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Characterizations of three homogeneous real hypersurfaces in a complex projective space

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Abstract. In an n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature $c(< 0)$, the horosphere HS, which is defined by HS $= \lim_{r\to\infty} G(r)$, is one of nice examples in the class of real hypersurfaces. Here, $G(r)$ is a geodesic sphere of radius r $(0 < r < \infty)$ in $\mathbb{C}H^n(c)$. The second author ([14]) gave a geometric characterization of HS. In this paper, motivated by this result, we study real hypersurfaces M^{2n-1} isometrically immersed into an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c(> 0)$.

Key words: geodesic spheres, homogeneous real hypersurfaces of types (A_2) and type B, complex projective spaces, contact form, exterior derivative, geodesics, extrinsic geodesics, circles, characteristic vector fields.

1. Introduction

We denote by $\widetilde{M}_n(c)$ a complex n-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature $c(\neq 0)$, namely it is holomorphically isometric to either $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$ according as c is positive or negative, which is called an n-dimensional *nonflat complex space form* of constant holomorphic sectional curvature c.

We consider a real hypersurface M^{2n-1} (with Riemannian metric g) in a nonflat complex space form $\widetilde{M}_n(c), n \geq 2$ through an isometric immersion. In the theory of real hypersurfaces in $\widetilde{M}_n(c)$, Hopf hypersurfaces all of whose principal curvatures are constant are fundamental examples (for the definition of Hopf hypersurfaces see Section 2). They are homogeneous in the ambient space $\widetilde{M}_n(c)$, namely they are orbits of some subgroups of the full isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of $\widetilde{M}_n(c)$.

The horosphere HS is a typical example of a Hopf hypersurface with constant principal curvatures in $\mathbb{C}H^n(c)$. The second author gave the following characterization of the horosphere HS in $\mathbb{C}H^n(c)$:

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Theorem A ([14]) *For a real hypersurface M^{2n-1} isometrically immersed into $\mathbb{C}H^n(c), n \geq 2$, the following three conditions are mutually equivalent*:

(1) *M is locally congruent to the horosphere* HS (*i.e., a homogeneous real hypersurface of type* (A_0));
(2) *At every point $p \in M$, there exist orthonormal vectors $v_1, \ldots, v_{2n-2}$ orthogonal to the characteristic vector ξ_p such that all geodesics $\gamma_i = \gamma_i(s)\,(1 \leq i \leq 2n-2)$ satisfying the initial condition that $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ are mapped to a circle of the same positive curvature $\sqrt{|c|}/2$ in the ambient space $\mathbb{C}H^n(c)$*;
(3) *M satisfies either $d\eta(X,Y) = (\sqrt{|c|}/2)g(X, \phi Y)$ for all $X, Y \in TM$ or $d\eta(X,Y) = -(\sqrt{|c|}/2)g(X, \phi Y)$ for all $X, Y \in TM$, where $d\eta$ is the exterior derivative of the contact form η on M and ϕ is the structure tensor on M induced from the Kähler structure J of $\mathbb{C}H^n(c)$.*

Here, $d\eta$ is given by

$$d\eta(X,Y) = (1/2)\{X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])\} \quad \text{for } X, Y \in TM. \tag{1.1}$$

Inspired by Theorem A, in this paper we establish the following four theorems on real hypersurfaces in $\mathbb{C}P^n(c)$:

Theorem 1 *A real hypersurface M^{2n-1} isometrically immersed into $\mathbb{C}P^n(c)$, $n \geq 2$ is locally congruent to either a geodesic sphere $G(\pi/(2\sqrt{c}\,))$ of radius $\pi/(2\sqrt{c}\,)$ (i.e., a homogeneous real hypersurface of type* (A_1) *of radius $\pi/(2\sqrt{c}\,)$) or a tube $T_1(\pi/(2\sqrt{c}\,))$ of radius $\pi/(2\sqrt{c}\,)$ around a complex ℓ-dimensional totally geodesic submanifold $\mathbb{C}P^\ell(c)\,(1 \leq \ell \leq n-2)$ (i.e., a homogeneous real hypersurface of type* (A_2) *of radius $\pi/(2\sqrt{c}\,)$) if and only if at every point $p \in M$, there exist orthonormal vectors $v_1, \ldots, v_{2n-2}$ orthogonal to the characteristic vector ξ_p such that all geodesics $\gamma_i = \gamma_i(s)\,(1 \leq i \leq 2n-2)$ satisfying the initial condition that $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ are mapped to a circle of the same positive curvature $\sqrt{c}/2$ in the ambient space $\mathbb{C}P^n(c)$.*

Theorem 2 *A real hypersurface M^{2n-1} isometrically immersed into $\mathbb{C}P^n(c)$, $n \geq 2$ is locally congruent to a geodesic sphere $G(\pi/(2\sqrt{c}\,))$ of radius $\pi/(2\sqrt{c}\,)$ if and only if at every point $p \in M$, there exist orthonormal vectors $v_1, \ldots, v_{2n-2}$ orthogonal to the characteristic vector ξ_p such that all geodesics $\gamma_i = \gamma_i(s)\,(1 \leq i \leq 2n-2)$ satisfying the initial condition that*

$\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ are mapped to a circle of the same positive curvature $\sqrt{c}/2$ in $\mathbb{C}P^n(c)$ and there exists just one extrinsic geodesic on M (i.e., this geodesic is also a geodesic in $\mathbb{C}P^n(c)$) with respect to the full isometry group $\mathrm{I}(M)$ of M.

Theorem 3 *A real hypersurface M^{2n-1} isometrically immersed into $\mathbb{C}P^n(c)$, $n \geq 2$ is locally congruent to either a geodesic sphere $G(\pi/(2\sqrt{c}\,))$ of radius $\pi/(2\sqrt{c}\,)$ or a tube $T_2(r)$ of radius r with $\cot(\sqrt{c}\,r/2) = \sqrt{2}\,+1$ around a complex hyperquadric $\mathbb{C}Q^{n-1}$ (i.e., a homogeneous real hypersurface of type (B) of radius $(2/\sqrt{c}\,)\cot^{-1}(\sqrt{2}\,+1)$ if and only if M satisfies either $d\eta(X,Y) = (\sqrt{c}/2)g(X,\phi Y)$ for all $X,Y \in TM$ or $d\eta(X,Y) = -(\sqrt{c}/2)g(X,\phi Y)$ for all $X,Y \in TM$, where $d\eta$ is the exterior derivative of the contact form η on M and ϕ is the structure tensor on M induced from the Kähler structure J of $\mathbb{C}P^n(c)$.*

Theorem 4 *A real hypersurface M^{2n-1} isometrically immersed into $\mathbb{C}P^n(c)$, $n \geq 2$ is locally congruent to a geodesic sphere $G(\pi/(2\sqrt{c}\,))$ of radius $\pi/(2\sqrt{c}\,)$ if and only if M satisfies either $d\eta(X,Y) = (\sqrt{c}/2)g(X,\phi Y)$ for all $X,Y \in TM$ or $d\eta(X,Y) = -(\sqrt{c}/2)g(X,\phi Y)$ for all $X,Y \in TM$ and M is positively curved at some point $x \in M$ (i.e., every sectional curvature of M is positive at $x \in M$).*

We remark that for a real hypersurface M^{2n-1} isometrically immersed into a nonflat complex space form $\widetilde{M}_n(c), n \geq 2$ the following hold:

(1) There does not exist a real hypersurface M all of whose geodesics are mapped to circles in $\widetilde{M}_n(c)$.
(2) There does not exist a real hypersurface M satisfying $d\eta \equiv 0$ on M.

Weakening the above two conditions, we establish all of our results Theorems A, 1, 2, 3 and 4.

In section 8, we will show that a real hypersurface M isometrically immersed in $\mathbb{C}P^n(4)$ is locally congruent to a geodesic hypersphere $G(r)$ of radius $r \in (\pi/4, \pi/2)$ if and only if there exists $\alpha \in (0,\pi)$, $\alpha \neq \pi/2$ such that for each point $p \in M$ and each unit tangent vector $X_p \in T_p(M)$ with $g(X_p, \xi_p) = \cos\alpha$, the geodesic γ of M satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = X$ is an extrinsic geodesic (see Theorem 5).

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2. Terminologies and fundamental results on real hypersurfaces

Let M^{2n-1} be a real hypersurface with unit normal local vector field $\mathcal{N}$ of a nonflat complex space form $\widetilde{M}_n(c), n \geq 2$. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by the following:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N} \tag{2.1}$$

and

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for all vector fields X and Y on M, where g denotes the metric induced from the standard Riemannian metric of $\widetilde{M}_n(c)$ and A is the shape operator of M in $\widetilde{M}_n(c)$ associated with $\mathcal{N}$. On M an almost contact metric structure (ϕ, ξ, η, g) associated with $\mathcal{N}$ is canonically induced from the Kähler structure J of the ambient space $\widetilde{M}_n(c)$. They are defined by

$$g(\phi X, Y) = g(JX, Y),\ \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

It follows from the Gauss formula (2.1), the Weingarten formula (2.2) and the property $\widetilde{\nabla} J = 0$ that

$$\nabla_X \xi = \phi AX \tag{2.3}$$

and

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.4}$$

for each $X \in TM$. We denote by R the curvature tensor of M. Then R is given by

$$\begin{aligned} g((R(X,Y)Z, W) = (c/4)\{&g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &+ g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) \\ &- 2g(\phi X, Y)g(\phi Z, W)\} \\ &+ g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W). \end{aligned} \tag{2.5}$$

The following is called the equation of Codazzi.

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi). \quad (2.6)$$

Let K be the sectional curvature of M. That is, K is defined by $K(X,Y) = g(R(X,Y)Y,X)$, where X and Y are orthonormal vectors on M. Then it follows from (2.5) that

$$K(X,Y) = (c/4)\big(1+3g(\phi X,Y)^2\big) + g(AX,X)g(AY,Y) - g(AX,Y)^2. \quad (2.7)$$

We call eigenvalues and eigenvectors of the shape operator A *principal curvatures* and *principal curvature vectors* of M in $\widetilde{M}_n(c)$, respectively. Here and in the following, we set $V_\lambda := \{X \in TM | AX = \lambda X\}$. We usually call M a *Hopf hypersurface* if the characteristic vector ξ of M is a principal curvature vector at each point of M. The following lemma clarifies fundamental properties of principal curvatures of a Hopf hypersurface M in $\widetilde{M}_n(c)$ (for examples, see [17]).

Lemma A *Let M be a Hopf hypsurface of a nonflat complex space form $\widetilde{M}_n(c), n \geq 2$. Then the following hold.*

(1) *If a nonzero vector $v \in TM$ orthogonal to ξ satisfies $Av = \lambda v$, then $(2\lambda - \delta)A\phi v = (\delta\lambda + (c/2))\phi v$, where δ is the principal curvature associated with ξ. In particular, when $c > 0$, we have $A\phi v = \big((\delta\lambda + (c/2))/(2\lambda - \delta)\big)\phi v$.*
(2) *The principal curvature δ associated with ξ is constant locally on M.*

Remark 1 When $c < 0$, the horosphere HS in $\mathbb{C}H^n(c)$ shows that we must consider the case of $2\lambda - \delta = \delta\lambda + (c/2) = 0$ in Lemma A(1) (see the following table of the principal curvatures in the case of $c < 0$).

We here recall the classification theorems of Hopf hypersurfaces with constant principal curvatures in a nonflat complex space form $\widetilde{M}_n(c), n \geq 2$.

Theorem B ([18], [12]) *For real hypersurface M^{2n-1} in $\mathbb{C}P^n(c)$ $(n \geq 2)$, the following three conditions are mutually equivalent.*

(1) *M is homogeneous in $\mathbb{C}P^n(c)$.*
(2) *M is locally congruent to a Hopf hypersurface all of whose principal curvatures are constant.*
(3) *M is locally congruent to one of the following*:
   (A_1) *a geodesic sphere of radius r, where $0 < r < \pi/\sqrt{c}$;*

(A_2) *a tube of radius r around a totally geodesic* $\mathbb{C}P^{\ell}(c)$ $(1 \leq \ell \leq n-2)$, *where* $0 < r < \pi/\sqrt{c}$;

(B) *a tube of radius r around a complex hyperquadric* $\mathbb{C}Q^{n-1}$, *where* $0 < r < \pi/(2\sqrt{c}\,)$;

(C) *a tube of radius r around the Segre embedding of* $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, *where* $0 < r < \pi/(2\sqrt{c}\,)$ *and* $n\ (\geq 5)$ *is odd*;

(D) *a tube of radius r around the Plüker embedding of a complex Grassmannian* $\mathbb{C}G_{2,5}$, *where* $0 < r < \pi/(2\sqrt{c}\,)$ *and* $n = 9$;

(E) *a tube of radius r around a Hermitian symmetric space* $\mathrm{SO}(10)/\mathrm{U}(5)$, *where* $0 < r < \pi/(2\sqrt{c}\,)$ *and* $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Unifying real hypersurfaces of types (A_1) and (A_2), we call them hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows:

	(A_1)	(A_2)	(B)	(C, D, E)
λ_1	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r\bigr)$	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r\bigr)$	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\bigr)$	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\bigr)$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan\bigl(\frac{\sqrt{c}}{2}r\bigr)$	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r+\frac{\pi}{4}\bigr)$	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r+\frac{\pi}{4}\bigr)$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r\bigr)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan\bigl(\frac{\sqrt{c}}{2}r\bigr)$
δ	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$

Theorem C ([8]) *Let M be a connected Hopf hypersurface all of whose principal curvatures are constant in* $\mathbb{C}H^n(c)$ $(n \geq 2)$. *Then M is locally congruent to one of the following homogeneous real hypersurfaces*:

(A_0) *the horosphere* HS *in* $\mathbb{C}H^n(c)$;

($A_{1,0}$) *a geodesic sphere* $G(r)$ *of radius* r $(0 < r < \infty)$;

($A_{1,1}$) *a tube of radius r around a totally geodesic* $\mathbb{C}H^{n-1}(c)$, *where* $0 < r < \infty$;

(A_2) *a tube of radius r around a totally geodesic* $\mathbb{C}H^{\ell}(c)$ $(1 \leq \ell \leq n-2)$, *where* $0 < r < \infty$;

(B) *a tube of radius r around a totally real totally geodesic* $\mathbb{R}H^n(c/4)$, *where* $0 < r < \infty$.

Remark 2 There exist many non-Hopf homogeneous real hypersurfaces M in $\mathbb{C}H^n(c)$, $n \geq 2$ (see Theorem 4.4 in [10]). Needless to say, these homogeneous real hypersurfaces have constant principal curvatures (for details, see [9]).

Here, type (A_1) means either type ($A_{1,0}$) or type ($A_{1,1}$). Unifying real hypersurfaces of types (A_0), (A_1) and (A_2), we call them hypersurfaces of type (A). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$ has two distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3}\,)$. Except for this real hypersurface, the numbers of distinct principal curvatures of Hopf hypersurfaces with constant principal curvatures are $2, 2, 2, 3, 3$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}H^n(c)$ are given as follows (see [7]):

	(A_0)	($A_{1,0}$)	($A_{1,1}$)	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$
δ	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}\,r)$

For the later use we prepare the following lemma (cf. [15], [17]):

Lemma B *For a real hypersurface M isometrically immersed into a nonflat complex space form $\widetilde{M}_n(c), n \geq 2$ the following three conditions are mutually equivalent*:

(1) *M is of type* (A);
(2) $\phi A = A\phi$;
(3) $g((\nabla_X A)Y, Z) = (c/4)\{-\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y)\}$ *for all X, Y and $Z \in TM$.*

3. Circles in Riemannian geometry

First of all we review the definition of the congruency for a smooth real curve $\gamma = \gamma(s)$ parametrized by its arclength s on a Riemannian manifold N. Two curves γ_1 and γ_2 are *congruent* if there exists an isometry φ on N with $\gamma_2(s) = (\varphi \circ \gamma_1)(s + s_0)$ for each s and some s_0.

Before proving Theorem 1 we recall the definition of circles in Riemannian geometry and the congruency theorem on circles in a nonflat complex space form $\widetilde{M}_n(c)$, $n \geq 2$.

Let $\gamma = \gamma(s)$ be a smooth real curve parametrized by its arclength s

on a Riemannian manifold N with Riemannian metric g. If the curve γ satisfies the following ordinary differential equations with some nonnegative constant k:

$$\nabla_{\dot\gamma}\dot\gamma = kY_s \quad \text{and} \quad \nabla_{\dot\gamma}Y_s = -k\dot\gamma, \tag{3.1}$$

where $\nabla_{\dot\gamma}$ is the covariant differentiation along γ with respect to ∇ of N and Y_s is the so-called the unit principal normal vector of γ, we call γ a circle of curvature k on N. We regard a geodesic as a circle of null curvature. It is known that Equation (3.1) is equivalent to

$$\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) + g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma)\dot\gamma = 0. \tag{3.2}$$

By virtue of the existence and the uniqueness of solutions to ordinary differential equations we can see that for each point $p \in N$, an arbitrary positive constant k and every pair of orthonormal vectors X and Y of T_pN, there exists locally the unique circle $\gamma = \gamma(s)$ on N satisfying the initial condition that $\gamma(0) = p, \dot\gamma(0) = X$ and $Y_0 = Y$.

Let $\gamma = \gamma(s)$ be a circle of positive curvature k on $\widetilde{M_n}(c)$. For the curve γ we set $\rho_\gamma := g(\dot\gamma(s), JY_s)$. Then it follows from Equation (3.1) and the equality $\widetilde\nabla J = 0$ that $\dot\gamma\rho_{\dot\gamma} = 0$ (see [5], [3]). So, ρ_γ is a constant along γ with $-1 \leq \rho_\gamma \leq 1$. In the following, we call ρ_γ the *structure torsion* of γ. The congruency theorem for circles in $\widetilde{M_n}(c)$ is stated as follows:

Lemma C ([5], [3]) *In a nonflat complex space form $\widetilde{M_n}(c), n \geq 2$, two circles $\gamma_i = \gamma_i(s)$ of curvature k_i and the structure torsion ρ_{γ_i} are congruent if and only if one of the following two conditions holds*:

(1) $k_1 = k_2 = 0$;
(2) $k_1 = k_2 > 0$ *and* $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$.

We remark that in Lemma C(2) when $\rho_{\gamma_1} = \rho_{\gamma_2}$ (resp. $\rho_{\gamma_1} = -\rho_{\gamma_2}$) circles γ_1 and γ_2 of the same positive curvature are congruent by a holomorphic (resp. an anti-holomorphic) isometry of a nonflat complex space form. For a circle γ of positive curvature we call γ a *Kähler circle* (resp. *totally real circle*) when $\rho_\gamma = \pm 1$ (resp. $\rho_\gamma = 0$).

4. Proof of Theorem 1

($\Longleftarrow$) Take orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ at any fixed point p of a real hypersurface M in $\mathbb{C}P^n(c), n \geq 2$ satisfying the assumption. Then, from (3.2) those curves $\gamma_i = \gamma_i(s)$ $(1 \leq i \leq 2n-2)$ satisfy

$$\widetilde{\nabla}_{\dot{\gamma}_i}(\widetilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i) = -(c/4)\dot{\gamma}_i. \tag{4.1}$$

On the other hand, from Gauss formula (2.1) and the Weingarten formula (2.2) we have

$$\widetilde{\nabla}_{\dot{\gamma}_i}(\widetilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i) = g((\nabla_{\dot{\gamma}_i}A)\dot{\gamma}_i, \dot{\gamma}_i)\mathcal{N} - g(A\dot{\gamma}_i, \dot{\gamma}_i)A\dot{\gamma}_i. \tag{4.2}$$

Comparing the tangential components of (4.1) and (4.2), we find that

$$g(A\dot{\gamma}_i, \dot{\gamma}_i)A\dot{\gamma}_i = (c/4)\dot{\gamma}_i,$$

so that at $s = 0$ we get

$$g(Av_i, v_i)Av_i = (c/4)v_i \quad \text{for } 1 \leq i \leq 2n-2.$$

This implies that

$$Av_i = (\sqrt{c}/2)v_i \quad \text{or} \quad Av_i = -(\sqrt{c}/2)v_i \quad \text{for } 1 \leq i \leq 2n-2. \tag{4.3}$$

Hence ξ is a principal curvature vector because $g(A\xi, v_i) = g(\xi, Av_i) = 0$ for $1 \leq i \leq 2n-2$. Then M is a Hopf hypersurface with at most three distinct constant principal curvatures $\sqrt{c}/2, -\sqrt{c}/2$ and $\delta = g(A\xi, \xi)$. Therefore in view of Theorem B and the table of the principal curvatures of case $c > 0$ we can see that our real hypersurface M is a real hypersurface of type (A) of radius $\pi/(2\sqrt{c}\,)$ or a certain real hypersurface of type (B). But, every real hypersurface M of type (B) does not have such principal curvatures $\pm\sqrt{c}/2$. In fact,

$$\begin{aligned} \lambda_1 &= \frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right) < -\frac{\sqrt{c}}{2}, \\ 0 < \lambda_2 &= \frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right) < \frac{\sqrt{c}}{2}, \end{aligned} \tag{4.4}$$

since $0 < r < \pi/(2\sqrt{c}\,)$. Thus we can see that M is of type (A) of radius $\pi/(2\sqrt{c}\,)$.

($\Longrightarrow$) Our aim here is to prove the following lemma:

Lemma 1 *For every real hypersurface M of type* (A) *in a nonflat commplex space form $\widetilde{M}_n(c), n \geq 2$, take a unit principal curvature vector v with principal curvature λ which is perpendicular to ξ_p at an arbitrary fixed point $p \in M$. Then the geodesic $\gamma = \gamma(s)$ with initial condition that $\gamma(0) = p$ and $\dot\gamma(0) = v$ is mapped to a totally real circle of positive curvature $|\lambda|$ in the ambient space $\widetilde{M}_n(c)$.*

Proof of Lemma 1. Let $\gamma = \gamma(s)$ be a geodesic satisfying the assumption of Lemma 1. We call $\rho_\gamma := g(\dot\gamma(s), \xi_{\gamma(s)})$ the structure torsion of the geodesic γ on a real hypersurface of type (A). Then ρ_γ is constant along γ. Indeed, from (2.3), Lemma B, the symmetry of A and the skew symmetry of ϕ we have $\dot\gamma\rho_\gamma = \nabla_{\dot\gamma}(g(\dot\gamma, \xi)) = g(\dot\gamma, \nabla_{\dot\gamma}\xi) = g(\dot\gamma, \phi A\dot\gamma) = g(\dot\gamma, A\phi\dot\gamma) = -g(\phi A\dot\gamma, \dot\gamma) = 0$. This, together with the hypothesis $g(\dot\gamma(0), \xi_p) = g(v, \xi_p) = 0$, implies that our geodesic $\gamma = \gamma(s)$ is orthogonal to the characteristic vector field $\xi_{\gamma(s)}$ along the curve γ. Furthermore, from the above fact and Lemma B we obtain $\dot\gamma\big(\|A\dot\gamma(s) - \lambda\dot\gamma(s)\|^2\big) = 0$, which, combined with $A\dot\gamma(0) = Av = \lambda v = \lambda\dot\gamma(0)$, implies that our geodesic γ satisfies $A\dot\gamma(s) = \lambda\dot\gamma(s)$ for every s. Thus, by virtue of Gauss formula (2.1) and the Weingarten formula (2.2) we have $\widetilde\nabla_{\dot\gamma}\dot\gamma = \lambda\mathcal{N}$, $\widetilde\nabla_{\dot\gamma}\mathcal{N} = -\lambda\dot\gamma$ and $\rho_\gamma = g(\dot\gamma(s), J\mathcal{N}) = -g(\dot\gamma(s), \xi_{\gamma(s)}) = 0$. Therefore we obtain the desired conclusion of Lemma 1. □

We next return to the discussion in the proof of Theorem 1. Since our real hypersurface M is of type (A) of radius $\pi/(2\sqrt{c}\,)$, from Lemma 1 and the table of the principal curvatures in the case of $c > 0$, at any fixed point $p \in M$ we can see that all geodesics $\gamma_i = \gamma_i(s)\,(1 \leq i \leq 2n-2)$ on M with initial condition that $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$ are mapped to the totally real circle of curvature $\sqrt{c}/2$ in $\mathbb{C}P^n(c)$, where $v_1, v_2, \ldots, v_{2n-2}$ are orthonormal principal curvature vectors orthogonal to the characteristic vector ξ_p. Hence we have proved Theorem 1.

5. Proof of Theorem 2

We first recall the congruence theorem for geodesics on a real hypersurface M of type (A) in a nonflat complex space form. For a geodesic $\gamma = \gamma(s)$ on a real hypersurface M^{2n-1} of type (A) in a nonflat complex space form $\widetilde{M}_n(c), n \geq 2$, we call $\rho_\gamma = g(\dot\gamma, \xi)$ the structure torsion of γ. Similarly, by the discussion in the proof of Lemma C and (2.3) we know that ρ_γ is constant along γ. Indeed, from Lemma B we have $\dot\gamma\rho_\gamma = \nabla_{\dot\gamma}(g(\dot\gamma, \xi)) = g(\dot\gamma, \phi A\dot\gamma) = -g(\phi A\dot\gamma, \dot\gamma) = 0$.

For geodesics on a real hypersurface which is either of type (A_0) or type (A_1), we can classify them by means of their structure torsions (see Proposition 2.3 in [6]).

Lemma D *On a real hypersurface M which is either of type* (A_0) *or type* (A_1) *in a nonflat complex space form $\widetilde{M}_n(c), n \geq 2$, two geodesics γ_1, γ_2 are congruent to each other with respect to the full isometry group* $\mathrm{I}(M)$ *of M if and only if their structure torsions ρ_{γ_1} and ρ_{γ_2} satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$.*

To obtain a congruence theorem for geodesics on a real hypersurface M of type (A_2) in $\widetilde{M}_n(c)$, we need another invariant. For a geodesic γ on a real hypersurface of type (A) in $\widetilde{M}_n(c)$ we define its *normal curvature* κ_γ by $\kappa_\gamma = g(A\dot\gamma, \dot\gamma)$. By Lemma B we have $\nabla_{\dot\gamma}\kappa_{\dot\gamma} = g((\nabla_{\dot\gamma(s)}A)\dot\gamma(s), \dot\gamma(s)) = 0$, whic yields that κ_γ is constant along γ. The following lemma shows that geodesics on real hypersurface of type (A_2) are classified by means of their structure torsions and normal curvatures (see Theorem 2 in [4]).

Lemma E *On a real hypersurface M of type* (A_2) *in a nonflat complex space form $\widetilde{M}_n(c), n \geq 2$, two geodesics γ_1, γ_2 are congruent to each other with respect to the full isometry group* $\mathrm{I}(M)$ *of M if and only if their structure torsions and normal curvatures satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$ and $\kappa_{\gamma_1} = \kappa_{\gamma_2}$.*

Next, we recall the notion of extrinsic geodesics. For a Riemannian manifold M^n isometrically immersed into anothe Riemannian manifold $\widetilde{M}^{n+p}$ through an isometric immersion f, a smooth curve $\gamma = \gamma(s)$ on M is an *extrinsic geodesic* on M if the curve $f \circ \gamma$ is a geodesic in the ambient space $\widetilde{M}$. In order to prove Theorem 2 we shall establish the following proposition which is a key in this section.

Proposition 1 *Let M be a real hypersurface of type* (A) *in a nonflat complex space form $\widetilde{M}_n(c), n \geq 2$. Then the number of congruency classes of extrinsic geodesics on M with respect to the full isometry group* $\mathrm{I}(M)$ *of M is as follows*:

(1) *In $\mathbb{C}P^n(c)$,*

1_a) *Every geodesic sphere $G(r)$ $(0 < r < \pi/(2\sqrt{c}\,))$ has no extrinsic* *geodesics*;

1_b) *Every geodesic sphere* $G(r)\,(\pi/(2\sqrt{c}\,) \leq r < \pi/\sqrt{c}\,))$ *has just one congruency class of extrinsic geodesics*;

1_c) *Every real hypersurface* M *of type* (A_2) *of radius* $(0 < r < \pi/\sqrt{c}\,)$ *has uncountably infinite congruency classes of extrinsic geodesics.*

(2) *In* $\mathbb{C}H^n(c)$, *every real hypersurface* M *of type* (A) *has no extrinsic geodesics.*

Proof of Proposition 1. First of all by virtue of Lemma B and the above discussion we know that a geodesic $\gamma = \gamma(s)$ on a real hypersurface M of type (A) is an extrinsic geodesic if and only if the initial vector $\dot\gamma(0)$ of the curve γ satisfies

$$g(A\dot\gamma(0), \dot\gamma(0)) = 0. \tag{5.1}$$

(1) Let M be a geodesic sphere $G(r)\,(0 < r < \pi/\sqrt{c}\,)$. For a geodesic $\gamma = \gamma(s)$ of $G(r)$, the initial vector $\dot\gamma(0)$ is written as:

$$\dot\gamma(0) = \rho_\gamma \xi_{\gamma(0)} + \sqrt{1 - \rho_\gamma^2}\, u, \tag{5.2}$$

where ρ_γ is the structure torsion of γ and u is a unit vector orthogonal to $\xi_{\gamma(0)}$. Then in view of Equation (5.2) and equalities $A\xi_{\gamma(0)} = \sqrt{c}\,\cot(\sqrt{c}\,r)\cdot\xi_{\gamma(0)}$, $Au = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)u$, $\sqrt{c}\cot(\sqrt{c}\,r) = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2) - (\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$ we have $\rho_\gamma^2 = \cot^2(\sqrt{c}\,r/2)$. This, combined with $0 \leq |\rho_\gamma| \leq 1$, shows that $r \geq \pi/(2\sqrt{c}\,)$. Thus we get the statement 1_a). Furthermore, from Lemma D for a geodesic sphere $G(r)$ of radius $r\,(\pi/(2\sqrt{c}\,) \leq r < \pi/\sqrt{c}\,)$ we obtain the statement 1_b).

Let M be a real hypersurface of type (A_2) of radius $r\,(0 < r < \pi/\sqrt{c}\,)$. For a geodesic $\gamma = \gamma(s)$ of M, the initial vector $\dot\gamma(0)$ can be expressed as:

$$\dot\gamma(0) = \sqrt{1 - a^2 - b^2}\, \xi_{\gamma(0)} + au + bv, \tag{5.3}$$

where a, b are nonnegative constants, $A\xi_{\gamma(0)} = \sqrt{c}\cot(\sqrt{c}\,r)\xi_{\gamma(0)}, Au = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)u$ and $Av = -(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)v$. These, together with Equation (5.1), yields

$$\left(\cot\left(\frac{\sqrt{c}\,r}{2}\right) - \tan\left(\frac{\sqrt{c}\,r}{2}\right)\right)(1-a^2-b^2) + a^2\cot\left(\frac{\sqrt{c}\,r}{2}\right) - b^2\tan\left(\frac{\sqrt{c}\,r}{2}\right) = 0.$$

So, setting $x = \cot(\sqrt{c}\, r/2)$ (> 0), we have

$$x - \frac{1}{x} + \frac{a^2}{x} - b^2 x = 0,$$

so that

$$\cot\left(\frac{\sqrt{c}\, r}{2}\right) = \sqrt{\frac{1-a^2}{1-b^2}} \quad \text{with } 0 \leq a^2 + b^2 < 1.$$

Thus, from lemma E we get the statement 1_c).

(2) Since all principal curvatures of real hypersurfaces of type (A) are positive (see the table of the principal curvatures in the case of $c < 0$) and the equality $\sqrt{|c|}\coth(\sqrt{|c|}\, r) = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\, r/2) + (\sqrt{|c|}/2)\tanh(\sqrt{|c|}\, r/2)$, by the discussion in (1) we get the statement (2). □

As an immediate consequence of Theorem 1 and Proposition 1 we can establish Theorem 2. □

6. Proof of Theorem 3

Before proving Theorem 3 we comment on the condition that either $d\eta(X,Y) = (\sqrt{c}/2)g(X,\phi Y)$ for all $X, Y \in TM$ or $d\eta(X,Y) = -(\sqrt{c}/2)g(X,\phi Y)$ for all $X, Y \in TM$. In general, by changing $\mathcal{N}$ into $-\mathcal{N}$ we know that every real hypersurface M has two almost contact metric structures (ϕ, ξ, η, g) and $(\phi, -\xi, -\eta, g)$ on M. From this viewpoint $d\eta(X,Y)$ depends on the choice of the unit normal vector $\mathcal{N}$, but $g(X,\phi Y)$ does *not* depend on $\mathcal{N}$. Hence the equality $d\eta(X,Y) = (\sqrt{c}/2)g(X,\phi Y)$ is *not* well-defined. So, in Theorems 3 and 4 we suppose these equalities.

It follows from (1.1) that

$$\begin{aligned} d\eta(X,Y) &= (1/2)\{X(g(\xi,Y)) - Y(g(\xi,X)) - g(\nabla_X Y - \nabla_Y X, \xi)\} \\ &= (1/2)\{g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X)\} \\ &= (1/2)\{g(\phi AX, Y) - g(\phi AY, X)\} \\ &= (1/2)g((\phi A + A\phi)X, Y). \end{aligned}$$

So, the hypothesis that $d\eta(X,Y) = \pm(\sqrt{c}/2)g(X,\phi Y)$ is equivalent to

$$\phi A + A\phi = \mp\sqrt{c}\,\phi. \tag{6.1}$$

By Equation (6.1) we first know that ξ is principal. So we can take a principal curvature vector X with $AX = \lambda X$ orthogonal to ξ. It follows from Lemma A and Equation (6.1) that

$$\lambda + \frac{\delta\lambda + (c/2)}{2\lambda - \delta} = \mp\sqrt{c}, \tag{6.2}$$

which implies that the λ satisfies the quadratic equation:

$$2\lambda^2 \pm 2\sqrt{c}\,\lambda + (c/2) \mp \delta\sqrt{c} = 0, \tag{6.3}$$

where the signatures take the same order. Hence our Hopf hypersurface M has at most three distinct constant principal curvatures λ_1, λ_2 which are solutions to Equation (6.3) and $\delta = g(A\xi, \xi)$. Then M is either of type (A) or type (B) (see Theorem B).

We shall check (6.1) one by one for real hypersurfaces of types (A) and (B). Let M be of type (A). Since $\phi A = A\phi$ (see Lemma B), Equation (6.1) is reduced to

$$AX = (\sqrt{c}/2)X \quad \text{for } \forall X(\perp \xi) \quad \text{or} \quad AX = -(\sqrt{c}/2)X \quad \text{for } \forall X(\perp \xi).$$

This shows that M is locally congruent to a geodesic sphere $G(\pi/(2\sqrt{c}\,))$. Next, let M be of type (B). Note that $\phi V_{\lambda_1} = V_{\lambda_2}$ (see Lemma A and the table of the principal curvatures in the case of $c > 0$). So we have only to solve the following equation:

$$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right) + \frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right) = \mp\sqrt{c}.$$

By putting $x = \cot(\sqrt{c}\,r/2)$, the above equation can be rewritten as:

$$\frac{1+x}{1-x} + \frac{x-1}{1+x} \pm 2 = 0,$$

so that $x = 1 \pm \sqrt{2}$ or $x = -1 \pm \sqrt{2}$. Since $x > 1$, we get $x = 1 + \sqrt{2}$, so that $r = (2/\sqrt{c}\,)\cot^{-1}(\sqrt{2} + 1)$. Therefore we obtain the conclusion.

7. Proof of Theorem 4

We shall investigate the sectional curvatures K for all homogeneous real hypersurfaces in $\mathbb{C}P^n(c), n \geq 2$.

Proposition 2 (1) *For every real hypersurface M of type* (A_1), *the sectional curvature K of M satisfies* $(c/4)\cot^2\big(\sqrt{c}\,r/2\big) \leq K \leq c + (c/4)\cot^2\big(\sqrt{c}\,r/2\big)$.

(2) *For every real hypersurface M of type* (A_2), *the sectional curvature K of M satisfies* $0 \leq K \leq c + \max\big\{(c/4)\cot^2\big(\sqrt{c}\,r/2\big), (c/4)\tan^2\big(\sqrt{c}\,r/2\big)\big\}$.

(3) *For every real hypersurface M of either type* (B), *type* (C), *type* (D) *or type* (E), *the sectional curvature K satisfies $K(\pi_1) < 0$ for some plane π_1 and $K(\pi_2) > 0$ for some plane π_2.*

Proof of Proposition 2. The authors ([16]) already proved the statements (1) and (2). But we here give the complete proof of the statement (1) for readers.

(1) We take an arbitrary pair of orthonormal vectors X and Y, which are orthogonal to the characteristic vector ξ of M. In order to estimate sectional curvatures K, from (2.7) we have the following

$$K(\sin\theta \cdot X + \cos\theta \cdot \xi, Y) = \frac{c}{4}\big\{\sin^2\theta\big(1 + 3g(\phi X, Y)^2\big) + \cot^2\big(\sqrt{c}\,r/2\big)\big\}.$$

Hence we find that sectional curvatures K of M satisfy

$$(c/4)\cot^2\big(\sqrt{c}\,r/2\big) \leq K \leq c + (c/4)\cot^2\big(\sqrt{c}\,r/2\big).$$

This yields that M has positive sectional curvature at its each point. Note that these estimations are sharp. Indeed,

$$K(X,\xi) = \frac{c}{4}\cot^2\left(\frac{\sqrt{c}\,r}{2}\right) \quad \text{and} \quad K(X,\phi X) = c + \frac{c}{4}\cot^2\left(\frac{\sqrt{c}\,r}{2}\right)$$

for each unit vector X perpendicular to ξ.

(2) See [16]. We remark that

$$K(X,Y) = 0,$$

$$K(X,\phi X)=c+\frac{c}{4}\cot^2\left(\frac{\sqrt{c}\,r}{2}\right) \text{ and } K(Y,\phi Y)=c+\frac{c}{4}\tan^2\left(\frac{\sqrt{c}\,r}{2}\right)$$

for all unit vectors X of $V_{(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)}$ and all unit vectors Y of $V_{-(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)}$. We emphasize that the estimations in the statement (2) are sharp.

(3) Let M be of either type(B), type (C), type (D) or type (E). Then every real hypersurface M has two common principal curvatures λ_1 and λ_2 satisfying (4.4). Setting $x=\cot(\sqrt{c}\,r/2)\,(>1)$, the principal curvatures δ,λ_1 and λ_2 are expressed as:

$$\delta=\frac{\sqrt{c}}{2}\left(x-\frac{1}{x}\right),\ \lambda_1=\frac{\sqrt{c}}{2}\frac{x+1}{1-x} \quad\text{and}\quad \lambda_2=\frac{\sqrt{c}}{2}\frac{x-1}{1+x}. \tag{7.1}$$

By virtue of (2.7) and (7.1) we find that

$$K(X,\xi)=\frac{c}{4}-\frac{c}{4}\frac{(1+x)^2}{x}<0 \quad \text{for each unit } X\in V_{\lambda_1}$$

and

$$K(Y,\xi)=\frac{c}{4}+\frac{c}{4}\frac{(x-1)^2}{x}>0 \quad \text{for each unit } Y\in V_{\lambda_2}.$$

Thus we obtain the statement (3).

As an immediate consequence of Theorem 3 and Proposition 2 we can establish Theorem 4. □

8. Extrinsic geodesics on real hypersurfaces in $\mathbb{C}P^n$

The class of ruled surfaces in $\mathbb{R}^3$ is an interesting subject in surface geometry. So if a submanifold M satisfies that through every point of M there is an extrinsic geodesic that lies on M, then M is considered as a generalization of ruled surface.

Now we study extrinsic geodesics on a *geodesic hypersphere* in $\mathbb{C}P^n$. In this section we assume that $c=4$. Let $G(r)$ be a geodesic hypersphere in $\mathbb{C}P^n(4)$ $(n\geq 2)$ with radius r $(\pi/4<r<\pi/2)$. $G(r)$ is realized as image of Riemannian product of a $(2n-1)$-sphere $S^{2n-1}(\sin r)$ and a circle $S^1(\cos r)$ under Hopf fibration $\pi: S^{2n+1}(1)\to\mathbb{C}P^n(4)$. We denote $M_r=S^{2n-1}(\sin r)\times S^1(\cos r)\subset S^{2n+1}(1)$. Let

$$p_0 = ((\sin r, 0, \dots, 0), \cos r) \in M_r \subset \mathbb{C}^n \times \mathbb{C}^1$$

be a point in M_r. Then a unit normal vector $\mathcal{N}_{p_0}$ of M_r in $S^{2n+1}(1)$ at p_0 and a horizontal lift ξ'_{p_0} of structure vector $\xi_{\pi(p_0)}$ of $G(r)$ in $\mathbb{C}P^n$ are given by

$$\mathcal{N}_{p_0} = ((-\cos r, 0, \dots, 0), \sin r) \text{ and}$$
$$\xi'_{p_0} = -i\mathcal{N}_{p_0} = ((i\cos r, 0, \dots, 0), -i\sin r),$$

respectively. We put

$$X_\pm := X_\pm(z_1, \dots, z_{n-1}) = ((\pm i \cot r \cos r, z_1, \dots, z_{n-1}), \mp i \cos r) \in \mathbb{C}^n \times \mathbb{C},$$

where $|z_1|^2 + \cdots + |z_{n-1}|^2 = 1 - \cot^2 r$. Then we have $X_\pm \in T_{p_0}(M_r)$ with $\|X_\pm(z_1, \dots, z_{n-1})\| = 1$ and $X_\pm(z_1, \dots, z_{n-1}) \perp ip_0$. So if we put

$$\gamma_\pm(t; z_1, \dots, z_{n-1}) = \cos t p_0 + \sin t X_\pm(z_1, \dots, z_{n-1}),$$

then we see that $t \mapsto \gamma_\pm(t; z_1, \dots, z_{n-1})$ is a horizontal great circle in S^{2n+1} and lies on M_r such that $\gamma_\pm(0; z_1, \dots, z_{n-1}) = p_0$ and $\dot\gamma_\pm(0; z_1, \dots, z_{n-1}) = X_\pm(z_1, \dots, z_{n-1})$. Hence $t \mapsto \pi(\gamma_\pm(t; z_1, \dots, z_{n-1}))$ is an *extrinsic geodesic* on the geodesic hypersphere $G(r)$ through $\pi(p_0)$. Note that

$$g(\xi'_{p_0}, X_\pm(z_1, \dots, z_{n-1})) = \pm \cot r,$$

and

$$\begin{aligned}&\{X \in T_{p_0}(M_r) |\ g(\xi'_{p_0}, X) = \pm \cot r\} \\ &\quad = \{X_\pm(z_1, \dots, z_{n-1}) \in T_{p_0}(M_r) |\ |z_1|^2 + \cdots + |z_{n-1}|^2 = 1 - \cot^2 r\}\end{aligned}$$

hold. Since $G(r)$ is a homogeneous real hypersurface, we have:

Proposition 3 *Let $G(r)$ be a geodesic hypersphere of radius r ($\pi/4 < r < \pi/2$) in $\mathbb{C}P^n(4)$. Then for each point $p \in G(r)$ and each unit tangent vector $X_p \in T_p(G(r))$ with $g(X_p, \xi_p) = \pm \cot r$, the geodesic γ of $G(r)$ satisfying $\gamma(0) = p$ and $\dot\gamma(0) = X$ is an extrinsic geodesic.*

Conversely we obtain:

Theorem 5 *Let M^{2n-1} be a real hypersurface isometrically immersed in $\mathbb{C}P^n(4)$. Suppose that there exists $\alpha \in (0,\pi)$, $\alpha \neq \pi/2$ such that for each point $p \in M$ and each unit tangent vector $X_p \in T_p(M)$ with $g(X_p, \xi_p) = \cos\alpha$, the geodesic γ of M satisfying $\gamma(0) = p$ and $\dot\gamma(0) = X$ is an extrinsic geodesic. Then M is locally congruent to a geodesic hypersphere $G(r)$ of radius $r \in (\pi/4, \pi/2)$ with $\cot r = |\cos\alpha|$.*

Proof. For $p \in M^{2n-1}$ and $\alpha \in (0,\pi)$, $\alpha \neq \pi/2$, we put

$$S_p(\alpha) := \{X_p \in T_p(M) |\ \|X\| = 1,\ g(X_p, \xi_p) = \cos\alpha\}.$$

Let Y_p and Z_p be unit tangent vectors at p satisfying $g(Y_p, \xi_p) = g(Z_p, \xi_p) = g(Y_p, Z_p) = 0$. For $t \in \mathbb{R}$, if we put

$$X(t; Y_p, Z_p) = \cos\alpha\xi_p + \sin\alpha(\cos tY_p + \sin tZ_p),$$

then we have $X(t; Y_p, Z_p) \in S_p(\alpha)$. Hence by the assumption of the Theorem, we can compute

$$\begin{aligned}
0 &= g(AX(t; Y_p, Z_p), X(t; Y_p, Z_p)) \\
&= \cos^2\alpha g(A\xi_p, \xi_p) + \sin^2\alpha\left(\frac{1+\cos 2t}{2} g(AY_p, Y_p) + \frac{1-\cos 2t}{2} g(AZ_p, Z_p)\right) \\
&\quad + \sin^2\alpha \sin 2t g(AY_p, Z_p) + \sin 2\alpha(\cos t g(A\xi_p, Y_p) + \sin t g(A\xi_p, Z_p)).
\end{aligned}$$

Since the above equation is valid for any $t \in \mathbb{R}$, we obtain

$$g(A\xi_p, Y_p) = g(A\xi_p, Z_p) = g(AY_p, Z_p) = 0,\ g(AY_p, Y_p) = g(AZ_p, Z_p), \quad (8.1)$$

$$\cos^2\alpha g(A\xi_p, \xi_p) + \sin^2\alpha g(AY_p, Y_p) = 0. \quad (8.2)$$

It follows from (8.1) that M is η-umbilic at its each point p, namely our real hypersurface M is locally congruent to a geodesic hypersphere $G(r)$ of radius (, say) r with $r \in (0, \pi/2)$. Furthermore, by virtue of (8.2) we find that $g(A\xi_p, \xi_p) = \pm 2\cot(2r) = \pm(\cot r - \tan r)$ and $g(AY_p, Y_p) = \pm\cot r$, where these signatures take the same orders. Therefore we have $\cot^2 r = \cos^2\alpha$ and $r \in (\pi/4, \pi/2)$. Thus we have proved Theorem 5. □

Remark 3 In Theorem 5, a real hypersurface M in $\mathbb{C}P^n$ satisfies the assumption with $\alpha = \pi/2$ (resp. $\alpha = 0$ or $\alpha = \pi$) if and only if M is a ruled real hypersurface (resp. a real hypersurface satisfies $A\xi = 0$).

Remark 4 We here explain the feature of real hypersurfaces of type (A) in $\mathbb{C}P^n(4)$, $n \geq 2$. We first consider the so-called Clifford hypersurface

$$M_{p,q}(r_1, r_2) := S^{2p+1}(r_1) \times S^{2q+1}(r_2)$$

in a unit sphere $S^{2n+1}(1)$, where $r_1^2 + r_2^2 = 1, p + q = n - 1$ and $0 \leq q \leq p \leq n - 1$. $M_{p,q}(r_1, r_2)$ has two distinct constant principal curvatures r_2/r_1 with multiplicity $2p + 1$ and $-r_1/r_2$ with multiplicity $2q + 1$ in the ambient space $S^{2n+1}(1)$. We here set $M_{p,q}^{\mathbb{C}} := \pi(M_{p,q}(r_1, r_2))$, where $\pi : S^{2n+1}(1) \to \mathbb{C}P^n(4)$ is the Hopf fibration. The manifold $M_{p,q}^{\mathbb{C}}$ is a real hypersurface of type (A) in $\mathbb{C}P^n(4), n \geq 2$. $M_{p,0}^{\mathbb{C}}$ is a Hopf hypersurface having two distinct principal curvatures $(r_2/r_1) - (r_1/r_2)$ with multiplicity 1 and r_2/r_1 with multiplicity $2n - 2$, which is congruent to a geodesic sphere $G(r)$ $(0 < r < \pi/2)$ with $\cot r = r_2/r_1$. When $pq \neq 0$, $M_{p,q}^{\mathbb{C}}$ is a Hopf hypersurface having three distinct constant principal curvatures $(r_2/r_1) - (r_1/r_2)$ with multiplicity 1 and r_2/r_1 with multiplicity $2p$ and $-r_1/r_2$ with multiplicity $2q$, which is a congruent to a tube of radius r $(0 < r < \pi/2)$ with $\tan r = r_1/r_2$ around a totally geodesic $\mathbb{C}P^q(4)$ in the ambient space $\mathbb{C}P^n(4)$.

A surface is *doubly ruled* if through each point there are two distinct lines that lie on the surface. The hyperbolic paraboloid and the hyperboloid of one sheet are doubly ruled surfaces. The plane is the only surface which contains at least three distinct lines through each point. On the other hand, the *minimal* Clifford torus T in 3-sphere satisfies that through each point p there are two distinct great circles that lie on T such that two great circles meet *orthogonally* at p.

In general, the following hold:

Proposition 4 *Let M be an n-dimensional submanifold in a Riemannian manifold $\widetilde{M}$. Suppose that at each point $p \in M$, there exist n extrinsic geodesics γ_i $(i = 1, 2, \ldots, n)$ of M through p such that $\gamma_1, \gamma_2, \ldots, \gamma_n$ meet orthogonally at p. Then M is a minimal submanifold of $\widetilde{M}$.*

In fact, since each γ_i is an extrinsic geodesic of $\widetilde{M}$, we have $\sigma(\dot{\gamma}_i(0), \dot{\gamma}_i(0)) = 0$ where we put $\gamma_i(0) = p$ and σ denotes the second fundamental tensor of M. Since $\dot{\gamma}_1(0), \dot{\gamma}_2(0) \ldots, \dot{\gamma}_n(0)$ form an orthonormal basis of $T_p(M)$, the mean curvature vector of M in $\widetilde{M}$ vanishes. Of course, every totally geodesic submanifold satisfies the condition. So it seems that submanifolds satisfying the conditon of the above proposition are geometrically good one among minimal submanifolds.

Now we consider the *minimal* geodesic hypersphere in $\mathbb{C}P^2$. A geodesic hypersphere $G(r)$ with radius r $(0 < r < \pi/2)$ is realized as $\pi(S^3(\sin r) \times S^1(\cos r))$, where $\pi : S^5 \to \mathbb{C}P^2$ is the Hopf fibration and $S^3(\sin r) \times S^1(\cos r)$ is a hypersurface in $S^5(1)$. Also $G(r)$ is minimal in $\mathbb{C}P^2$ if and only if $S^3(\sin r) \times S^1(\cos r)$ is minimal in $S^5(1)$, and we can see that $G(r)$ is minimal if and only if $r = \pi/3$. Hence $M := S^3(\sqrt{3}/2) \times S^1(1/2)$ (resp. $G(\pi/3)$) is a minimal hypersurface in $S^5(1)$ (resp. $\mathbb{C}P^2(4)$).

We define

$$\gamma(t; p_1, v_1, p_2, v_2) = \left(\frac{\sqrt{3}}{2}(\cos t p_1 + \sin t v_1), \frac{1}{2}(\cos t p_2 + \sin t v_2) \right),$$

where $p_1 \in S^3(1)$, $v_1 \in T_{p_1}(S^3(1))$ $(|v_1| = 1)$, $p_2 \in S^1(1)$ and $v_2 \in T_{p_2}(S^1(1))$ $(|v_2| = 1)$. Then $\gamma(t; p_1, v_1, p_2, v_2)$ is a great circle in $S^5(1)$ and lies on M with

$$\gamma(0; p_1, v_1, p_2, v_2) = \left(\frac{\sqrt{3}}{2} p_1, \frac{1}{2} p_2 \right)$$

$$\text{and} \quad \dot{\gamma}(0; p_1, v_2, p_2, v_2) = \left(\frac{\sqrt{3}}{2} v_1, \frac{1}{2} v_2 \right).$$

We put

$$p_1 = (1, 0) \in S^3(1) \subset \mathbb{C}^2, \quad p_2 = 1 \in S^1(1) \subset \mathbb{C}^1$$

$$\text{and } p_0 = \left(\left(\frac{\sqrt{3}}{2}, 0 \right), \frac{1}{2} \right) \in M.$$

Then unit tangent vectors $v_1 \in T_{p_1}(S^3(1))$ and $v_2 \in T_{p_2}(S^1(1))$ are written as

$$v_1 = (iy, z) \ \ (y \in \mathbb{R},\ z \in \mathbb{C},\ |y|^2 + |z|^2 = 1) \quad \text{and} \quad v_2 = \pm i,$$

respectively.

For $\theta \in \mathbb{R}$, we put

$$\begin{aligned}
\gamma_1(t;\theta) &= \gamma\big(t; p_1, \big(-i/3, \sqrt{8}e^{i\theta}/3\big), p_2, i\big) \\
&= \bigg(\bigg(\frac{\sqrt{3}}{2}\cos t - \frac{i\sin t}{\sqrt{12}}, \sqrt{\frac{2}{3}}e^{i\theta}\sin t\bigg), \frac{e^{it}}{2}\bigg), \\
\gamma_2(t;\theta) &= \gamma\big(t; p_1, \big(i/3, \sqrt{8}e^{i(\theta+\pi/3)}/3\big), p_2, -i\big) \\
&= \bigg(\bigg(\frac{\sqrt{3}}{2}\cos t + \frac{i\sin t}{\sqrt{12}}, \sqrt{\frac{2}{3}}e^{i(\theta+\pi/3)}\sin t\bigg), \frac{e^{-it}}{2}\bigg), \\
\gamma_3(t;\theta) &= \gamma\big(t; p_1, \big(-i/3, \sqrt{8}e^{i(\theta+2\pi/3)}/3\big), p_2, i\big) \\
&= \bigg(\bigg(\frac{\sqrt{3}}{2}\cos t - \frac{i\sin t}{\sqrt{12}}, \sqrt{\frac{2}{3}}e^{i(\theta+2\pi/3)}\sin t\bigg), \frac{e^{it}}{2}\bigg).
\end{aligned}$$

Then $\gamma_1(t;\theta)$, $\gamma_2(t;\theta)$ and $\gamma_3(t;\theta)$ are all *horizontal* great circles in $S^5(1)$ which lie on M with $\gamma_1(0;\theta) = \gamma_2(0;\theta) = \gamma_3(0;\theta) = p_0$. Hence $\pi(\gamma_1(t;\theta))$, $\pi(\gamma_2(t;\theta))$ and $\pi(\gamma_3(t;\theta))$ are *extrinsic geodesics* on the minimal geodesic hypersphere $G(\pi/3)$ through $\pi(p_0)$. Furthermore, we have

$$\begin{aligned}
\dot\gamma_1(0;\theta) &= \bigg(\bigg(-\frac{i}{\sqrt{12}}, \sqrt{\frac{2}{3}}e^{i\theta}\bigg), \frac{i}{2}\bigg), \\
\dot\gamma_2(0;\theta) &= \bigg(\bigg(\frac{i}{\sqrt{12}}, \sqrt{\frac{2}{3}}e^{i(\theta+\pi/3)}\bigg), -\frac{i}{2}\bigg), \\
\text{and}\quad \dot\gamma_3(0;\theta) &= \bigg(\bigg(-\frac{i}{\sqrt{12}}, \sqrt{\frac{2}{3}}e^{i(\theta+2\pi/3)}\bigg), \frac{i}{2}\bigg).
\end{aligned}$$

Consequently these 3 extrinsic geodesics meet *orthogonally* at $\pi(p_0)$. Since $G(r)$ is a homogeneous hypersurface in $\mathbb{C}P^2$, the same phenomena occur at each point of $G(\pi/3)$.

9. Viewpoint from the contact geometry

For real hypersurfaces M in $\widetilde{M}_n(c)$ we recall some notions in the contact geometry. We first say that every real hypersurface M has two almost contact metric structures (ϕ, ξ, η, g) and $(\phi, -\xi, -\eta, g)$ (see Section 6). A real hypersurface M is a *Sasakian manifold* if and only if the structure tensor ϕ of M satisfies either the equation $(\nabla_X\phi)Y = g(X,Y)\xi - \eta(Y)X$ for all vectors

$X, Y \in TM$ or $(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X$ for all vectors $X, Y \in TM$. A Sasakian manifold manifold M is called a *Sasakian space form* if every ϕ-sectional curvature $K(u, \phi u) := g(R(u, \phi u)\phi u, u)$ associated to a unit vector $u(\in TM)$ orthogonal to ξ does not depend on the choice of u, where R is the curvature tensor of M. A real hypersurface M is called a *contact manifold* if the exterior differentiation of the contact form η on M satisfies either $d\eta(X, Y) = g(X, \phi Y)$ for all $X, Y \in TM$ or $d\eta(X, Y) = -g(X, \phi Y)$ for all $X, Y \in TM$. When M is contact and $L_\xi g = 0$, M is called a *K-contact manifold*, where L is the Lie derivative on M. In the contact geometry, Sasakian always means K-contact. In general the converse does not hold (cf. [11]). But, in the theory of real hypersurfaces the following hold:

Proposition A ([13]) *For a real hypersurface M isometically immersed into a nonflat complex space form $\widetilde{M}_n(c)$, $n \geq 2$, the following three conditions are mutually equivalent*:
(1) *M is a Sasakian space form.*
(2) *M is a Sasakian manifold.*
(3) *M is a K-contact manifold.*
In Condition (1)*, M has automatically ϕ-sectional curvature $c + 1$.*

It is well-known that a Sasakian space form of constant ϕ-sectional curvature 1 is realized as a real hypersurface $S^{2n-1}(1)$ of a flat complex space form $\mathbb{C}^n$. J. Berndt showed that every Sasakian space form of constant ϕ-sectional curvature $c(\neq 1)$ can be a realized as a real hypersurface in a nonflat complex space form through an isometric immersion.

Proposition B ([7]) *Let M^{2n-1} be a connected real hypersurface isometrically immersed into a nonflat complex space form $\widetilde{M}_n(c), n \geq 2$. Suppose that M is a Sasakian space form. Then M is locally congruent to one of the following real hypersurfaces in the ambient space $\widetilde{M}_n(c)$:*

- i) *a geodesic sphere $G(r)$ of radius r with* $\cot(\sqrt{c}\, r/2) = 2/\sqrt{c}\ (0 < r < \pi/\sqrt{c}\,)$ *in* $\mathbb{C}P^n(c)$;
- ii) *the horosphere in* $\mathbb{C}H^n(-4)$;
- iii) *a geodesic sphere $G(r)$ of radius r with* $\coth(\sqrt{|c|}\, r/2) = 2/\sqrt{|c|}\ (0 < r < \infty)$ *in* $\mathbb{C}H^n(c)\ (-4 < c < 0)$;
- iv) *a tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$ with* $\coth(\sqrt{|c|}\, r/2) = \sqrt{|c|}/2\ (0 < r < \infty)$ *in* $\mathbb{C}H^n(c)\ (c < -4)$.

In these cases, M has constant ϕ-sectional curvature $c+1$.

The following are classification theorems of *contact* real hypersurfaces in a nonflat complex space form.

Proposition C ([2]) *Let M^{2n-1} be a connected real hypersurface isometrically immersed into $\mathbb{C}P^n(c), n \geq 2$. Suppose that M is a contact manifold. Then M is locally congruent to one of the following homogeneous real hypersurfaces in the ambient space $\mathbb{C}P^n(c)$:*

1) *a geodesic sphere $G(r)$ of radius r with $\cot(\sqrt{c}\, r/2) = 2/\sqrt{c}\, (0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$;*
2) *a tube of radius $r = (2/\sqrt{c}\,)\cot^{-1}((\sqrt{c+4} + \sqrt{c}\,)/2)$ around a complex hyperquadric $\mathbb{C}Q^{n-1}$, $0 < r < \pi(2\sqrt{c}\,)$.*

Proposition D ([2]) *Let M^{2n-1} be a connected real hypersurface isometrically immersed into $\mathbb{C}H^n(c), n \geq 2$. Suppose that M is a contact manifold. Then M is locally congruent to one of the following homogeneous real hypersurfaces in the ambient space $\mathbb{C}H^n(c)$:*

1) *the horosphere* HS *in $\mathbb{C}H^n(c)\,(c = -4)$;*
2) *either a geodesic sphere $G(r)$ of radius $r = (1/\sqrt{|c|}\,)\{\log(2+\sqrt{|c|}\,) - \log(2-\sqrt{|c|}\,)\}$ or a tube of radius $r = (1/(2\sqrt{|c|}\,))\{\log(2+\sqrt{|c|}\,) - \log(2-\sqrt{|c|}\,)\}$ around a totally real totally geodesic $\mathbb{R}H^n(c/4)\,(-4 < c < 0)$,*
3) *a tube of radius $r = (1/\sqrt{|c|}\,)\{\log(\sqrt{|c|}+2) - \log(\sqrt{|c|}-2)\}$ around a totally geodesic $\mathbb{C}H^{n-1}(c)\,(c < -4)$.*

In consideration of Propositions B and C we can see our real hypersurfaces in Theorems A, 2 and 3 from the viewpoint of the contact geometry.

(1) The horosphere HS in $\mathbb{C}H^n(c)$ is a Sasakian space form (of constant ϕ-sectional curvature -3) if and only if $c = -4$.
(2) The geodesic sphere $G(\pi/(2\sqrt{c}\,))$ in $\mathbb{C}P^n(c)$ is a Sasakian space form (of constant ϕ-sectional curvature 5) if and only if $c = 4$.
(3) The tube $T_2(r)$ of radius r with $\cot(\sqrt{c}\, r/2) = \sqrt{2}+1$ around a complex hyperquadric $\mathbb{C}Q^{n-1}$ in $\mathbb{C}P^n(c)$ is a contact manifold in $\mathbb{C}P^n(c)$ if and only if $c = 4$.

10. The length spectrum on the geodesic sphere $G(\pi/4)$ in $\mathbb{C}P^n(4)$

We first recall the fact that in $\mathbb{C}P^n(c)$ every geodesic sphere $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ has *countably infinite* congruency classes of closed geodesics with respect to $\mathrm{I}(G(r))$ (cf. [6]) and every real hypersurface M of type (A_2) of radius $r\,(0 < r < \pi/\sqrt{c}\,)$ has *uncountably infinite* congruency classes of closed geodesics with respect to $\mathrm{I}(M)$ (see the discussion in the proof of Theorem 2 and [1]). Note that every geodesic γ of each real hypersurface M of type (A) is a simple curve.

In the last section, we state some fundamental results in the length spectrum $\mathrm{Lspec}(G(\pi/4))$, which is the set of lengths (on a real line $\mathbb{R}$) of all closed geodesics on a Sasakian space form $G(\pi/4)$ (of constant ϕ-sectional curvature 5) in $\mathbb{C}P^n(4), n \geq 2$ (for details, see [6]).

(1) The length of every integrable curve γ of the characteristic vector field (i.e., $\rho_\gamma = \pm 1$) is the first length spectrum given by π. The length of every geodesic γ with structure torsion $\rho_\gamma = 0$ is the second length spectrum given by $\sqrt{2}\,\pi$.

$\mathrm{Lspec}(G(\pi/4))$ is expressed as:

$$\begin{aligned}\mathrm{Lspec}(G(\pi/4)) = \big\{&\pi, \sqrt{2}\,\pi, \sqrt{5}\,\pi, \sqrt{10}\,\pi, \sqrt{13}\,\pi, \sqrt{17}\,\pi, 5\pi, \sqrt{26}\,\pi,\\ &\sqrt{29}\,\pi, \sqrt{34}\,\pi, \sqrt{37}\,\pi, \sqrt{41}\,\pi, \sqrt{50}\,\pi, \sqrt{53}\,\pi,\\ &\sqrt{58}\,\pi, \sqrt{61}\,\pi, \sqrt{65}\,\pi, \sqrt{73}\,\pi, \dots\big\}.\end{aligned}$$

Note that the multiplicity of $\sqrt{65}\,\pi$ is two, namely it is the common length of geodesics of structure torsions $3/\sqrt{65}$ and $7/\sqrt{65}$. Every spectrum which is shorter than $\sqrt{65}\,\pi$ is simple, i.e., its multiplicity is one.

(2) $\mathrm{Lspec}(G(\pi/4))$ is a discrete unbouded subset in the real line $\mathbb{R}$.

We here denote by $m_{G(\pi/4)}(\lambda)$ the number of congruency classes of closed geodesics on $G(\pi/4)$ with length λ, that is, the multiplicity of $\lambda \in \mathrm{Lspec}(G(\pi/4))$. Then $m_{G(\pi/4)}(\lambda)$ is finite for each $\lambda \in \mathrm{Lspec}(G(\pi/4))$. But it is not uniformly bounded, i.e., $\limsup_{\lambda\to\infty} m_{G(\pi/4)} = \infty$. In this case, the growth order of $m_{G(\pi/4)}$ is not so rapid. It satisfies $\lim_{\lambda\to\infty} \lambda^{-\delta} m_{G(\pi/4)}(\lambda) = 0$ for every positive δ.

(3) We denote by $n_{G(\pi/4)}(\lambda)$ the number of congruency classes of closed

geodesics on $G(\pi/4)$ whose length λ is *not* longer than λ. Then we obtain $\lim_{\lambda\to\infty}(n_{G(\pi/4)}(\lambda)/\lambda^2) = 3/4\pi^3$.

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