On the propagation speed of hyperbolic operator with mixed boundary conditions

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction and results.

We are concerned in this paper with the propagation speeds of solutions of the mixed problem:

$$
(1.1)\qquad
\begin{aligned}
&P(X, D)u=f \text{ in } x_0>0,\ x_1>0, \\
&B_j(X, D)u=g_j \text{ in } x_0>0,\ x_1=0 \qquad (j=1, 2\cdots l), \\
&D_{x_0}^k u=h_k \text{ in } x_0=0,\ x_1>0 \qquad (k=0, 1, \cdots, m-1).
\end{aligned}
$$

Here $X=(x_0, x_1, \cdots, x_n)$, $\sqrt{-1}\ D_{x_i}=\dfrac{\partial}{\partial x_i}$, $P(X, D)$ is a x_0-strictly hyperbolic operator of order m, l is the number of roots λ with positive imaginary part of $p_m(X, \tau, \lambda, \sigma)=0$ with $Im\ \tau>0$, $\sigma=(\sigma_2, \sigma_3\cdots, \sigma_n)\in R^{n-1}$, $B_j(j=1, 2\cdots, l)$ are differential operators of order m_j. Furthermore we assume that $m_i \lneqq m$, $m_i \neq m_j$ $(i\neq j)$ and that P, $B_j(j=1, 2, \cdots, l)$ are non-characteristic with respect to the hyperplane $x_1=0$.

Throughout the paper we assume that the coefficients of P and B_j be constant, unless the contrary is explicitly stated. We say that $\rho(-(\omega_1, \omega))$ is the propagation speed in the direction $-(\omega_1, \omega)$ with $\omega_1\leq 0$ of P under the mixed boundary conditions if $\rho(-(\omega_1, \omega))$ is the minimum of $\rho(\geqq 0)$ with the following property:

$$
(1.2)\qquad
\begin{aligned}
&\max_{(x, y)}\left\langle \text{supp } (u(t, ., .)), -(\omega_1, \omega)\right\rangle \\
&\leqq \rho t+\max_{(x, y)}\left\langle \text{supp } (u(0, ., .)), -(\omega_1, \omega)\right\rangle
\end{aligned}
$$

for any $t\in[0, T]$ and for any solution $u\in C^m([0\ T]\times R_+^n)$ of (1. 1) with $f=0$ and $g_j=0\,(j=1, 2, \cdots, l)$. Where $t=x_0$, $x=x_1$, $y_i=x_i\,(i=2, \cdots, n)$, $R_+^n=\{x, y)|x\geqq 0\}$ and T is an arbitrary, but fixed positive number.

Let $R(\tau, \sigma)$ be the Lopatinski determinant for (1. 1) with $R_0(\tau, \sigma)$ as its principal part. Put $N=(1, 0, 0)$. We denote by $\Gamma(P, N)$ the connected component containing N in R^{n+1} of the complement of the zeros of $p_m(\xi)$.

The aim of the present paper is to prove the following theorems.

THEOREM 1. *Let* $(\omega_1, \omega)\in S^{n-1}$. *Set* $\rho_0(\omega_1, \omega)=l.u.b.\{\rho>0|(1, \rho\omega_1, \rho\omega)\in \Gamma(P, N)\}$. *Moreover let* $\omega\in S^{n-2}$. *Set* $\rho_1(\omega)=min\{\rho\geqq 0|R_0(1, \rho\omega)=0\}$. *Then we have*

$$\rho(-(\omega_1, \omega))=max\{\rho_0(\omega_1, \omega)^{-1},\ \rho_1(\omega/\|\omega\|)^{-1}\|\omega\|\}$$

for (1.1) *with* $R_0(1, 0)\neq 0$. *Where we assume* $-1<\omega_1\leqq 0$.

THEOREM 2. *Let* P *and* $B_j(j=1, 2, \cdots, l)$ *be homogeneous operators. Assume problem* (1.1) *with homogeneous boundary conditions be* L^2-*well posed, i.e., there are constants* $C>0$ *and* $T>0$ *such that for any* $f\in H^1([-\infty, T]\times R^n_+)$ *with supp* $(f)\subset[0, T]\times R^n_+$ *there exists a solution* $u\in H^m([-\infty, T]\times R^n_+)$ *with supp* $(u)\subset[0, T]\times R^n_+$ *enjoining the following inequality*:

$$\|u\|_{m-1}([0, T]\times R^n_+)\leqq C\|f\|_0([0, T]\times R^n_+),$$

furthermore we assume that such solution be unique.
Then the propagation speed in the direction $-(\omega_1, \omega)$ *with* $-1\leqq\omega_1\leqq 0$ *coincides with that of solutions for Cauchy proboem with respect to the operator* P.

Theorem 2 is a direct consequence of Theorem 1 and the following

THEOREM 3. *Under the assumptions of Theorem 2, the Hersh's condition is valid for the problem* (1, 1), *i.e.,* $R(\tau, \sigma)$ *is not zero for any* (τ, σ) *with Im* $\tau<0$ *and* $\sigma\in R^{n-1}$. *Furthermore* $R(\tau, \sigma)$ *does not vanish, whenever* $(\tau, \lambda, \sigma)\in\Gamma(P, N)$ *for some real* λ.

It is not difficult to see that the above theorems are extended to the mixed problems for systems of operators of the first order. In fact by R. M. Lewis' results [9] we were suggested the assertion of Theorem 3. Moreover our results are also extended to the operator P such that the hyperplane $x_1=0$ is characteristic. Therefore it seems to us that our results will be interesting for further investigations of energy inequalities and wave propagations for mixed problems of hyperbolic systems.

§2. The proofs of Theorem 2 and 3.

Under the assumptions of Theorem 2 for the problem (1.1) the author and Agemi [1] proved the following

LEMMA 2.1. *i) Let* V *be the set* $\{(\tau, \sigma)|\ \mathrm{Im}\ \tau<0,\ \sigma\in R^{n-1},\ R(\tau, \sigma)=0\}$. *Then* $S(\tau)=\{\sigma|(\tau, \sigma)\in V\}$ *is independent of* τ *and its Lebesgue measure is zero. ii) Let* $(\tau_0, \sigma_0)\in S^{n-1}$ *such that the roots* λ *of* $P(\tau_0, \lambda, \sigma_0)=0$ *are separated.*

Then there is a neighborhood $U(\tau_0, \sigma_0)$ *such that for any* $(\tau, \sigma)\in V^c\cap U(\tau_0, \sigma_0)$ *with Im* $\tau<0$, $|\tau|^2+|\sigma|^2=1$ *and for any* $j=1, 2, \cdots, l$, $k=l+1, \cdots, m$

(2. 1) $$|C_j(\tau, \lambda_k^-(\tau, \sigma), \sigma)|^2 \leqq C(\tau_0, \sigma_0)|\,Im\ \lambda_j^+(\tau, \sigma)||\,Im\ \lambda_k^-(\tau, \sigma)||\,Im\ \tau|^{-2},$$

where $\lambda_i^{\pm}(\tau, \sigma)$ *are roots of* $P(\tau, \lambda, \sigma)=0$ *with* $Im\ \lambda_i^+(\tau, \sigma)>0$ *and* $Im\ \lambda_i^-(\tau, \sigma)<0$ *respectively,* $C(\tau_0, \sigma_0)$ *is a positive constant and* $C_j(\tau, \lambda_k^-(\tau, \sigma), \sigma)$

$$=\left|\begin{pmatrix} B_h(\tau, \lambda_i^+(\tau, \sigma), \sigma) \\ i\rightarrow \quad 1, 2, \cdots, l \\ h\downarrow \end{pmatrix}\right|^{-1} \cdot \left|\begin{matrix}\text{The matrix replacing } \lambda_j^+(\tau, \sigma) \\ \text{in the left one by } \lambda_k^-(\tau, \sigma)\end{matrix}\right|.$$

Now let $\theta=(\tau_0, \lambda_0, \sigma_0)\in\Gamma(P, N)$. Then for any lower order term Q, $(P+Q)(t\theta+se_1)\in hyp(\theta)$ where $e_1=(0, 1, 0, \cdots, 0)$. Therefore $P(t\theta+se_1)\in Hyp_0(\theta)$ and since the surface $x_1=0$ is noncharacteristic with respect to P, we see that the roots $t_k(s)$ are real, distinct and non-zero whenever s is real and not zero. Hence it implies that the roots λ of $P(\tau_0, \lambda, \sigma_0)=0$ are the form $t_k(1)^{-1}\cdot(t_k(1)\cdot\lambda_0+1)$ which are also real and distinct. Thus for $(\tau_0', \sigma_0')=(\tau_0, \sigma_0)((|\tau_0|^2+|\sigma_0|^2)^{-\frac{1}{2}})$ it satisfies the condition ii) of Lemma 2. 1. Furthermore it follows from the fact that λ_k are all real and distinct that for some neighborhood $U(\tau_0', \sigma_0')$ and for all $(\tau', \sigma')\in V^c\cap U(\tau_0', \sigma_0')$ with $Im\ \tau'<0$

$$|Im\ \lambda_j^{\pm}(\tau', \sigma')|=0(|Im\ \tau'|).$$

Thus from (2. 1) we see that for such (τ', σ')

$$|C_j(\tau', \lambda_k^-(\tau', \sigma'), \sigma')|\leqq K'<\infty.$$

Furthermore by the homogeneity of P and B_j we see that for some neighborhood $U(\tau_0, \sigma_0)$, for any $(\tau, \sigma)\in V^c\cap U(\tau_0, \sigma_0)$ with $Im\ \tau<0$ and for any $j=1, \cdots, l$; $k=l+1, \cdots, m$

(2. 2) $$|C_j(\tau, \lambda_k^-(\tau, \sigma, \sigma)|\leqq K<\infty.$$

Since $S=S(\tau)$ is independent of τ, we can select a real analytic curve $\sigma(\eta)=\sigma(\tau)$ for $\eta\in[-\varepsilon, \varepsilon]$ $(\varepsilon>0)$ such that

$$\tau=\tau_0+i\eta,\ \sigma(0)=\sigma_0 \text{ and } \sigma(\eta)\notin S \text{ for } \eta\in[-\varepsilon, 0).$$

Now let $F(\tau, \lambda)$ be

$$\left|\begin{matrix} B_1(\tau, \lambda, \sigma(\tau)), & B_1(\tau, \lambda_2^+(\tau, \sigma(\tau)), \sigma(\tau)), \cdots, & B_1(\tau, \lambda_l^+(\tau, \sigma(\tau)), \sigma(\tau)) \\ \vdots & \vdots & \vdots \\ B_l(\tau, \lambda, \sigma(\tau)), & B_l(\tau, \lambda_2^+(\tau, \sigma(\tau)), \sigma(\tau)), \cdots, & B_l(\tau, \lambda_l^+(\tau, \sigma(\tau)), \sigma(\tau)) \end{matrix}\right|.$$

Then $R(\tau, \sigma(\tau))\cdot\prod_{i>j}(\lambda_i^+(\tau, \sigma(\tau))-\lambda_j^+(\tau, \sigma(\tau)))=F(\tau, \lambda_1^+(\tau, \sigma(\tau))$ which we denote by $R(\tau)$. Since for $\eta<0$ and for $\tau=\tau_0+i\eta\,(\tau, \sigma(\tau))\notin V$, $R(\tau)\neq0$.
Therefore if $R(\tau_0)=0$, there is an integer $k\geqq1$ such that for some $a_k\neq0$, $R(\tau)=a_k\ \eta^k(1+0(\eta))$, i. e.,

(2. 3) $$F(\tau, \lambda_1^+(\tau, \sigma(\tau))) = 0(|\eta|^k).$$

Then from (2. 2) and (2. 3) it follows that

$$F(\tau, \lambda_i^-(\tau, \sigma(\tau))) = 0(|\eta|^k) \qquad \text{for } i=l+1, \cdots, m,$$

and obviously we see that

$$F(\tau, \lambda_i^+(\tau, \sigma(\tau))) \equiv 0 \qquad \text{for } j=2, \cdots, l.$$

Since degree$_\lambda$ $F(\tau, \lambda) = \max\limits_{i=1,\cdots,l} m_i < m$, by the above equalities we see that

$$F(\tau_0, \lambda) \equiv 0.$$

Furthermore since $F(\tau, \lambda) = B_1(\tau, \lambda, \sigma(\tau))\, A_{11}(\tau) + \cdots + B_l(\tau, \lambda, \sigma(\tau)) \cdot A_{1l}(\tau)$, where $A_{11}, \cdots, A_{1l}$ are $(l-1, l-1)$ cofactors of $R(\tau)$, and by hypotheses in §1 $B_i(\tau_0, \lambda, \rho_0)$ $(i=1, 2, \cdots, l)$ are linearly independent as functions of λ, we obtain that

$$A_{1i}(\tau_0) = 0 \qquad (i=1, 2, \cdots, l).$$

By the same method used above we also see that

(2. 4) $$A_{ij}(\tau_0) = 0 \qquad (i, j=1, 2, \cdots, l).$$

If $k \geqq 2$, using (2, 4) and differentiating $F(\lambda, \tau)$ with respect to τ,

$$B_1(\tau, \lambda_i(\tau, \sigma(\tau)), \sigma(\tau))\, A'_{11}(\tau) + \cdots + B_l(\tau, \lambda_i(\tau, \sigma(\tau)), \sigma(\tau)) \cdot A'_{1l}(\tau) = 0(|\eta|) \qquad \text{for } i=1, 2, \cdots, m.$$

Therefore from the same consideration used above it follows that

$$|A_{ij}(\tau)| = 0(|\eta|^2) \qquad (i, j=1, 2, \cdots, l).$$

By the induction with respect to k we conclude that

(2. 5) $$|A_{ij}(\tau)| = 0(|\eta|^k) \qquad (i, j=1, 2, \cdots, l).$$

Finally by simple calculation with respect to determinant and from (2. 5) it implies that

$$R(\tau)^{l-1} = |A_{ij}(\tau) \;{}^{i\rightarrow}_{j\downarrow}\; 1, 2, \cdots, l| \leqq 0(|\eta|^{kl}).$$

Therefore from (2. 3) we see that $(l-1)\, k \geqq kl$ which is contradiction. Thus we have the fact that the Lopatinski determinant $R(\tau, \sigma)$ is not zero whenever $(\tau, \lambda, \sigma) \in \Gamma(P, N)$ for some λ. In particular $(\tau, 0, \mathbf{0}) = \tau N \in \Gamma(P, N)$, hence $R(1, \mathbf{0}) \neq 0$. Therefore by corollary 3. 3 in our paper [1] we see that $R(\tau, \sigma) \neq 0$ for (τ, σ) with $Im\ \tau < 0$ and $\sigma \in R^{n-1}$, *i. e.*, V is empty. Thus we complete our proof of Theorem 3.

Now we show that Theorem 1 and 3 imply Theorem 2. To show this we have only to consider the case where $-1 < \omega_1 \leqslant 0$. Let $(1, \rho\omega_1, \rho\omega) \in \Gamma(P, N)$. Then by Theorem 3 we see that $R(1, \rho\omega) \neq 0$. Therefore by the definitions described in Theorem 1 we obtain that

$$\rho_0(\omega_1, \omega) \leqq \rho_1(\omega/\|\omega\|) \cdot \|\omega\|^{-1}.$$

Hence by Theorem 1 we see that

$$\rho(-(\omega_1, \omega)) = \rho_0(\omega_1, \omega)^{-1},$$

which is the propagation speed with respect to the solutions of Cauchy problem for P in the direction $-(\omega_1, \omega)$.

§ 3. The proof of Theorem 1.

In section 2 we deal only with L^2-sense-solutions, but hereafter we treat C^m-solutions of problems (1. 1) which is not always well posed. For this purpose we use the following

LEMMA 3. 1. *Let coefficients of P, B_j be real analytic and $f=h_k=0$ $(k=0, \cdots, m-1)$ and $g_i=\tilde{r}_i \cdot x_0^{m-m_i} \cdot H(x_0)$ $(i=1, \cdots, l)$ where $\tilde{r}_i$ are analytic in complex neighborhood $U(\mathbf{0})$ of the origin and let $H(x_0)$ be the Heaviside function with respect to x_0. Assume $R_0(1, \mathbf{0}) \neq 0$ where R_0 is the principal part of Lopatinski determinant with respect to the constant coefficients problem* (1. 1) *resulting from freezing the coefficients at the origin.*

Then there exist a neighborhood $U_1(\mathbf{0})$ independents of $\tilde{r}_i$ $(i=1, 2, \cdots, l)$ and a piecewise real analytic solution $u(X)$ of (1, 1) *defined in $U_1(\mathbf{0})$ with $x_1 \geqq 0$ such that snpp $(u(X))$ in $U_1(\mathbf{0})$ with $x_1 \geqq 0$ is contained in $R_+ \times R_+^n$.*

We can prove Lemma 3. 1 by a simple modification of Lax's consideration and Mizohata's estimate (See also Hamada [4]).

Using Lemma 3. 1 and Hörmander-Hersh's results [5] we obtain the following

LEMMA 3. 2. *Let the coefficients of P, B_j $(j=1, \cdots, l)$ be constant and let $R_0(\tau, \omega)$ be not identically zero. Then in order that* (1. 1) *have a non-trivial null solution it is necessary and sufficient that*

$$R_0(1, \mathbf{0}) = 0.$$

Now we proceed to prove Theorem 1. Under the assumption in Theorem 1, let $\xi=(1, \rho\omega_1, \rho\omega)$ with $\rho<\rho_0(\omega_1, \omega)$. Then by the definition of ρ_0, $\xi \in \Gamma(P, N)$. Now we consider the case $\rho_1(\omega/\|\omega\|) \cdot \|\omega\|^{-1} < \rho_0(\omega_1, \omega)$. If $\rho < \rho_1(\omega/\|\omega\|)\|\omega\|^{-1}$, $R_0(1, \rho\omega) = R_0(1, \rho\|\omega\| \, \omega/\|\omega\|) \neq 0$. Then by the coordinate transformation

$$(3.1) \qquad \begin{cases} t' = t + \sum_{i=1}^n \rho\omega_i \cdot y_i, \\ y_i' = y_i \qquad\qquad (i=2, 3, \cdots, n), \\ x' = x, \end{cases}$$

it follows that

$$P(D_t, D_x, D_y) = P(D_{t'}, D_{x'} + \rho\omega_1 D_{t'}, D_{y'} + \rho\omega D_{t'}),$$
$$B_j(D_t, D_x, D_y) = B_j(D_{t'}, D_{x'} + \rho\omega_1 D_{t'}, D_{y'} + \rho\omega D_{t'})$$

which we denote by $P'(D_{t'}, D_{x'}, D_{y'})$, $B'_j(D_{t'}, D_{x'}, D_{y'})$ respectively. Then $P'(1, \lambda, \mathbf{0}) = P(1, \lambda + \rho\omega_1, \rho\omega) = P((1, \rho\omega_1, \rho\omega) + \lambda e_1)$. Since $\xi \in \Gamma(P, N)$, as in the proof of Theorem 3, we see that the number of negative roots λ of $P'(1, \lambda, \mathbf{0}) = 0$ is l and the Lopatinski determinant $R_0(P', B'_j;\ 1, \mathbf{0})$ corresponding to the homogeneous operators P' B'_j are well defined and is equal to $R_0(1, \rho\omega) \neq 0$. Furthermore it is easy to see that all the assumptions in the introduction are valid for P', B'_j. Hence from Lemma 3. 1 with respect to its dual problem it follows that the Holmgren uniqueness theorem with respect to P, B_j with the initial surface $t + \rho\omega_1 \cdot x + \rho < \omega, y > = 0$ is true. From the fact that P, B_j are of constant coefficients and by translating the dependence domain of solutions, we see that $\rho(-(\omega_1, \omega)) \geqq \rho_1(\omega/\|\omega\|)^{-1} \|\omega\|$.

On the other hand if $\rho = \rho_1(\omega/\|\omega\|) \cdot \|\omega\|^{-1}$, then by the coordinate transformation analogous to (3. 1) the operators P, B_j are transformed to P', B'_j respectively such that $R_0(P', B'_j;\ \tau, \omega)$ does not vanish identically, but that

$$R_0(P', B'_j;\ 1, \mathbf{0}) = 0.$$

Therefore from Lemma 3. 2 we see that there exists a non-trivial solution $u(x)$ of

$$\begin{aligned} &Pu(X) = 0 \text{ in } x_1 > 0, \\ &B_j u(X) = 0 \text{ in } x_1 = 0 \qquad (j = 1, 2, \cdots, l), \\ &u(X) = 0 \text{ in } t + \rho\omega_1 x_1 + \rho < \omega, y > \leqq 0. \end{aligned} \tag{3. 2}$$

Then it follows from (3. 2) that

$$\operatorname*{Max}_{x,\, y} \left\langle supp\ u(0, x, y),\ -(\omega_1, \omega)\right\rangle = 0.$$
$$\operatorname*{Max}_{x,\, y} \left\langle supp\ u(t, x, y),\ -(\omega_1, \omega)\right\rangle = t\rho^{-1}$$

which implies $\rho(-(\omega_1, \omega)) \leqq \rho_1(\omega/\|\omega\|)^{-1} \cdot \|\omega\|$. Here we use, if necessary, translations of a non-trivial null solution.

Finally we must consider the case where $\rho_1(\omega/\|\omega\|)\, \|\omega\|^{-1} \geqq \rho_0(\omega_1, \omega)$, but we have already known that $\rho_0(\omega_1, \omega)^{-1}$ is the propagation speed of Cauchy problem for P in the direction $-(\omega_1, \omega)$. Therefore it is not difficult to see that $\rho(-(\omega_1, \omega)) = \rho_0(\omega_1, \omega)^{-1}$.

§ 4. Example.

Let $P(D_t, D_x, D_y) = D_t^2 - D_x^2 - D_y^2$ and $B(D_t, D_x, D_y) = D_x + bD_y + cD_t$, where

b and c are real.

Then if $|b|\leqq -c(c<0)$ or $b^2+1<c^2(c>0)$, for any (ω_1, ω) $\rho(-(\omega_1, \omega))=\rho_0(\omega_1, \omega)^{-1}$.

If $c=1$, $R(1, \mathbf{0})=0$. Finally in the other case $\rho(-(\omega_1, \omega))>\rho_0(\omega_1, \omega)^{-1}$ for some (ω_1, ω), i.e., there exists at least one supersonic wave (see Duff [3]).

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