On Riemannian Manifolds Satisfying the Condition $R(X, Y)R=0$

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 0. Introduction and Preliminaries.

Let $(M^n, g)(n \geqq 2)$ be an n-dimensional Riemannian manifold[1] with a positive definite metric tensor g. We denote by R and ∇ the curvature tensor and the Riemannian connection defined by g, respectively. For tangent vectors X and Y, we consider $R(X, Y)$ as a derivation of the tensor algebra at each point of (M^n, g). A conjecture by K. Nomizu [5][2] is the following.

CONJECTURE. *Let $(M^n, g)(n \geqq 3)$ be a complete irreducible Riemannian manifold. If (M^n, g) satisfies the condition*

$(*)$ $R(X, Y)R=0$ *for all tangent vectors X and Y,*

(M^n, g) is locally symmetric.

The purpose of this note is to show that under some additional conditions the above conjecture is true. Let R, Ric, S and C be the curvature tensor, the Ricci tensor, the scalar curvature and the Weyl's conformal curvature tensor of (M^n, g), respectively. With respect to local coordinates, we denote by $R^h{}_{ijk}$, R_{ij} and $C^h{}_{ijk}$ the components of R, Ric and C, respectively. (M^n, g) is said to be locally symmetric if $\nabla R=0$, Ricci symmetric if $\nabla Ric=0$ and conformally symmetric [2] if $\nabla C=0$.

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§ 1. Conformally symmetric Riemannian manifolds.

K. Yamauchi [8] proved the following.

THEOREM A. *Let (M^4, g) be a conformally flat Riemannian manifold with non-zero constant scalar curvature. If (M^4, g) satisfies the condition $(*)$, (M^4, g) is locally symmetric.*

The first purpose of this note is to give a generalization of Theorem A.

THEOREM 1. *Let $(M^n, g)(n \geqq 4)$ be a conformally symmetric Riemannian*

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1) Throughout the paper, we assume that (M^n, g) is connected.

2) Numbers in brackets refer to references at the end of this paper.

manifold with constant scalar curvature. If (M^n, g) *satisfies the condition*

(**) $R(X, Y)Ric=0$ *for all tangent vectors* X *and* Y,

(M^n, g) *is locally symmetric.*

From the definition of C, we have

$$\nabla_l C^h{}_{ijk}=\nabla_l R^h{}_{ijk}-\frac{1}{n-2}(\delta^h_k\nabla_l R_{ij}-\delta^h_j\nabla_l R_{ik}+g_{ij}\nabla_l R^h{}_k -g_{ik}\nabla_l R^h{}_j)+\frac{1}{(n-1)(n-2)}\nabla_l S(\delta^h_k g_{ij}-\delta^h_j g_{ik})\,, \tag{1}$$

where δ is the Kronecker's delta, and since

$$\nabla_m\nabla_l S-\nabla_l\nabla_m S=0$$

for a scalar S, we have

$$\begin{aligned}\nabla_m\nabla_l C^h{}_{ijk}-\nabla_l\nabla_m C^h{}_{ijk}&=\nabla_m\nabla_l R^h{}_{ijk}-\nabla_l\nabla_m R^h{}_{ijk}\\ &-\frac{1}{n-2}\Big\{\delta^h_k(\nabla_m\nabla_l R_{ij}-\nabla_l\nabla_m R_{ij})-\delta^h_j(\nabla_m\nabla_l R_{ik}-\nabla_l\nabla_m R_{ik})\\ &+g_{ij}(\nabla_m\nabla_l R^h{}_k-\nabla_l\nabla_m R^h{}_k)-g_{ik}(\nabla_m\nabla_l R^h{}_j-\nabla_l\nabla_m R^h{}_j)\Big\}.\end{aligned} \tag{2}$$

If $\nabla C=0$ and $\nabla Ric=0$, from (1), we have $\nabla R=0$. From (2), if (M^n, g) satisfies (**) and $\nabla C=0$, we have (*). Therefore, we obtain the following.

LEMMA 1. *Let* $(M^n, g)\,(n\geqq 3)$ *be a conformally symmetric Riemannian manifold. On* (M^n, g), (**) *is equivalent to* (*), *and* (M^n, g) *is locally symmetric if and only if* (M^n, g) *is Ricci symmetric.*

LEMMA 2. *Let* (M^n, g), $(n\geqq 4)$ *be a Riemannian manifold satisfying the following condition*

$$\nabla_l C^h{}_{ijk}+\nabla_k C^h{}_{ilj}+\nabla_j C^h{}_{ikl}=0. \tag{3}$$

Then, a necessary and sufficient condition that

$$\nabla_k R_{ij}-\nabla_j R_{ik}=0 \tag{4}$$

is that S *is constant.*

When $\nabla C=0$, M. C. Chaki and B. Gupta [2] proved Lemma 2.

PROOF. From the Bianchi's identity and $C^h{}_{ijh}=0$, it follows that (3) is equivalent to

$$\nabla_k R_{ij}-\nabla_j R_{ik}-\frac{1}{2(n-1)}(g_{ij}\nabla_k S-g_{ik}\nabla_j S)=0\,. \tag{5}$$

Substituting $\nabla_k S=0$ in (5), we have (4). Conversely, contracting (4) by g^{ij}, we can see that S is constant.

LEMMA 3. *A necessary and sufficient condition for a Riemannian manifold* $(M^n, g)(n\geqq 2)$ *to be Ricci symmetric is that* (M^n, g) *satisfies* $(**)$ *and* (4).

PROOF. Applying the Ricci's identity to $(**)$, we have

$$(6)\qquad R_{hj}R^h{}_{ikl}+R_{ih}R^h{}_{jkl}=0\,.$$

Derivating (6) by ∇_m and contracting by $g^{ik}g^{jm}$, we have

$$(7)\qquad R^{mj}\nabla_j R_{ml}+R^m{}_l\nabla^j R_{mj}+R^{mji}{}_l\nabla_j R_{im}-R^{im}\nabla_j R^j{}_{mil}=0\,.$$

From (4), we have

$$R^{mj}\nabla_j R_{ml}=R^{mj}\nabla_l R_{mj}=\frac{1}{2}\nabla_l(R_{mj}R^{mj})\,,$$

and from the Bianchi's identity and (4), it follows that

$$\nabla^j R_{mj}=\frac{1}{2}\nabla_m S=0\,,$$

and

$$\nabla_j R^j{}_{mil}=\nabla_l R_{mi}-\nabla_i R_{ml}=0\,.$$

Since $R^{mji}{}_l$ is skew-symmetric and $\nabla_j R_{im}$ is symmetric with respect to j and m, respectively, we have

$$R^{mji}{}_l\nabla_j R_{im}=0\,.$$

Thus, from (7), we have

$$\nabla_l(R_{mj}R^{mj})=0\,.$$

On the other hand, from $(**)$, (4) and the Bianchi's identity, it follows that

$$g^{kl}\nabla_l\nabla_k R_{ij}=g^{kl}\nabla_l\nabla_j R_{ik}=g^{kl}\nabla_j\nabla_l R_{ik}=\frac{1}{2}\nabla_j\nabla_i S=0\,.$$

Hence, from the above results and the following equation

$$g^{kl}\nabla_k\nabla_l(R_{ij}R^{ij})=2(\nabla_k R_{ij})(\nabla^k R^{ij})+2R^{ij}g^{kl}\nabla_l\nabla_k R_{ij}\,,$$

we have that (M^n, g) is Ricci symmetric. The converse is obvious.

PROOF of Theorem 1. Since $\nabla C=0$ and S is constant, from Lemma 2, we have (4). Hence, using Lemma 3, we have that (M^n, g) is Ricci symmetric. Therefore, from Lemma 1, our theorem is proved.

Since C is identically zero for $n=3$, we have Lemma 1 without assumption of $\nabla C=0$. Lemma 2 can be proved for 3-dimensional conformally flat Riemannian manifold. Therefore, we have

COROLLARY 1. *Let* $(M^n, g)(n\geqq 3)$ *be a conformally flat Riemannian* *manifold with constant scalar curvature. If* (M^n, g) *satisfies* $(**)$, (M^n, g) *is locally symmetric.*

COROLLARY 2. *Let* $(M^n, g)(n\geqq 3)$ *be an irreducible conformally flat Riemannian manifold with constant scalar curvature. If* (M^n, g) *satisfies* $(**)$, (M^n, g) *is of constant curvature.*

§ 2. Riemannian manifolds admitting projective transformations.

Let (M^n, g) and $(\bar{M}^n, \bar{g})(n\geqq 2)$ be two Riemannian manifolds. A necessary and sufficient condition for a correspondence f from (M^n, g) to $(\bar{M}^n, \bar{g})$ to be projective is that there exists a vector field ψ_i defined on (M^n, g) such that

$$\left\{\begin{matrix}\tilde{h}\\ ij\end{matrix}\right\} = \left\{\begin{matrix}h\\ ij\end{matrix}\right\} + \delta_i^h\psi_j + \delta_j^h\psi_i\,,$$

where $\left\{\begin{matrix}h\\ ij\end{matrix}\right\}$ and $\left\{\begin{matrix}\tilde{h}\\ ij\end{matrix}\right\}$ are coefficients of connections defined by g and $f^*\bar{g}$, respectively. Particularly, if ψ_i is non zero vector on (M^n, g), f is said to be proper, and if ψ_{ij} is non zero tensor on (M^n, g), f is said to be strictly proper, where $\psi_{ij}=\nabla_j\psi_i-\psi_i\psi_j$. f is not strictly proper if and only if f preserves the curvature tensor. Our purpose of this section is to prove the following theorem.

THEOREM 2. *Let* $(M^n, g)(n\geqq 2)$ *be a Riemannian manifold admitting a strictly proper projective transformation. If* (M^n, g) *satisfies* $(*)$, (M^n, g) *is an Einstein manifold.*

We already have

LEMMA 4 (N. S. Sinyukov [7]). *Let* (M^n, g) *and* $(\bar{M}^n, \bar{g})(n\geqq 2)$ *be two Riemannian manifolds such that there exists a projective correspondence between* (M^n, g) *and* $(\bar{M}^n, \bar{g})$. *If* $(\bar{M}^n, \bar{g})$ *satisfies* $(*)$, *one of the following two conditions must be satisfied.*

$$R_{ij}-\frac{1}{n}Sg_{ij}=0\,. \tag{8}$$

$$\Box\psi\cdot g_{ij}-n\psi_{ij}=0\,, \tag{9}$$

where $\Box\psi=g^{ij}\psi_{ij}$.

From LEMMA 4, we have

LEMMA 5. *Let* $(M^n, g)(n\geqq 2)$ *be a Riemannian manifold satisfying* $(**)$ *and* $(\bar{M}^n, \bar{g})$ *a Riemannian manifold satisfying* $(*)$. *If there exists a projective correspondence f between* (M^n, g) *and* $(\bar{M}^n, \bar{g})$, (M^n, g) *is an Einstein manifold, or f preserves the curvature tensor.*

PROOF. Let $\tilde{g}$ be a metric tensor induced from $\bar{g}$ by f and $\tilde{R}$ the curvature tensor defined by $\tilde{g}$. Then, we have

$$\tilde{R}(X, Y)\tilde{R}=f^*\left(\bar{\boldsymbol{R}}(\bar{X}, \bar{Y})\bar{\boldsymbol{R}}\right)=0\,, \tag{10}$$

where $\bar{X}$ and $\bar{Y}$ are tangent vectors on $(\bar{M}^n, \bar{g})$ defined by f from X and Y. If we denote by $\tilde{\nabla}$ the connection defined by $\tilde{g}$,

$$\begin{aligned}\tilde{\nabla}_l\tilde{\nabla}_k\tilde{R}_{ij}-\tilde{\nabla}_k\tilde{\nabla}_l\tilde{R}_{ij} &= \nabla_l\nabla_k R_{ij}-\nabla_k\nabla_l R_{ij}\\ &\quad -(n-1)(\nabla_l\nabla_k\psi_{ij}-\nabla_k\nabla_l\psi_{ij})\\ &\quad -\psi_{ik}R_{lj}+\psi_{il}R_{kj}-\psi_{jk}R_{il}+\psi_{jl}R_{ik}\,.\end{aligned} \tag{11}$$

Substituting $\tilde{R}(X, Y)\tilde{R}ic=R(X, Y)Ric=0$ and (9) in (11), we have

$$\frac{1}{n}\Box\psi(g_{ik}R_{lj}-g_{il}R_{kj}+g_{jk}R_{il}-g_{jl}R_{ik})=0\,. \tag{12}$$

Contracting (12) by g^{ik}, we have

$$\frac{1}{n}\Box\psi\left(R_{ij}-\frac{1}{n}Sg_{ij}\right)=0. \tag{13}$$

Thus, from (8), (9) and (13), we have Lemma 5.

The proof of Theorem 2 is evident from Lemma 5.

COROLLARY 3. *Let $(M^n, g)(n\geqq 2)$ be a compact Riemannian manifold admtting a strictly proper projective transformation. If (M^n, g) satisfies $(*)$, (M^n, g) is of constant curvature.*

This corollary is proved by Theorem 2 and the following two theorems.

THEOREM B (A. Lichnerowicz [4]). *Let $(M^n, g)(n\geqq 2)$ be a compact Ricci symmetric Riemannian manifold. If (M^n, g) satisfies $(*)$, (M^n, g) is locally symmetric.*

THEOREM C (N. S. Sinyukov [7]). *Let $(M^n, g)(n\geqq 2)$ be an arbitrary Riemannian manifold and $(\bar{M}^n, \bar{g})$ a locally symmetric Riemannian manifold. If there exists a proper projective correspondence between (M^n, g) and $(\bar{M}^n, \bar{g})$, (M^n, g) and $(\bar{M}^n, \bar{g})$ are of constant curvature.*

For an infinitesimal projective transformation, we heve Lemma 6 by the same way as Lemma 4.

LEMMA 6. *Let $(M^n, g)(n\geqq 2)$ be a Riemannian manifold admitting an infinitesimal projective transformation. If (M^n, g) satisfies $(*)$, one of the following two conditions must be satisfied*

$$R_{ij}-\frac{1}{n}Sg_{ij}=0\,. \tag{14}$$

$$g^{kl}\nabla_l\psi_k\cdot g_{ij}-n\nabla_j\psi_i=0\,, \tag{15}$$

where ψ_i *is the vector field on* (M^n, g) *such that* $\mathfrak{L}_X \left\{ {h \atop ij} \right\} = \delta_i^h \psi_j + \delta_j^h \psi_i$ *for an infinitesimal projective transformation* X *and* $\mathfrak{L}_X$ *denotes the Lie derivative with respect to* X.

From Lemma 6, we can obtain the following.

THEOREM 3. *Let* $(M^n, g)\,(n \geqq 3)$ *be a compact Riemannian manifold admitting a proper infinitesimal projective transformation* (*i.e.*, $\psi_i \neq 0$). *If* (M^n, g) *satisfies* $(*)$, (M^n, g) *is of constant curvature.*

In order to prove this theorem, we use the following three theorems.

THEOREM D (S. Ishihara and Y. Tashiro [3]). *Let* $(M^n, g)\,(n \geqq 2)$ *be a complete Riemannian manifold. In order for* (M^n, g) *to admit a non-constant solutiou* ρ *for the system of partial differential equations*

$$\nabla_j \nabla_i \rho = \frac{1}{n} g^{kl} \nabla_l \nabla_k \rho \cdot g_{ij} ,$$

it is necessary and sufficient that (M^n, g) *be conformal to a sphere in the* $(n+1)$*-dimensional Euclidean space.*

THEOREM E (K. Sekigawa and H. Takagi [6]). *Let* $(M^n, g)\,(n \geqq 3)$ *be a complete conformally flat Riemannian manifold. If* (M^n, g) *satisfies* $(**)$, (M^n, g) *is locally symmetric.*

THEOREM F (K. Yano and T. Nagano [9]). *Let* $(M^n, g)\,(n \geqq 3)$ *be a locally symmetric Riemannian manifold. If* (M^n, g) *admits a proper infinitesimal projective transformation,* (M^n, g) *is of constant curvature.*

PROOF of Theorem 3. Since ψ_i is a gradient vector, from Theorem D, we have that (M^n, g) satisfying (15) is conformal to a sphere. Therefore, (M^n, g) is conformally flat. Applying this fact to Theorem E, it follows that (M^n, g) satisfying (15) is locally symmetric. Thus, (M^n, g) is Ricci symmetric all over. From Theorem B and $(*)$, (M^n, g) is locally symmetric all over. Then, from Theorem F, our theorem is proved.

§ 3. Conformally symmetric Riemannian manifolds admitting projective transformations.

The following lemma is a generalization of the result obtained by T. Adati and T. Miyazawa [1].

LEMMA 7. *Let* $(M^n, g)\,(n \geqq 4)$ *be a Riemannian manifold satisfying* (3). *If* (M^n, g) *admits a vector field* v_i such that

(16) $$\nabla_j v_i - \alpha v_i v_j = 0 \qquad \textit{for a certain constant } \alpha,$$

then, (M^n, g) *is with constant scalar curvature, or* v_i *is a zero vector field.*

Proof. Applying the Ricci's identity to (16), we have

$$v_h R^h{}_{ijk}=0\,. \tag{17}$$

Derivating (17) by ∇_l and using (16), we get

$$v_h\nabla_l R^h{}_{ijk}=0\,,\quad v_h\nabla_l R^h{}_k=0 \quad\text{and}\quad v^h\nabla_h S=0\,. \tag{18}$$

Applying the Bianchi's identity to the first equation of (18) and substituting the second into it, we hvae

$$v^h\nabla_h R_{ij}=0\,. \tag{19}$$

On the other hand, contracting (5) by v^h, we have

$$v^h\nabla_j R_{ih}-v^h\nabla_h R_{ij}-\frac{1}{2(n-1)}(v_i\nabla_j S-g_{ij}v^h\nabla_h S)=0\,. \tag{20}$$

Substituting (18) and (19) into (20), we have

$$v_i\nabla_j S=0\,.$$

Therefore, S is constant, or v_i is a zero vector field.

THEOREM 4. *Let $(M^n, g)(n\geqq 4)$ be a conformally symmetric Riemannian manifold satisfying* $(**)$ *and $(\bar M^n, \bar g)$ a Riemannian manifold satisfying* $(*)$. *If there exists a proper projective correspondence between (M^n, g) and $(\bar M^n, \bar g)$, (M^n, g) and $(\bar M^n, \bar g)$ are of constant curvature.*

PROOF. From Lemma 5, we have that (M^n, g) is an Einstein manifold or $\psi_{ij}=0$. When $\psi_{ij}=0$, from Lemma 7, (M^n, g) is with constant scalar curvature, because a given correspondence is proper. Therefore, from Theorems C and 1, we have this theorem.

COROLLARY 4. *Let $(M^n, g)(n\geqq 4)$ be a conformally symmetric Riemannian manifold admitting a proper projective transformation. If (M^n, g) satisfies* $(**)$, *(M^n, g) is of constant curvature.*

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