphi,\lambda\mu^{-p}} \to \mathcal{K} \to \mathcal{Z}_{\varphi,\mu} \to 0$$

where $p = [\log|\lambda|/\log|\mu|] \in \mathbb{N}$, *where* $[\,\cdot\,]$ *is the greatest integer function.*

Proof. $\mathcal{K}$ is regarded as the space of solutions to Condition (L). If $\tau = 0$ and $\lambda = \mu$, $\mathcal{K}$ is the set of functions (a^1_{00}, a^2_{00}) in Case 1, and if $\tau = 0$ and $\lambda \neq \mu$, it is the set of functions $(a^1_{00}, \ldots, a^1_{0q}, a^2_{00})$ ($q = p-1$ or p) in Cases 2 and 3. By assigning β_2 to an element $(\beta_{1,0}, \ldots, \beta_{1,q}, \beta_2) \in \mathcal{K}$, we obtain the projection $P_2 : \mathcal{K} \to \mathcal{Z}_{\varphi,\mu}$. Then, the kernel of P_2 is $\mathcal{Z}_{\varphi,\mu^p} \times \mathcal{Z}_{\varphi,\mu^{p-1}} \times \cdots \times \mathcal{Z}_{\varphi,\mu}$ in Case 2, and $\mathcal{Z}_{\varphi,\lambda} \times \mathcal{Z}_{\varphi,\lambda\mu^{-1}} \times \cdots \times \mathcal{Z}_{\varphi,\lambda\mu^{-p}}$ in Case 3. □

4.2.2 Nondiagonal case ($\tau \neq 0$)

Assume $\tau \neq 0$. This case is divided into two cases.

Case 4. $\tau \neq 0$ and $\lambda = \mu$.

Let S be the diffeomorphism of $\tilde{R}$ given by $S(z_1, z_2, x) = (z_1, \tau z_2, x)$. Then, for any $(z_1, z_2, x) \in \tilde{R}$,

$$S \circ T \circ S^{-1}(z_1, z_2, x) = (\lambda z_1 + z_2, \lambda z_2, \varphi(x)).$$

Therefore, the LC Reeb component $R = \tilde{R}/T^{\mathbb{Z}}$ is leafwise holomorphic foliated diffeomorphic to the LC Reeb component $R' = \tilde{R}/(S \circ T \circ S^{-1})^{\mathbb{Z}}$, so we may assume that τ is equal to 1.

If $\tau = 1$ and $\lambda = \mu$, Condition (L) is written as

$$\text{(L1)}\quad \xi_1(\lambda z_1 + z_2,\, \lambda z_2,\, \varphi(x)) = \lambda \xi_1(z_1, z_2, x) + \xi_2(z_1, z_2, x),$$

$$\text{(L2)}\quad \xi_2(\lambda z_1 + z_2,\, \lambda z_2,\, \varphi(x)) = \lambda \xi_2(z_1, z_2, x).$$

First, we determine the function ξ_2 which satisfies (L2). By using the series expansion (4.1) of ξ_2, we rewrite (L2) as

$$\sum_{k+l\geq 0} a^2_{kl}(\varphi(x))(\lambda z_1 + z_2)^k (\lambda z_2)^l = \sum_{k+l\geq 0} \lambda a^2_{kl}(x) z_1^k z_2^l.$$

Now, we compare coefficients of $z_1^k z_2^l$'s terms on both sides.

On $z_1^0 z_2^0$'s term ($k = l = 0$), the equation is

$$a^2_{00}(\varphi(x)) = \lambda a^2_{00}(x), \;\; i.e.\; a^2_{00} \in \mathcal{Z}_{\varphi,\lambda}.$$

On $z_1^1 z_2^0$'s term ($k = 1$, $l = 0$), the equation is

$$\lambda a^2_{10}(\varphi(x)) = \lambda a^2_{10}(x).$$

Then, by (4.2) and Proposition 4.8 1), $a^2_{10}(x) = a^2_{10}(0) = 0$. On $z_1^0 z_2^1$'s term ($k = 0$, $l = 1$), the equation is

$$a^2_{10}(x) + \lambda a^2_{01}(\varphi(x)) = \lambda a^2_{01}(x).$$

Since a^2_{10} is equal to 0, this equation becomes

$$a^2_{01}(\varphi(x)) = a^2_{01}(x).$$

By (4.2) and Proposition 4.8 1), $a^2_{01}(x) = a^2_{01}(0) = 1$. If $k + l \geq 2$, the functional equation of $z_1^k z_2^l$'s term is

$$\sum_{j=0}^{l} \binom{k+l-j}{k} \lambda^{k+j} a^2_{k+l-j,\,j}(\varphi(x)) = \lambda a^2_{kl}(x).$$

In particular, if $k \geq 2$ and $l = 0$, this equation is

$$\lambda^k a^2_{k0}(\varphi(x)) = \lambda a^2_{k0}(x).$$

Then, by Proposition 4.8 2), $a^2_{k0}(x) = 0$. Now, we use an induction. If

$a^2_{k+l,0}, \dots, a^2_{k+2,l-2}$ and $a^2_{k+1,l-1}$ are identically equal to 0, then the equation of $z_1^k z_2^l$'s term is

$$\lambda^{k+l} a^2_{kl}(\varphi(x)) = \lambda a^2_{kl}(x).$$

By Proposition 4.8 2), $a^2_{kl}(x) = 0$. Thus, by induction on l, a^2_{kl} is identically equal to 0 for all $k + l \geq 2$. Therefore, the function ξ_2 satisfying (L2) takes the form

$$\xi_2(z_1, z_2, x) = z_2 + a^2_{00}(x), \quad a^2_{00} \in \mathcal{Z}_{\varphi,\lambda}.$$

Next, we determine the function ξ_1 which satisfies (L1). By using the series expansion (4.1) of ξ_1, we rewrite (L1) as

$$\sum_{k+l\geq 0} a^1_{kl}(\varphi(x))(\lambda z_1 + z_2)^k (\lambda z_2)^l = \sum_{k+l\geq 0} \lambda a^1_{kl}(x) z_1^k z_2^l + (z_2 + a^2_{00}(x)).$$

On $z_1^0 z_2^0$'s term ($k = l = 0$), the equation is

$$a^1_{00}(\varphi(x)) = \lambda a^1_{00}(x) + a^2_{00}(x).$$

On $z_1^1 z_2^0$'s term ($k = 1$, $l = 0$), the equation is

$$\lambda a^1_{10}(\varphi(x)) = \lambda a^1_{10}(x).$$

By (4.2) and Proposition 4.8 1), $a^1_{10}(x) = a^1_{10}(0) = 1$. On $z_1^0 z_2^1$'s term ($k = 0$, $l = 1$), the equation is

$$a^1_{10}(\varphi(x)) + \lambda a^1_{01}(\varphi(x)) = \lambda a^1_{01}(x) + 1.$$

Since a^1_{10} is identically equal to 1, this equation becomes

$$a^1_{01}(\varphi(x)) = a^1_{01}(x).$$

By (4.2) and Proposition 4.8 1), $a^1_{01}(x) = a^1_{01}(0) = 0$. If $k + l \geq 2$, the equation of $z_1^k z_2^l$'s term is

$$\sum_{j=0}^{l} \binom{k+l-j}{k} \lambda^{k+j} a^1_{k+l-j,\,j}(\varphi(x)) = \lambda a^1_{kl}(x),$$

which is the same as the equation of a^2_{kl} for $k+l \geq 2$, so that a^1_{kl} is identically equal to 0 for all $k+l \geq 2$. Therefore, g belongs to $Aut\,(\tilde{R}, \tilde{H}; T)$ iff it takes the following form.

$$g(z_1, z_2, x) = \big(z_1 + a^1_{00}(x),\ z_2 + a^2_{00}(x),\ \eta(x)\big),$$

where $a^1_{00},\ a^2_{00} \in C^\infty([0,\infty); \mathbb{C})$ satisfy the functional equations

$$a^1_{00}(\varphi(x)) = \lambda a^1_{00}(x) + a^2_{00}(x), \quad a^2_{00}(\varphi(x)) = \lambda a^2_{00}(x)$$

i.e. $(a^1_{00}, a^2_{00}) \in \mathcal{S}_{\varphi,\lambda}$ (see Section 3) and $\eta \in Z_\varphi$.

Case 5. $\tau \neq 0$ and $\lambda = \mu^p$ for some integer $p \geq 2$.

Let S be the diffeomorphism of $\tilde{R}$ given by $S(z_1, z_2, x) = (z_1, \tau^{1/p} z_2, x)$. Then, for any $(z_1, z_2, x) \in \tilde{R}$,

$$S \circ T \circ S^{-1}(z_1, z_2, x) = (\mu^p z_1 + z_2^p, \mu z_2, \varphi(x)).$$

Therefore, the LC Reeb component $R = \tilde{R}/T^{\mathbb{Z}}$ is leafwise holomorphic foliated diffeomorphic to the LC Reeb component $R' = \tilde{R}/(S \circ T \circ S^{-1})^{\mathbb{Z}}$, so we may assume that τ is equal to 1.

If $\tau = 1$ and $\lambda = \mu^p$, Condition (L) is written as

$$\text{(L1)} \quad \xi_1(\mu^p z_1 + z_2^p,\ \mu z_2,\ \varphi(x)) = \mu^p \xi_1(z_1, z_2, x) + \xi_2(z_1, z_2, x)^p,$$

$$\text{(L2)} \quad \xi_2(\mu^p z_1 + z_2^p,\ \mu z_2,\ \varphi(x)) = \mu \xi_2(z_1, z_2, x).$$

First, we determine the function ξ_2 which satisfies (L2). By using the series expansion (4.1) of ξ_2, we rewrite (L2) as

$$\sum_{k+l \geq 0} a^2_{kl}(\varphi(x))(\mu^p z_1 + z_2^p)^k (\mu z_2)^l = \sum_{k+l \geq 0} \mu a^2_{kl}(x) z_1^k z_2^l.$$

Now, we compare coefficients $z_1^k z_2^l$'s terms on both sides.

Let p and r be the integers such that $l = pq + r$, $q \geq 0$, $0 \leq r \leq q-1$. If $k \geq 1$ and $l \geq 0$, then the functional equation of $z_1^k z_2^l$'s term is

$$\sum_{j=0}^{q} \binom{k+j}{j} \mu^{pk} \mu^{p(q-j)+r} a^2_{k+j,p(q-j)+r}(\varphi(x)) = \mu a^2_{k,pq+r}(x).$$

In particular, if $k \geq 1$ and $q = 0$ ($l = r$), then the above equation is

$$\mu^{pk} \mu^{r} a^2_{kr}(\varphi(x)) = \mu a^2_{kr}(x).$$

Then, by Proposition 4.8 2), $a^2_{kr}(x) = 0$ for $k \geq 1$ and $0 \leq r \leq p-1$. Now, we use an induction. For $k \geq 1$ and $q \geq 1$, if $a^2_{k+q,r}$, $a^2_{k+q-1,q+r}, \dots$ and $a^2_{k+1,p(q-1)+r}$ are identically equal to 0, then the equation of $z_1^k z_2^{pq+r}$'s term is

$$\mu^{pk} \mu^{pq+r} a^2_{k,pq+r}(\varphi(x)) = \mu a^2_{k,pq+r}(x).$$

By Proposition 4.8 2), $a^2_{k,pq+r}(x) = 0$. Thus, by induction on q, a^2_{kl} is identically equal to 0 for $k \geq 1$ and $l \geq 0$. If $k = 0$ and $l \geq 0$, then the equation of $z_1^0 z_2^l$'s term is

$$\sum_{j=0}^{q} \mu^{p(q-j)+r} a^2_{j,p(q-j)+r}(\varphi(x)) = \mu a^2_{0,pq+r}(x).$$

If $q = r = 0$, then the above equation is

$$a^2_{00}(\varphi(x)) = \mu a^2_{00}(x), \ \ i.e.\ a^2_{00} \in \mathcal{Z}_{\varphi,\mu}.$$

If $q = 0$ and $r \neq 0$, then the equation of $z_1^0 z_2^r$'s term is

$$\mu^r a^2_{0r}(\varphi(x)) = \mu a^2_{0r}(x).$$

If $r = 1$, then $a^2_{01}(x) = a^2_{01}(0) = 1$ by (4.2) and Proposition 4.8 1), and if $2 \leq r \leq p-1$, then $a^2_{0r}(x) = 0$ by Proposition 4.8 2). If $q \geq 1$, $a^2_{1,p(q-1)+r}, \dots, a^2_{q-1,p+r}$ and a^2_{qr} are identically equal to 0, and so the equation of $z_1^0 z_2^{pq+r}$'s term is

$$\mu^{pq+r} a^2_{0,pq+r}(\varphi(x)) = \mu a^2_{0,pq+r}(x).$$

By Proposition 4.8 2), $a^2_{0,pq+r}(x) = 0$. Thus, a^2_{0l} is identically equal to 0 for $l \geq 0$. Therefore, the function ξ_2 satisfying (L2) takes the form

$$\xi_2(z_1, z_2, x) = z_2 + a^2_{00}(x), \quad a^2_{00} \in \mathcal{Z}_{\varphi,\mu}.$$

Next, we determine the function ξ_1 satisfying (L1). By using the series expansion (4.1) of ξ_1, we rewrite (L1) as

$$\sum_{k+l\geq 0} a^1_{kl}(\varphi(x))(\mu^p z_1 + z_2^p)^k(\mu z_2)^l = \sum_{k+l\geq 0} \mu^p a^1_{kl}(x) z_1^k z_2^l + (z_2 + a^2_{00}(x))^p.$$

If $k \geq 1$ and $l \geq 0$, then the functional equation of $z_1^k z_2^l$' term is

$$\sum_{j=0}^{q} \binom{k+j}{j} \mu^{pk} \mu^{p(q-j)+r} a^1_{k+j,p(q-j)+r}(\varphi(x)) = \mu^p a^1_{k,pq+r}(x).$$

In particular, if $k \geq 1$ and $q = 0$ $(l = r)$, then the above equation is

$$\mu^{pk}\mu^r a^1_{kr}(\varphi(x)) = \mu^p a^1_{kr}(x).$$

Thus, if $(k,r) = (1,0)$, $a^1_{10}(x) = a^1_{10}(0) = 1$ by (4.2) and Proposition 4.8 1), and if $(k,r) \neq (1,0)$, $a^1_{kr}(x) = 0$ by Proposition 4.8 2). Now, we use an induction. For $k \geq 1$ and $q \geq 1$, if $a^1_{k+q,r}, \ldots, a_{k+2,p(q-2)+r}$ and $a^1_{k+1,p(q-1)+r}$ are identically equal to 0, then the equation of $z_1^k z_2^{pq+r}$'s term is

$$\mu^{pk}\mu^{pq+r} a^1_{k,pq+r}(\varphi(x)) = \mu^p a^1_{k,pq+r}(x).$$

By Proposition 4.8 2), $a^1_{k,pq+r}(x) = 0$. Thus, by induction on q, a^1_{kl} is identically equal to 0 for $k \geq 1$, $l \geq 0$ and $k + l \geq 2$. If $k = 0$ and $l \geq 0$, then the functional equation of $z_1^0 z_2^l$'s term is the following.

$$\mu^l a^1_{0l}(\varphi(x)) = \mu^p a^1_{0l}(x) + \binom{p}{l} a^2_{00}(x)^{p-l} \quad (\text{if } 0 \leq l \leq p-1)$$

$$a^1_{10}(\varphi(x)) + \mu^p a^1_{0p}(\varphi(x)) = \mu^p a^1_{0p}(x) + 1 \quad (\text{if } l = p)$$

$$\sum_{j=0}^{q} \mu^{p(q-j)+r} a^1_{j,p(q-j)+r}(\varphi(x)) = \mu^p a^2_{0,pq+r}(x) \quad (\text{if } l \geq p+1)$$

If $0 \leq l \leq p-1$, by dividing both sides by μ^l, the equation is reduced to

$$a^1_{0l}(\varphi(x)) = \mu^{p-l} a^1_{00}(x) + c_l a^2_{00}(x)^{p-l}, \quad c_l = \binom{p}{l} \mu^{-l}.$$

If $l = p$, the equation is reduced to

$$a^1_{0p}(\varphi(x)) = a^1_{0p}(x),$$

because a^1_{10} is identically equal to 1. By (4.2) and Proposition 4.8 1), $a^1_{0p}(x) = a^1_{0p}(0) = 0$. If $l \geq p+1$, q is not equal to 0. Then, $a^1_{1,p(q-1)+r}$, $\ldots, a^1_{q-1,p+r}$ and a^1_{qr} are identically equal to 0, and so the equation is reduced to

$$\mu^{pq+r} a^1_{0,pq+r}(\varphi(x)) = \mu^p a^1_{0,pq+r}(x).$$

By Proposition 4.8 2), $a^1_{0,pq+r}(x) = 0$ for $l = pq + r \geq p + 1$. Therefore, g belongs to $Aut\,(\tilde{R}, \tilde{H}; T)$ iff it takes the following form.

$$g(z_1, z_2, x) = \left(z_1 + \sum_{j=0}^{p-1} a^1_{0j}(x) z_2^j,\ z_2 + a^2_{00}(x),\ \eta(x) \right),$$

where $\eta \in Z_\varphi$ and a^1_{0j} $(j = 0, 1, \ldots, p-1)$, $a^2_{00} \in C^\infty([0,\infty); \mathbb{C})$ satisfy the functional equations

$$a^1_{0j}(\varphi(x)) = \mu^{p-j} a^1_{0j}(x) + c_j a^2_{00}(x)^{p-j}, \quad a^2_{00}(\varphi(x)) = \mu a^2_{00}(x),$$

where $c_j = \binom{p}{j} \mu^{-j}$.

Let us consider the following system of functional equations on $\beta_{1,0}, \ldots, \beta_{1,p-1}$ and $\beta_2 \in C^\infty([0,\infty); \mathbb{C})$ concerning φ and μ.

$$\text{Equation (III)}: \quad \begin{aligned} \beta_{1,0}(\varphi(x)) &= \mu^p \beta_{1,0}(x) + c_0 \{\beta_2(x)\}^p, \\ \beta_{1,1}(\varphi(x)) &= \mu^{p-1} \beta_{1,1}(x) + c_1 \{\beta_2(x)\}^{p-1}, \\ &\vdots \\ \beta_{1,p-1}(\varphi(x)) &= \mu \beta_{1,p-1}(x) + c_{p-1}\, \beta_2(x), \\ \beta_2(\varphi(x)) &= \mu \beta_2(x). \end{aligned}$$

where $c_0, \ldots, c_{p-1} \in \mathbb{C}$ are non-zero constants. Let us take the space $\mathcal{S}_{\varphi,\mu}(\mathbf{c})$ of solutions to Equation (III) on $(0,\infty)$, where $\mathbf{c} = (c_0, \ldots, c_{p-1}) \in \mathbb{C}^p$. If we assign β_2 to a solution $(\beta_{1,0}, \ldots, \beta_{1,p-1}, \beta_2) \in \mathcal{S}_{\varphi,\mu}(\mathbf{c})$, we obtain the projection $P_2 : \mathcal{S}_{\varphi,\mu}(\mathbf{c}) \to \mathcal{Z}_{\varphi,\mu}$. Here the kernel of P_2 is nothing but $\mathcal{Z}_{\varphi,\mu^p} \times \cdots \times \mathcal{Z}_{\varphi,\mu}$. If we fix any solution $\beta^* \in \mathcal{Z}_{\varphi,\mu}$ which never vanishes, we also see that the projection P_2 is surjective because for any $\beta_2 \in \mathcal{Z}_{\varphi,\mu}$

$$\beta_{1,j}(x) = \frac{c_j}{\mu^{p-j} \log \mu^{p-j}} \beta_2(x)^{p-j} \log\{\beta^*(x)^{p-j}\} \quad (j = 0, 1, \ldots, p-1)$$

gives a solution $(\beta_{1,0}, \ldots, \beta_{1,p-1}, \beta_2) \in \mathcal{S}_{\varphi,\mu}(\mathbf{c})$, where for $\log\{\beta^*(x)^{p-j}\}$ any smooth branch can be taken. Therefore, as a vector space, $\mathcal{S}_{\varphi,\mu}(\mathbf{c})$ has a structure such that

$$0 \to \mathcal{Z}_{\varphi,\mu^p} \times \cdots \times \mathcal{Z}_{\varphi,\mu} \to \mathcal{S}_{\varphi,\mu}(\mathbf{c}) \to \mathcal{Z}_{\varphi,\mu} \to 0$$

is a short exact sequence.

Any solution $(\beta_{1,0}, \ldots, \beta_{1,p-1}, \beta_2) \in \mathcal{S}_{\varphi,\mu}(\mathbf{c})$ extends to $[0, \infty)$ so as to be smooth functions which are flat at $x = 0$ because $(\beta_{1,j},\, c_j \beta_2^{p-j}) \in \mathcal{S}_{\varphi,\mu^{p-j}}$ for $j = 0, 1, \ldots, p-1$ (see Theorem 3.1 in [HM]).

Consequently, we obtain the following theorem.

Theorem 4.10 1) *If $\tau = 1$ and $\lambda = \mu$, then $Aut\,(R, H)$ is isomorphic to a semi-direct product*

$$\mathcal{S}_{\varphi,\lambda} \rtimes Z_\varphi$$

where $\eta \in Z_\varphi$ acts on $(\beta_1, \beta_2) \in \mathcal{S}_{\varphi,\lambda}$ by $(\beta_1(x), \beta_2(x)) \mapsto (\beta_1(\eta(x)), \beta_2(\eta(x)))$.

2) *If $\tau = 1$ and $\lambda = \mu^p$ for some integer $p \geq 2$, then the kernel $\mathcal{K}$ of the surjective homomorphism $Aut\,(R, H) \to Z_\varphi$ is isomorphic to the infinite dimensional vector space $\mathcal{S}_{\varphi,\mu}(\mathbf{c})$, where $\mathbf{c} = (c_0, \ldots, c_{p-1}) \in \mathbb{C}^p$, $c_j = \binom{p}{j}\mu^{-j}$ $(j = 0, \ldots, p-1)$.*

5. Higher dimensional Reeb components

To close this article, we make some remarks on the automorphisms of higher dimensional LC Reeb components. Let $\tilde{R}$ be $\mathbb{C}^n \times [0, \infty) \setminus \{(O, 0)\}$ and $\varphi \in \mathit{Diff}^\infty([0, \infty))$ be a diffeomorphism which is tangent to the identity to the infinite order at 0 and satisfies $\varphi(x) - x > 0$ for $x > 0$. Also take an automorphism $G \in Aut\,(\mathbb{C}^n, \{O\})$ which is expanding. Let T be a diffeomorphism given by $T = G \times \varphi$. Then we obtain an LC Reeb component

$(R, \mathcal{F}, J) = (\tilde{R}, \tilde{\mathcal{F}}, J_{\mathrm{std}})/T^{\mathbb{Z}}$ as the quotient, as well as the boundary Hopf manifold $H = \mathbb{C}^n \backslash \{O\}/G^{\mathbb{Z}}$.

Let $Aut\, R$ be the group of leafwise holomorphic smooth automorphisms of R and $Aut\, H$ be the group of holomorphic automorphisms of the boundary Hopf manifold H. Then, by the similar argument, we can obtain the claims as Proposition 4.2, 4.3 and 4.7.

Proposition 5.1 *For $n \geq 1$, $Aut\, R$ and $Aut\,(R, H)$ are isomorphic to $Aut\,(\tilde{R}; T)/T^{\mathbb{Z}}$ and $Aut\,(\tilde{R}, \tilde{H}; T)$ respectively.*

Proposition 5.2 *For $n \geq 2$, the restriction map $r_H : Aut\, R \to Aut\, H$ is surjective.*

If G is in some normal forms like in Theorem 2.2, then we can most probably compute the group $Aut\, H$ and the kernel $Aut\,(R, H)$ of the restriction map r_H for $n > 2$. However, in order to determine these groups in all cases, we need to classify Hopf manifolds and to give normal forms like in Theorem 2.2.

Example 5.3 Let us look at the simplest case, where G is a diagonal matrix $\lambda I_n \in GL(n; \mathbb{C})$ with $|\lambda| > 1$. Then the automorphism group $Aut\, R$ admits the following sequence of extensions.

$$1 \to Aut\,(R, H) \to Aut\, R \to Aut\, H \cong GL(n; \mathbb{C})/\{\lambda I_n\}^{\mathbb{Z}} \to 1,$$
$$0 \to (\mathcal{Z}_{\varphi,\lambda})^n \to Aut(R, H) \cong (\mathcal{Z}_{\varphi,\lambda})^n \rtimes Z_\varphi \to Z_\varphi \to 1$$

where $\eta \in Z_\varphi$ acts on $(\beta_1, \ldots, \beta_n) \in (\mathcal{Z}_{\varphi,\lambda})^n$ by $(\beta_1(x), \ldots, \beta_n(x)) \mapsto (\beta_1(\eta(x)), \ldots, \beta_n(\eta(x)))$.

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Tomohiro Horiuchi
Department of Mathematics
Chuo University
1-13-27 Kasuga Bunkyo-ku, Tokyo 112-8551, Japan
E-mail : horiuchi@gug.math.chuo-u.ac.jp