A remark on the Steenrod representation of $B(Z_p \times Z_p)$

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction

For a topological space X, $z \in H_n(X; Z)$ is Steenrod representable if there exists a closed oriented smooth n-manifold M and a continuous map $f: M \to X$ such that $f_*(\sigma)=z$, where σ is a fundamental homology class of M. In [4], Thom showed that for a finite polyhedron X any $z \in H_n(X; Z)$ is representable if $n \leqq 6$, but if $n \geqq 7$ not everything is representable. He exhibited a class in $H_7(L^7(3) \times L^7(3); Z)$ which was not, where $L^7(3)$ is 7-dimesional lens space mod 3. Moreover Burdick [1] extended to $B(Z_3 \times Z_3)$, classifying space of $Z_3 \times Z_3$, and computed all representable elements. He dermined E^∞ terms of bordism spectral sequence of $B(Z_3 \times Z_3)$ and used necessary condition of representability of Thom [4].

In this note we show the case $p=2$ and any odd prime p. Latter case we use the same methods as Burdick's.
We have

THEOREM 1.
(*a*) *Every elements of* $H_*(B(Z_2 \times Z_2); Z)$ *are Steenrod representable.*
(*b*) *For* p *an odd prime the elements of* $H_*(B(Z_p \times Z_p); Z)$ *which are Steenrod representable are generated by* $e_0 \otimes e_0$, $e_{2i-1} \otimes e_{2j-1}$, $e_0 \otimes e_{2j-1}$, $e_{2i-1} \otimes e_0$, $\{(e_2 \otimes e_{2j-1} + e_1 \otimes e_{2j}) + (e_6 \otimes e_{2j-5} + e_5 \otimes e_{2j-4}) + \cdots\}$, *and* $\{(e_4 \otimes e_{2j-3} + e_3 \otimes e_{2j-2}) + (e_8 \otimes e_{2j-7} + e_7 \otimes e_{2j-6}) + \cdots\}$.

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§ 2. Homology groups of $B(Z_p \times Z_p)$

Let $X = B(Z_p \times Z_p)$, $Y = B(Z_p)$.
Case (a): $p=2$.

Let RP^n be the n dimensional real projective space, RP^∞ be the direct limit of it. Then we can consider $Y = RP^\infty$, and so $X = Y \times Y$. The cell structure of RP^n and its boundary operations are given as follows:

$$RP^n = e_0 \cup e_1 \cup \cdots \cup e_n,$$

(1. 1) $$\partial e_{2i}=2e_{2i-1}, \quad \partial e_{2i+1}=0,$$

where e_i is the i dimensional cell. Y is a CW complex with one cell e_i in each dimension. We will use the same symbol e_i for the homology class containing e_i.

Let $C_*(X)$ and $C_*(Y)$ be the chain complexes as CW complex X and Y respectively. $C_*(X)\cong C_*(Y)\otimes C_*(Y)$ by cross product, thus $C_n(X)=H_n(X^n, X^{n-1};Z)\cong\sum_{i+j=n}H(Y^i, Y^{i-1};Z)\otimes H_j(Y^j, Y^{j-1};Z)$, where X^n and Y^n are n-skeleton of X and Y respectively. Therefore $C_n(X)$ is generated by $e_i\otimes e_{n-i}$ for $i=0, 1, \cdots, n$ and $\partial_n: C_n(X)\to C_{n-1}(X)$ is given as follows:

(1. 2)
$$\partial_n(e_{2i-1}\otimes e_{2j-1})=0,$$
$$\partial_n(e_{2i}\otimes e_{2j-1})=2e_{2i-1}\otimes e_{2j-1},$$
$$\partial_n(e_{2i-1}\otimes e_{2j})=-2e_{2i-1}\otimes e_{2j-1},$$
$$\partial_n(e_{2i}\otimes e_{2j})=2e_{2i-1}\otimes e_{2j}+2e_{2i}\otimes e_{2j-1}.$$

Then we have

(1. 3) *$H_{2n}(X;Z)$ is generated by $e_{2i-1}\otimes e_{2n-2i+1}$ for $i=1, \cdots, n$ and $H_{2n-1}(X;Z)$ is generated by $e_{2i-1}\otimes e_{2n-2i}+e_{2i}\otimes e_{2n-1-2i}$ for $i=0, 1, \cdots, n$ and every elements are order 2. $H_0(X;Z)\cong Z$ generated by $e_0\otimes e_0$.*

Case (b): p is the odd prime.

Let S^{2n+1} be the unit $(2n+1)$-sphere. A point of S^{2n+1} is represented by a $(n+1)$-tuple of complex numbers $(z_0, z_1, \cdots, z_n)$ with $\sum_{i=0}^{n}|z_i|^2=1$. Let T be the rotation of S^{2n+1} defined by $T(z_0, z_1, \cdots, z_n)=(\lambda z_0, \lambda z_1, \cdots, \lambda z_n)$, where $\lambda=\exp(2\pi i/p)$. T generates a fixed point free topological transformation group of S^{2n+1} of order p, so we will say it Z_p action on S^{2n+1}. Then the lens space mod p is defined to be the orbit space $L^{2n+1}(p)=S^{2n+1}/Z_p$. This is the closed orientable $2n+1$ smooth manifold. For $m<n$ consider S^{2m+1} as contained in S^{2n+1} with $(z_0, \cdots, z_m)=(z_0, \cdots, z_m, 0, 0, \cdots)$. Then $L^1(p)\subset L^3(p)\subset\cdots$. Let $L^\infty(p)$ be the direct limit of this sequence, then we can consider $Y=L^\infty(p)$, and so $X=Y\times Y$. The cell structure of $L^{2n+1}(p)$, and its boundary relations are given as follows:

$$L^{2n+1}(p)=e_0\cup e_1\cup\cdots\cup e_{2n+1},$$
(1. 4) $$\partial e_{2i}=pe_{2i-1}, \quad \partial e_{2i+1}=0.$$

Y is a CW complex with one cell e_i in each dimension and the $(2n+1)$-skeleton is $L^{2n+1}(p)$. $C_n(X)$ is generated by $e_i\otimes e_{n-i}$ $(i=0, 1, \cdots, n)$ and ∂_n: $C_n(X)\to C_{n-1}(X)$ is given as follows:

(1.5) $$\partial_n(e_{2i-1}\otimes e_{2j-1})=0\,,$$
$$\partial_n(e_{2i}\otimes e_{2j-1})=pe_{2i-1}\otimes e_{2j-1}\,,$$
$$\partial_n(e_{2i-1}\otimes e_{2j})=-pe_{2i-1}\otimes e_{2j-1}\,,$$
$$\partial_n(e_{2i}\otimes e_{2j})=pe_{2i-1}\otimes e_{2j}+pe_{2i}\otimes e_{2j-1}\,.$$

Then we have

(1.6) $H_{2n}(X;Z)$ *is generated by* $e_{2i-1}\otimes e_{2n-2i+1}$ $(i=1,\cdots,n)$, $H_{2n-1}(X;Z)$ *is generated by* $e_{2i-1}\otimes e_{2n-2i}+e_{2i}\otimes e_{2n-2i-1}$ $(i=0,1,\cdots,n)$ *and every elements are order* p. $H_0(X;Z)\cong Z$ *generated by* $e_0\otimes e_0$.

§ 3. Theorems

Let $\Omega_n(X, A)$ be n-dimensional oriented bordism group of (X, A). There is a natural homomorphism $\mu:\Omega_n(X,A)\to H_n(X,A;Z)$. Given $[B^n,f]\in\Omega_n(X,A)$, let $\sigma_n\in H_n(B^n,\partial B^n;Z)$ denote the fundamental homology class of B^n. Then μ is defined $\mu[B^n,f]=f_*(\sigma_n)\in H_n(X,A;Z)$. The image of μ is the subgroup of integral homology classes representable in the sense of Steenrod. μ has following properties which are proved by Conner-Floyd.

THEOREM 2. (Conner-Floyd) ([2], (7.2))

The edge homomorphism $\Omega_n(X,A)=J_{n,0}\to E^\infty_{n,0}\to E^2_{n,0}=H_n(X,A;Z)$ *of the bordism spectral sequence coincides with the homomorphism* $\mu:\Omega_n(X,A)\to H_n(X,A;Z)$.

THEOREM 3. (Conner-Floyd) ([2], (15.1))

If (X, A) *is a CW pair then the bordism spectral sequence is trivial if and only if* $\mu:\Omega_n(X,A)\to H_n(X,A;Z)$ *is an epimorphism for all* $n\geqq 0$.

THEOREM 4. (Conner-Floyd) ([2], (15.2))

If (X, A) *is a CW pair such that each* $H_n(X,A;Z)$ *is finitely generated and has no odd torsion, then the bordism spectral sequence is trivial.*

Next theorem is useful to obtain the manifold with Z_p action.

THEOREM 5. (Conner-Floyd) ([2], (46.1))

Consider the generating set α_{2k-1}; $k=1,2,\cdots$ *for* $\Omega_*(Z_p)$, p *an odd prime, where* $\alpha_{2k-1}=[T,S^{2k-1}]$. *Then there exist closed oriented manifolds* M^{4k}, $k=1,2,\cdots$, *such that for each* k, $p\alpha_{2k-1}+[M^4]\alpha_{2k-5}+[M^8]\alpha_{2k-9}+\cdots=0$ *in* $\Omega_*(Z_p)$.

§ 4. Proof of Theorem 1.

Case (a): $p=2$.

This case follows immediately from Theorems 3 and 4. Because each

$H_n(B(Z_2 \times Z_2); Z)$ is finitely generated and has no odd torsion from (1. 3).

REMARK. $e_0 \otimes e_0$, $e_{2i-1} \otimes e_0$, $e_0 \otimes e_{2j-1}$ and $e_{2i-1} \otimes e_{2j-1}$ are explicitly represented by $RP^0 \times RP^0$, $RP^{2i-1} \times RP^0$, $RP^0 \times RP^{2j-1}$ and $RP^{2i-1} \times RP^{2j-1}$ respectively. $e_{2i-1} \otimes e_{2n-2i} + e_{2i} \otimes e_{2n-1-2i}$ is represented by $H_{2i,2n-2i}$ which is the subset in $RP^{2i} \times RP^{2n-2i}$ defined by the equation

$$x_0 y_0 + x_1 y_1 + \cdots + x_m y_m = 0 ,$$

where $m = \min(2i, 2n-2i)$, and $(x_0, \cdots, x_{2i})$ and $(y_0, \cdots, y_{2n-2i})$ are the standard homogeneous coordinates in RP^{2i} and RP^{2n-2i} respectively. It is a smooth submanifold of codimension 1, and orientable because its first *Stiefel-Whitney* class $w_1 = 0$. Consider the intersection of $H_{2i,2n-2i}$ and 1 cycles of $RP^{2i} \times RP^{2n-2i}$ we can see that $i_* : H_{2n-1}(H_{2i,2n-2i}; Z) \to H_{2n-1}(RP^{2i} \times RP^{2n-2i}; Z)$ is non-trivial, that is onto.

Case (b): p an odd prime.

By Theorem 5 there exists compact orientable $2n$ dimensional manifold V^{2n} with $\partial V^{2n} = pS^{2n-1} \cup M^4 \times S^{2n-5} \cup M^8 \times S^{2n-9} \cup \cdots$ and an action of Z_p restricted to $M^{4k} \times S^{2n-4k-1}$ is $id \times T$. We can chose following classifying maps from the property of classifying space:

$$f_{2n} : V^{2n}/Z_p \longrightarrow Y = B(Z_p) \qquad \text{such that}$$
$$f_{2n}(V^{2n}/Z_p) \subseteq Y^{2n}, \quad f_{2n}(M^{4k} \times S^{2n-4k-1}/Z_p) \subseteq Y^{2n-4k-1}$$

and $f_{2n*}(\sigma_{2n}) = e_{2n}$, where σ_{2n} is fundamental homology class of V^{2n}/Z_p.

Let $f_0 : V^0/Z_p \to Y^0$ and let $f_{2n-1} : S^{2n-1}/Z_p \to Y^{2n-1}$ be inclusion, then $f_{2n-1*}(\sigma'_{2n-1}) = e_{2n-1}$, where σ'_{2n-1} is fundamental class of S^{2n-1}/Z_p.

Next let $G = Z_p \times Z_p$ and choose classifying maps

$$g_j : S^{2j-1} \times S^{2n-2j+1}/G \longrightarrow X^{2n},$$
$$h_j : V^{2j} \times S^{2n-2j-1}/G \longrightarrow X^{2n-1},$$
$$k_j : S^{2j-1} \times V^{2n-2j}/G \longrightarrow X^{2n-1},$$
$$l_j : V^{2j} \times V^{2n-2j}/G \longrightarrow X^{2n}$$

such that
$$h_j(M^{4k} \times S^{2j-4k-1} \times S^{2n-2j-1}/G) \subseteq X^{2n-4k-2},$$
$$k_j(M^{4k} \times S^{2j-1} \times S^{2n-2j-4k-1}/G) \subseteq X^{2n-4k-2},$$

and
$$l_j\Big(\big\{(M^{4k} \times V^{2j} \times S^{2n-2j-4k-1}/G) \cup (M^{4k} \times S^{2j-4k-1} \times V^{2n-2j}/G)\big\}\Big) \subseteq X^{2n-4k-1}.$$

Then each fundamental class is mapped onto $e_{2j-1} \otimes e_{2n-2j+1}$, $e_{2j} \otimes e_{2n-2j-1}$, $e_{2j-1} \otimes e_{2n-2j}$ and $e_{2j} \otimes e_{2n-2j}$ by g_{j*}, h_{j*}, k_{j*} and l_{j*} respectively.

Let

$$\begin{aligned}\alpha_j^{2n}&=[g_j,\ S^{2j-1}\times S^{2n-2j+1}/G]\,, & j&=1,\cdots,n,\\ \delta_j^{2n}&=[l_j,\ V^{2j}\times V^{2n-2j}/G]\,, & j&=0,\cdots,n,\\ \beta_j^{2n-1}&=[h_j,\ V^{2j}\times S^{2n-2j-1}/G]\,, & j&=0,\cdots,n-1,\\ \gamma_j^{2n-1}&=[k_j,\ S^{2j-1}\times V^{2n-2j}/G]\,, & j&=1,\cdots,n.\end{aligned}$$

Then α_j^{2n} and δ_j^{2n} generate $\Omega_*(X^{2n},X^{2n-1})$ freely over Ω_*, and β_j^{2n-1} and γ_j^{2n-1} generate $\Omega_*(X^{2n-1},X^{2n-2})$ freely over Ω_*, because $\mu:\Omega_*(X^r,X^{r-1})\to H_*(X^r,X^{r-1};\Omega_*)$ is an Ω_* isomorphism.

LEMMA.

C^2-term of bordism spectral sequence of $X=B(Z_p\times Z_p)$ is generated over Ω_* by δ_0^0, α_i^{2n}, β_0^{2n-1}, γ_n^{2n-1}, and $(\beta_j^{2n-1}+\gamma_j^{2n-1})$, $(n=1,2,\cdots;\ i=1,\cdots,n;\ j=1,\cdots,n-1)$ and B^2-term is generated over Ω_* by $p\alpha_i^{2n}$, $p\beta_0^{2n-1}$, $p\gamma_n^{2n-1}$ and $p(\beta_j^{2n-1}+\gamma_j^{2n-1})$, $(n=1,2,\cdots;\ i=1,\cdots,n;\ j=1,\cdots,n-1)$.

PROOF.

$$\begin{aligned}C_*^2&=Ker(\partial:\Omega_*(X^r,X^{r-1})\longrightarrow\Omega_*(X^{r-1},X^{r-2}))\\ &=\mu^{-1}(Ker\ \partial:H_*(X^r,X^{r-1};\Omega_*)\longrightarrow H_*(X^{r-1},X^{r-2};\Omega_*))\end{aligned}$$

and

$$\partial\mu(\delta_j^{2n})=pe_{2j-1}\otimes e_{2n-2j}+pe_{2j}\otimes e_{2n-2j-1}\,,$$

$$\partial\mu(\alpha_j^{2n})=0\,,\quad \partial\mu(\beta_j^{2n-1})=pe_{2j-1}\otimes e_{2n-2j-1}$$

and

$\partial\mu(\gamma_j^{2n-1})=-pe_{2j-1}\otimes e_{2n-2j-1}$ therefore C^2-term follows.

$$\begin{aligned}B_*^2&=Im\,(\partial:\Omega_*(X^{r+1},X^r)\longrightarrow\Omega_*(X^r,X^{r-1}))\\ &=\mu^{-1}Im(\partial:H_*(X^{r+1},X^r;\Omega_*)\longrightarrow H_*(X^r,X^{r-1};\Omega_*))\end{aligned}$$

and $\mu^{-1}\partial(e_{2j}\otimes e_{2n+1-2j})=p\alpha_j^{2n}$, $\mu^{-1}\partial(e_{2j-1}\otimes e_{2n-2j+2})=-p\alpha_j^{2n}$, $\mu^{-1}\partial(e_{2j}\otimes e_{2n-2j})=p(\gamma_j^{2n-1}+\beta_j^{2n-1})$, $\mu^{-1}\partial(e_0\otimes e_{2n})=p\beta_0^{2n-1}$, $\mu^{-1}\partial(e_{2n}\otimes e_0)=p\gamma_n^{2n-1}$ and $\mu^{-1}\partial(e_{2j-1}\otimes e_{2n+1-2j})=0$, so we have B^2-term.

Next theorem essentially is the same as the case $p=3$ proved by Burdick [1].

THEOREM 6. *The bordism spectral sequence of $X=B(Z_p\times Z_p)$ is as follows:*

$$E^2\cong\cdots\cong E^5;\quad E^6\cong\cdots\cong E^\infty$$

E^∞ is generated by δ_0^0, α_i^{2n}, β_0^{2n-1}, γ_n^{2n-1} $(n=1,2,\cdots,\ i=1,2,\cdots,n)$, $\{(\beta_1^{2n-1}+\gamma_1^{2n-1})+(\beta_3^{2n-1}+\gamma_3^{2n-1})+\cdots\}$, and $\{(\beta_2^{2n-1}+\gamma_2^{2n-1})+(\beta_4^{2n-1}+\gamma_4^{2n-1})+\cdots\}$ with relations $[M^4][\alpha_i^{2n}-\alpha_{i-2}^{2n}]=0$, and every element except δ_0^0 has order p.

PROOF.

Every elements of $H_m(X;Z)$ have order p (odd prime). Ω_n is free group if $n\equiv 0$ $(mod\ 4)$ and 2-torsion groups if $n\not\equiv 0$ $(mod\ 4)$.

$$E_{m,n}^{2}=H_{m}(X;\Omega_{n})=H_{m}(X;Z)\otimes\Omega_{n}+H_{m-1}(X;Z)*\Omega_{n}\,.$$

Therefore we have $d^{2}=d^{3}=d^{4}=0$, so $E^{2}\cong\cdots\cong E^{5}$. Now recall the definition of $d_{m,n}^{r}$:

$$\begin{array}{ccccc}
 & & & & \Omega_{m+n}(X^{m-1},X^{m-r}) \\
 & & & \swarrow i''_{*} & \downarrow \partial_{3} \\
 & & \Omega_{m+n}(X^{m},X^{m-r}) & \xrightarrow{\partial_{2}} & \Omega_{m+n-1}(X^{m-r},X^{m-r-1}) \\
 & \nearrow i_{*} & j_{*}\downarrow & \Psi \searrow & \downarrow i'_{*} \\
\Omega_{m+n}(X^{m},X^{m-r-1}) & \xrightarrow[j'_{*}]{} & \Omega_{m+n}(X^{m},X^{m-1}) & \xrightarrow[\partial_{1}]{} & \Omega_{m+n-1}(X^{m-1},X^{m-r-1}),
\end{array}$$

where i,j,i',j' are inclusion maps and ∂_{1}, ∂_{2}, ∂_{3} are boundary homomorphisms of triple, then there exist homomorphism Ψ such that $\Psi=\partial_{1}\cdot j_{*}=i'_{*}\cdot\partial_{2}$, every triangles are commutative.

Let $C_{m,n}^{r}=Im\,j_{*}$, $C_{m,n}^{r+1}=Im\,j'_{*}$, $B_{m-r,n+r-1}^{r+1}=Im\,\partial_{2}$ and $B_{m-r,n+r-1}^{r}=Im\,\partial_{3}$. Then the definition of $d_{m,n}^{r}$ is composition of $d_{m,n}^{r}:E_{m,n}^{r}=C_{m,n}^{r}/B_{m,n}^{r}\xrightarrow[p_{r}]{}C_{m,n}^{r}/C_{m,n}^{r+1}\underset{\cong}{\xrightarrow{\partial_{1}}}$ $Im\,\Psi\underset{\cong}{\xrightarrow{i_{*}^{-1}}}B_{m-r,n+r-1}^{r+1}/B_{m-r,n+r-1}^{r}\longrightarrow C_{m-r,n+r-1}^{r}/B_{m-r,n+r-1}^{r}=E_{m-r,n+r-1}^{r}$. Here let $r=5$ then $d^{5}(\delta_{0}^{0})=d^{5}(\alpha_{i}^{2n})=d^{5}(\beta_{0}^{2n-1})=d^{5}(\gamma_{n}^{2n-1})=0$. Because δ_{0}^{0}, α_{i}^{2n}, β_{0}^{2n-1}, γ_{n}^{2n-1} are represented by closed manifolds ∂_{1} will kill them.

For $j=1,\cdots,n-1$ let N_{j}^{2n-1} be the manifold obtained from $V^{2j}\times S^{2n-2j-1}\cup S^{2j-1}\times V^{2n-2j}$ by joining $pS^{2j-1}\times S^{2n-2j-1}$ in $\partial(V^{2j}\times S^{2n-2j-1})$ to $-pS^{2j-1}\times S^{2n-2j-1}$ in $\partial(S^{2j-1}\times V^{2n-2j})$.

Then $\partial N_{j}^{2n-1}=M^{4}\times S^{2j-5}\times S^{2n-2j-1}\cup -M^{4}\times S^{2j-1}\times S^{2n-2j-5}$
$\cup M^{8}\times S^{2j-9}\times S^{2n-2j-1}\cup -M^{8}\times S^{2j-1}\times S^{2n-2j-9}\cup\cdots.$

There is an induced action of $G=Z_{p}\times Z_{p}$ on N_{j}^{2n-1}. Choose classifying maps $\phi_{j}:N_{j}^{2n-1}/G\longrightarrow X^{2n-1}$ such that $\phi_{j}(\partial(N_{j}^{2n-1}/G))\subseteq X^{2n-6}$ and such that

$$\begin{array}{ccc}
N_{j}^{2n-1}/G & \xrightarrow{\phi_{j}} & X^{2n-1} \\
\nwarrow & & \nearrow h_{j}\cup k_{j} \\
 & (V^{2j}\times S^{2n-2j-1}/G)\cup(S^{2j-1}\times V^{2n-2j}/G) &
\end{array}$$

commutes up to homotopy.

Then $\phi_{j*}(\sigma)=e_{2j}\otimes e_{2n-2j-1}+e_{2j-1}\otimes e_{2n-2j}$, where σ is a fundamental class of

N_j^{2n-1}/G. Thus $[\phi_j, N_j^{2n-1}/G]=\beta_j^{2n-1}+\gamma_j^{2n-1}$ in $\Omega_*(X^{2n-1}, X^{2n-2})$.
By the definition of d^5, $d^5(\beta_j^{2n-1}+\gamma_j^{2n-1})=[M^4][\alpha_{j-2}^{2n-6}-\alpha_j^{2n-6}]$. Therefore Ker d^5 is generated by δ_0^0, α_i^{2n}, β_0^{2n-1}, γ_n^{2n-1} and $\{(\beta_1^{2n-1}+\gamma_1^{2n-1})+(\beta_3^{2n-1}+\gamma_3^{2n-1})+\cdots\}$ and $\{(\beta_2^{2n-1}+\gamma_2^{2n-1})+(\beta_4^{2n-1}+\gamma_4^{2n-1})+\cdots\}$.
Let K_1^{2n-1} be the identification manifold obtained from

$$V^2\times S^{2n-3}\cup S^1\times V^{2n-2}\cup V^6\times S^{2n-7}\cup S^5\times V^{2n-6}\cup V^{10}\times S^{2n-11}\cup\cdots$$

by identifying pair-wise of boundary components of this manifold.

Then K_1^{2n-1} is an orientable closed $(2n-1)$-manifold with induced natural action of $G=Z_p\times Z_p$.
Let $\Psi_1 : K_1^{2n-1}/G\to X^{2n-1}$ be a classifying map, then $[\Psi_1, K_1^{2n-1}/G]=(\beta_1^{2n-1}+\gamma_1^{2n-1})+(\beta_3^{2n-1}+\gamma_3^{2n-1})+\cdots$ in $\Omega_*(X^{2n-1}, X^{2n-2})$. Likewise construct K_2^{2n-1} from

$$V^4\times S^{2n-5}\cup S^3\times V^{2n-4}\cup V^8\times S^{2n-9}\cup S^7\times V^{2n-8}\cup\cdots$$

and $\Psi_2 : K_2^{2n-1}/G\to X^{2n-1}$ with

$$[\Psi_2, K_2^{2n-1}/G]=(\beta_2^{2n-1}+\gamma_2^{2n-1})+(\beta_4^{2n-1}+\gamma_4^{2n-1})+\cdots$$

Therefore every generator of E^6 can be represented by a closed manifolds, so $d^6=d^7=\cdots=0$ and hence $E^6\cong\cdots\cong E^\infty$.

PROOF of THEOREM 1.

The classes listed in Theorem 6 really belong to $E_{*,0}^\infty$. Therefore from Theorems 2 and 6 $e_0\otimes e_0$, $e_0\otimes e_{2j-1}$, $e_{2i-1}\otimes e_0$ and $e_{2i-1}\otimes e_{2j-1}$ are represented by $V^0\times V^0/G$, $V^0\times S^{2j-1}/G$, $S^{2i-1}\times V^0/G$ and $V^{2i-1}\times V^{2j-1}/G$ respectively.

$$(e_2\otimes e_{2j-1}+e_1\otimes e_{2j})+(e_6\otimes e_{2j-5}+e_5\otimes e_{2j-4})+\cdots \quad \text{and}$$
$$(e_4\otimes e_{2j-3}+e_3\otimes e_{2j-2})+(e_8\otimes e_{2j-7}+e_7\otimes e_{2j-6})+\cdots \quad \text{are}$$

represented by K_1^{2j+1}/G and K_2^{2j+1}/G respectively.

The proof of Theorem 1 is completed.

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