On homotopically trivial links

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction.

Throughout this paper we shall only be concerned with the combinatorial category, consisting of simplicial complexes and piecewise linear maps (for the combinatorial categoryf see [3]). Zeeman [3] shows that if $(n, k, 2)$-*link* $L=(S^n \supset K_1^k \cup K_2^k)$ is homotopically trivial and if $2n \geqq 3k+4$, then L is geometrically trivial. And if $n=k+2$, it is well known that there exists a homopically trivial $(n, n-2, 2)$-*link* which is not geometrically trivial (for example, in $(3, 1, 2)$-*link*). For the case $n \geqq k+3$ and $2n \leqq 3k+3$ Zeeman says that there exist homotopically trivial links which are not geometrically trivial same as $(3, 1, 2)$-*link*. But there is no proof for these links be geometrically non-trivial. So we consider the relation between homotopically trivial links and geometrically trivial links under $n \geqq k+3$ and $2n \leqq 3k+3$. In this paper we obtain a geometrical sufficient condition for a homotopically trivial $(n, k, 2)$-*link* be geometrically trivial. I should like to express my sincere gratitude to the members of Kōbe and Hokkaidō topology seminars for many discussion of this problem.

§ 2. Notations and Definitions

S^n is a standard n-sphere and D^n is a standard n-cell. An $(n, k, 2)$-*link* L is a pair $(S^n \supset K_1^k \cup K_2^k)$ of an n-sphere S^n and a disjoint union of locally flatly embedded k-spheres K_i^k, $i=1, 2$ in S^n. ∂X and $\operatorname{Int} X$ mean the boundary and the interior of a manifold X. $X * Y$ denote the join of spaces X and Y. $\cong$ means "homeomorphic to". $\bigvee_{i=1}^{m} S_i^p$ means one point join of p-spheres $S_1^p, \cdots, S_m^p$ and $\mathrm{I}=[0, 1]$. For any manifolds X, Y such that X is a submanifold of Y, $U(X, Y)$ means a regular neighborhood of X in Y. And we always take $U(X, Y)$ to be a second derived neighborhood of X in Y for a suitable subdivision of X and Y unless otherwise stated.

DEFINITION 1. Let X, Y be subsets in an n-manifold Z. We say that X and Y *split each other* in Z if there exists an n-cell B^n in Z such that either $X \subset \operatorname{Int} B^n$, $Y \cap B^n = \phi$ or $Y \subset \operatorname{Int} B^n$, $X \cap B^n = \phi$.

DEFINITION 2. We say that an $(n, k, 2)$-*link* $L=(S^n \supset K_1^k K_2^k)$ is *geometrically trivial* or briefly *G-trivial* if there exist locally flatly embedded $(k+1)$-cells B_1^{k+1}, B_2^{k+1} in S^n such that $\partial B_i^{k+1}=K_i^k$, $i=1, 2$ and $B_1^{k+1}\cap B_2^{k+1}=\phi$.

REMARK 1. When $n\geqq k+3$ it is sufficient that there exists a locally flatly embedded $(k+1)$-cell B^{k+1} which is bounded by K_1^k or K_2^k in $S^n-K_2^k$ or $S^n-K_1^k$. For if $\partial B^{k+1}=K_1^k$ and $B^{k+1}\subset S^n-K_2^k$, by *Zeeman's Unknotting theorem* [4] we can find a locally flatly embedded $(k+1)$-cell B_2^{k+1} bounded by K_2^k in $S^n-U(B_1^{k+1}, S^n-K_2^k)\cong D^n$.

DEFINITION 3. We say that an $(n, k, 2)$-*link* $L=(S^n\supset K_1^k\cup K_2^k)$ is *homotopically trivial* or briefly *H-trivial* if K_1 is contractible in $S^n-\mathrm{Int}\, U(K_2, S^n)$ and if K_2 is contractible in $S^n-\mathrm{Int}\, U(K_1, S^n)$.

DEFINITION 4. We say that an $(n, k, 2)$-*link* $L=(S^n\supset K_1^k\cup K_2^k)$ is *weak H-trivial* if either K_1^k or K_2^k is contractible in the complementary of the other or equivalently if there exists a map $F:(S_1^k\cup S_2^k)\times I\to S^n$ such that

(1) $F_0(S_i^k)=K_i$, $i=1, 2$,

(2) $(S^n\supset F_1(S_1^k)\cup F_1(S_2^k))$ is a G-trivial link,

(3) $F_t(S_1^k)\cap F_t(S_2^k)=\phi$ for any $t\in[0, 1]$.

REMARK 2. G-trivial$\Longrightarrow$H-trivial$\Longrightarrow$weak H-trivial.

REMARK 3. If $2n\geqq 3k+4$

weak H-trivial$\Longleftrightarrow H$-trivial$\Longleftrightarrow G$-trivial (Zeeman [3]).

REMARK 4. If $n=k+2$

weak H-trivial$\not\Longleftrightarrow H$-trivial$\not\Longleftrightarrow G$-trivial

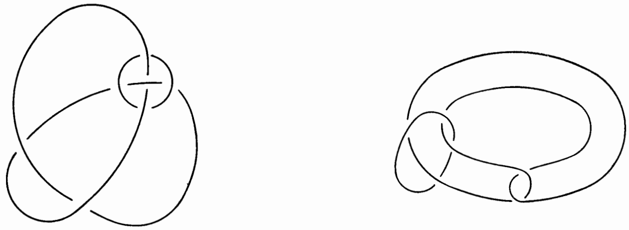

Fig. 1.

REMARK 5. Let $n=k+2$ and $(S^n\supset K_i^k)$, $i=1, 2$ be both trivial knots. And if $k\geqq 2$, any such $(k+2, k, 2)$-*link* is H-trivial. If $k=1$, for any such $(3, 1, 2)$-*link*, homologically trivial link$\Longleftrightarrow$weak H-trivial link$\Longleftrightarrow H$-trivial.

We assume $n\geqq k+3$ throughout the remaining of the paper unless otherwise is stated. Then the *knots* $(S^n\supset K_i^k)$, $i=1, 2$ are both trivial by [4]. Hence $K_1^k\subset S^n-\mathrm{Int}\, U(K_2^k, S^n)\cong S^{n-k-1}\times D^{k+1}$.

Condition $(n-k-1, k, m, \mu)$ (briefly *Cond.* (m, μ)). *For any link* $L=$

$(S^n \supset K_1^k \cup K_2^k)$ $(n \geqq k+3)$ *since* $K_1^k \subset S^n\text{-Int}\, U(K_2^k, S^n) \cong S^{n-k-1} \times D^{k+1}$ *we identify* $S^n - \text{Int}\, U(K_2^k, S^n)$ *and* $S^{n-k-1} \times D^{k+1}$ *and so we consider* $S^{n-k-1} \times D^{k+1} \supset K_1^k$. *We say that L satisfies Cond.* (m, μ) *if there exist a point* $p \in S^{n-k-1}$ *and a neighborhood* U_p *of* p *in* S^{n-k-1} *satisfying the following conditions;*

(1) $U_p \times D^{k+1} \cap K_1^k \cong \bigcup_{i=1}^{m} D_i^k$ *for some* m,

(2) $\bigcup_{i=1}^{m} D_i^k$ *properly embedded in* $U_p \times D^{k+1}$,

(3) $(\partial D_1^k \cup \cdots \cup \partial D_\mu^k)$ *and* $(p \times \partial D^{k+1}) \cup (\partial D_{\mu+1}^k \cup \cdots \cup \partial D_m^k)$

split each other in $\partial(U_p \times D^{k+1})$ *for some* $\mu \leqq m$.

PROPOSITION. *If a link* $L = (S^n \supset K_1^k \cup K_2^k)$ *satisfies Cond.* (m, m), L *is weak H-trivial.*

PROOF. Since $\partial D_1^k \cup \cdots \cup \partial D_m^k$ splits from $p \times \partial D^{k+1}$ in $\partial(U_p \times D^{k+1})$, $\bigcup_{i=1}^{m} D_i^k$ contracts into $\partial U_p \times D^{k+1}$. Next we push $\bigcup_{i=1}^{m} D_i^k$ into $(S^{n-k-1} - \text{Int}\, U_p) \times D^{k+1} \cong D^n$ using a collar of $\partial U_p \times D^{k+1}$ in $(S^{n-k-1} - \text{Int}\, U_p) \times D^{k+1}$. Hence K_1^k contracts into $(S^{n-k-1} - \text{Int}\, U_p) \times D^{k+1} \cong D^n$. So K_1^k is contractible in $S^{n-k-1} \times D^{k+1} = S^n - \text{Int}\, U(K_2^k, S^n)$. Therefore L is weak H-trivial.

THEOREM. *If an* $(n, k, 2)$*-link* $L = (S^n \supset K_1^k \cup K_2^k)$ $(n \geqq k+3)$ *satisfies Cond.* (m, m), L *is G-trivial.*

§ 3. Lemmas and Proof of Theorem.

LEMMA 1. *For a link* $L = (S^n \supset K_1^k \cup K_2^k)$, L *satisfies* (1), (2) *of Cond.* (m, μ) *and satisfies that* $\bigcup_{i=1}^{m} \partial D_i^k$ *bound a manifold* M_1 *in* $\partial U_p \times D^{k+1}$ *which is homeomorphic to* $S^k - \bigcup_{i=1}^{\mu} \text{Int}\, D_i^k$, *then* L *is G-trivial.*

PROOF. Let $M_2 = (S^{n-k-1} - \text{Int}\, U_p) \times D^{k+1} \cap K_1^k$. Then $(S^{n-k-1} - \text{Int}\, U_p) \times D^{k+1} \cap K_1^k \cong S^k - \bigcup_{i=1}^{\mu} \text{Int}\, D_i^k$ by (1), (2) of *Cond.* (m, μ). Hence $M_1 \cong M_2$. Since $(S^{n-k-1} - \text{Int}\, U_p) \times D^{k+1}$ has a collar [3], we push homeomorphically M_1 into $\text{Int}\, ((S^{n-k-1} - \text{Int}\, U_p) \times D^{k+1})$ keeping ∂M_1 fixed using the collar. We denote it M_0. Then the followings hold for M_0;

1) $n \geqq k+3$,

2) M_0 is homeomorphic to M_2 and $\partial M_0 = \partial M_1 = \partial M_2$,

3) M_0 is $(k-2)$-connected because $M_0 \cong S^k - \bigcup_{i=1}^{\mu} \text{Int}\, D_i^k$ has a homotopy type of $\bigvee_{i=1}^{\mu-1} S_i^{k-1}$,

4) M_0 is homotopic to M_2 keeping the boundary fixed because $(S^{n-k-1}$

$-\text{Int}\, U_p)\times D^{k+1}\cong D^n$.
So by *Zeeman's Unknotting Theorem* [3. chap. 8] M_0 and M_2 are ambient isotopic by an ambient isotopy of $S^{n-k-1}\times D^{k+1}$ keeping $\partial(S^{n-k-1}\times D^{k+1})\cup U_p\times D^{k+1}$ fixed. Therefore conversely we can consider M_2 ambient isotopic to M_1 through M_0 by an ambient isotopy of $S^{n-k-1}\times D^{k+1}$ keeping $\partial(S^{n-k-1}\times D^{k+1})\cup W_p\times D^{k+1}$ where $W_p=U_p-$(*collar of* ∂U_p). Next we push homeomorphically M_1 into $U_p\times D^{k+1}\cong D^n$ using a collar of $\partial U_p\times D^{k+1}$ in $U_p\times D^{k+1}$. Hence we can ambient isotop K_1^k into $\text{Int}\,(U_p\times D^{k+1})\cong\text{Int}\, D^n$ and so K_1^k bound a k-cell in $\text{Int}\,(U_p\times D^{k+1})\subset S^{n-k-1}\times D^{k+1}=S^n-\text{Int}\, U(K_2^k, S^n)$ because $n\geqq k+3$. Therefore L is G-trivial (see Remark 1).

LEMMA 2. *For any link* $L=(S^n\supset K^k\cup K_1^{k-1}\cup\cdots\cup K_m^{k-1})$ (*codimension is unrestricted*) *if* $(S^n\supset K_1^{k-1}\cup\cdots\cup K_\mu^{k-1})$ $(\mu\leqq m)$ *is G-trivial and if* $K_1^{k-1}\cup\cdots\cup K_\mu^{k-1}$ *and* $K^k\cup K_{\mu+1}^{k-1}\cup\cdots\cup K_m^{k-1}$ *split each other in* S^n, K_i^{k-1}, $1\leqq i\leqq\mu$ *bound disjoint locally flat k-cells in* $S^n-(K^k\cup K_{\mu+1}^{k-1}\cup\cdots\cup K_m^{k-1})$.

PROOF. Since $(S^n\supset K_1^{k-1}\cup\cdots\cup K_\mu^{k-1})$ is G-trivial K_i^{k-1}, $1\leqq i\leqq\mu$ bound locally flat disjoint k-cells D_i^k, $1\leqq i\leqq\mu$. And we may suppose that $\bigcup_{i=1}^{m} D_i^k\subset S^n-\text{Int}\,|St(v, S^n)|$ for some vertex v. Since $K_1^{k-1}\cup\cdots\cup K_\mu^{k-1}$ and $K^k\cup K_{\mu+1}^{k-1}\cup\cdots\cup K_m^{k-1}$ split each other, there exists an n-cell B_1^n such that $K^k\cup K_{\mu+1}^{k-1}\cup\cdots\cup K_m^{k-1}\subset\text{Int}\, B_1^n$ and $\bigcup_{i=1}^{\mu} K_i^{k-1}\cap B_1^n=\phi$. Let $B_2^n=S^n-\text{Int}\, U(B_1^n, S^n)$. Then by [1] there exists a PL homeomorphism $h: S^n\to S^n$ which is isotopic to the identity and $h(B_1^n)=|St(v, S^n)|$. Since h is isotopic to the identity and since $(S^n\supset K_1^{k-1}\cup\cdots\cup K_\mu^{k-1})$ is G-trivial, $(S^n\supset h(K_1^{k-1})\cup\cdots\cup h(K_\mu^{k-1}))$ is also G-trivial. Hence $h(K_i^{k-1})$, $1\leqq i\leqq\mu$ bound locally flat disjoint k-cells in $S^n-\text{Int}\,|St(v, S^n)|\subset S^n-\bigcup_{i=\mu+1}^{m} h(K_i^{k-1})\cup h(K^k)$. Hence K_i^{k-1}, $1\leqq i\leqq\mu$ bound locally flat disjoint k-cells in $S^n-K^k\cup K_{\mu+1}^{k-1}\cup\cdots\cup K_m^{k-1}$.

LAMMA 3. *If a link* $L=(S^n\supset K_1^k\cup K_2^k)$ *satisfies Cond.* (m, m), ∂D_i^k, $1\leqq i\leqq m$ *bound locally flat disjoint k-cells* $B_1^k,\cdots,B_m^k$ *in* $\partial U_p\times D^{k+1}$.

PROOF. Since $\partial D_i^k, 1\leqq i\leqq m$ bound locally flat disjoint k-cells $D_1^k,\cdots,D_m^k$ in $U_p\times D^{k+1}\cong D^n$, $(\partial(U_p\times D^{k+1})\supset\partial D_1^k\cup\cdots\cup\partial D_m^k)$ is G-trivial by [2. Th. 7 and Remark 1]. And since $\bigcup_{i=1}^{m}\partial D_i^k$ and $p\times\partial D^{k+1}$ split each other in $\partial(U_p\times D^{k+1})$ by *Cond.* (m, m), ∂D_i^k, $1\leqq i\leqq m$ bound locally flat disjoint k-cells in $\partial(U_p\times D^{k+1})-p\times\partial D^{k+1}$ by Lemma 2. But since $U_p\times\partial D^{k+1}$ is a regular neighborhood of $p\times\partial D^{k+1}$ in $\partial(U_p\times D^{k+1})$ and since the complement of $U_p\times\partial D^{k+1}$ in $\partial(U_p\times D^{k+1})$ is $\partial U_p\times D^{k+1}$, hence ∂D_i^k, $1\leqq i\leqq m$ bound locally flat disjoint k-cells $B_1^k,\cdots,B_m^k$ in $\partial U_p\times D^{k+1}$.

PROOF OF THEOREM. Since the *link* $L=(S^n\supset K_1^k\cup K_2^k)$ satisfies *Cond.* $(m\ m)$, ∂D_i^k, $1\leqq i\leqq m$ bound locally flat disjoint k-cells $B_1^k,\cdots,B_m^k$ in $\partial U_p\times D^{k+1}$ by Lemma 3. Since $n\geqq k+3$, there exist embeddings

$$h_i : I\times D^{k-1}\to\partial U_p\times D^{k+1},\quad 1\leqq i\leqq m-1$$

such that

(1) $h_i(I\times D^{k-1})\cap h_j(I\times D^{k-1})=\phi$,

(2) for all i, $h_i(I\times D^{k-1})\cap\bigcup_{j=1}^{m}B_j^k=h_i(\{0\}\times D^{k-1}\cup\{1\}\times D^{k-1})\subset B_1^k\cup B_i^k$

(3) $h_i(\{0\}\times D^{k-1})\subset B_1^k$, $h_i(\{1\}\times D^{k-1})\subset B_i^k$.

Let $M=(\bigcup_{j=1}^{m}B_j^k)\cup_{\cup h_i}(\bigcup_{i=1}^{m}h_i(I\times D^{k-1}))-\bigcup_{i=1}^{m-1}h_i(I\times(\mathrm{Int}\,D^{k-1}))$. Then M is a manifold homeomopphic to $S^k-\bigcup_{i=1}^{m}D_i^k$. Hence $\bigcup_{i=1}^{m}\partial D_i^k$ bounds a manifold homeomorphic to $S^k-\bigcup_{i=1}^{m}D_i^k$ in $\partial U_p\times D^{k+1}$ and L is G-trivial by Lemma 1.

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