Hokkaido Mathematical Journal Vol. 12 (1983) p. 461-471

Particle path length estimates for the Navier Stokes equations in three space dimensions

By G. F. D. DUFF

(Received January 14, 1983)

Abstract. Flows with finite energy of a viscous incompressible fiuid in a domain of three dimensional space are studied to estimate particle path lengths. In the general case a bound is given for the essential maximum path length as time $T\rightarrow\infty$. If the domain satisfies a Poincaré inequality, then as $T\rightarrow\infty$ all particle motions are essentially uniformly bounded. Some additional asymptotic results are also given.

1. Introduction

In this paper are given results on the path lengths of the motions of fluid elements or point masses, which we may refer to as particles.

For three space dimensions solutions of the Navier Stokes equations may be turbulent in the sense of Leray [7] and all calculations must be made with allowance for this possibility which remains not completely decided despite many advances in the mathematical theory of nonlinear fluid motions. Any turbulent solutions may develop singularities which however can only occur on sets of low dimension in space and time [2, 8]. One must therefore work with integrals that remain convergent in the presence of such possible singularities. After a sufficiently long time interval, the singularities can no longer appear, and emphasis is then on asymptotic behaviour. We show that motions generated by initial values with finite energy give rise to finite path lengths over bounded time intervals. An asymptotic estimate is given for large elapsed times in the general case of a domain in R^3 that satisfies a cone condition. When the domain also satisfies a Poincaré inequality, it is shown that the path lengths are bounded as $T\rightarrow\infty$. These results extend and complete those announced in [3] and are in turn based on a technique used by Foias, Guillopé and Temam in [4]. For domains with boundary we use orthogonal projection on the subspace of solenoidal vector functions that vanish on the boundary, and this is essentially our other necessary condition on the domains considered. The method yields an estimate free of boundary terms as stated in Theorem 1 below. This is applied to the path length estimations in Theorems 2, 3 and 4. For completeness we include proofs of certain key lemmas.

2. The Navier Stokes equations and the initial value problem

Let $\Omega \subseteq R^3$ be a region in which solutions $u=\{u_i(x,t)\}$ are defined for the Navier Stokes equations for $t \geq 0$. We assume such solutions exist, are globally weak and locally regular except on certain singular sets [2, 5, 6, 7, 9]. They satisfy

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=-\frac{\partial p}{\partial x_i}+\nu \Delta u_i, \qquad i=1,2,3$$

and
$$\frac{\partial u_k}{\partial x_k}=0,$$

where summation over repeated indices such as k is understood for $k=1,2,3$. Here ν is the constant viscosity and Δ the Laplacian differential operator. The boundary conditions of no slip are $u_i(x_j,t)=0$ for $\{x_j\} \in \partial\Omega$, while the initial conditions are

$$u_i(x_j,0)=u_{i0}(x_j). \qquad i=1,2,3.$$

We assume

$$||u_0||_2^2=\int_\Omega \sum_{i=1}^3 u_{i0}^2(x_j)\,dV < \infty$$

as the hypothesis of finite initial energy. We use the Lebesgue p-norm

$$||v||_p=\left[\int_\Omega \sum_{i=1}^3 |v_i|^p dV\right]^{1/p}, \qquad p \geq 1$$

for vectors v, and similar norms for derivatives.

The Sobolev inequality in three space dimensions is

$$||u||_{\frac{3p}{3-p}} \leq C\ ||\nabla u||_p, \qquad 1 \leq p < 3$$

where ∇u denotes the gradient $u_{i,j}=\dfrac{\partial u_i}{\partial x_j}$.

Note that the constant in this inequality is independent of the domain Ω, [6].

Let $L^2(\Omega)$ be the Hilbert space of vector functions u, v with inner product $\int_\Omega u\cdot v dV=\int_\Omega \sum_{i=1}^3 u_i v_i dV$. Consider the closure $\mathscr{L}^2$ of the divergence free (or solenoidal) vector functions of compact support in Ω. This set is a linear subspace of $\mathscr{L}^2$ that is orthogonal to the space of gradient vectors in Ω, for if $v_i=\dfrac{\partial f}{\partial x_i}$ and $u\in\mathscr{L}^2$ we have

$$
\begin{aligned}
(u, v) = \int_\Omega \sum_{i=1}^{3} u_i v_i dV &= \int_\Omega \sum_{i=1}^{3} u_i \frac{\partial f}{\partial x_i}\, dV \\
&= \int_\Omega \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(u_i f)\, dV \qquad (u_{i,i}=0) \\
&= \int_{\partial\Omega} u_i f n_i dS = 0
\end{aligned}
$$

since u_i is of compact support within Ω or is a limit of such vectors. Hence orthogonality of the subspaces will follow [6; 9, p. 141].

If P denotes orthogonal projection from L^2 onto $\mathscr{L}^2$, we may write $P\Delta=\tilde{\Delta}$ thus defining the solenoidal projection of the Laplacian with compact support. As shown in [9, p. 148] $\|\tilde{\Delta}u\|_2=0$ implies $u=0$ if $u\in C^2$, $u=0$ on $\partial\Omega$ and $u_{i,i}=0$. Indeed $0=(\tilde{\Delta}u, u)=(\Delta u, Pu)=(\Delta u, u)=-(\nabla u)_2^2$ by Green's Theorem. Hence $\nabla u=0$ in Ω and u is a constant, which must be zero. Hence $\|\tilde{\Delta}u\|_2$ is a norm.

Following [9, p. 148], the completion of C_*^∞ the set of smooth solenoidal vector fields vanishing on $\partial\Omega$, in the $\|\tilde{\Delta}u\|_2$ norm will be denoted by $\mathscr{H}^2$. We now show [9, p. 194] that a form of Sobolev's inequality holds, namely

LEMMA 2.1. $\|\nabla u\|_6\leq c\|\tilde{\Delta}u\|_2$ *for* $u\in\mathscr{H}^2$.

PROOF. Let $u\in C_*^\infty$, $v\in C_0^\infty\subset\mathscr{L}^2$. Then $u\in C^\infty(\Omega)\subset C^0(\bar{\Omega})$, $u=0$ on $\partial\Omega$ and $u_{i,i}=0$, so that

$$
\begin{aligned}
\left|(\nabla u, \nabla v)\right| = \left|(\Delta u, v)\right| &= \left|(\tilde{\Delta}u, v)\right| \\
&\leq \|\tilde{\Delta}u\|_2\|v\|_2 \\
&\leq c\|\tilde{\Delta}u\|_2\|\nabla v\|_{6/5}
\end{aligned}
$$

by Sobolev's inequality with $p=6/5$. Hence if $v\not\equiv 0$, $\dfrac{|(\nabla u, \nabla v)|}{\|\nabla v\|_{6/5}}\leq c\|\tilde{\Delta}u\|_2$.

By duality of $\mathscr{H}^p$ spaces, the dual of $\mathscr{H}_0^{1,6}$ is $\mathscr{H}_0^{1,6/5}$ so the supremum of the left side over all $v\in C_0^\infty$ is exactly the norm of u in $\mathscr{H}_0^{1,6}$. This proves the inequality for $u\in C_*^\infty$ and the result for $\mathscr{H}^2$ follows on closure since C_*^∞ is dense in $\mathscr{H}^2$.

3. The main estimate

We multiply the Navier Stokes equations by $\tilde{\Delta}u_i$ and integrate over Ω. The time derivative term yields

$$\int_{\Omega} \tilde{\Delta} u_i \cdot u_{i,t} dV = \int_{\Omega} \Delta u_i \cdot u_{i,t} dV$$

since $u_{i,t}$ is solenoidal and vanishes on the boundary. After integration by parts we obtain

$$-\int_{\Omega} \nabla u_i \nabla u_{i,t} dV = -\frac{1}{2} \frac{d}{dt} \int (\nabla u_i)^2 dV$$
$$= -\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 .$$

The pressure term is

$$-\int_{\Omega} p_{,i} \tilde{\Delta} u_i dV = 0$$

since $\tilde{\Delta} u_i$ is orthogonal to all gradients. The viscous term becomes

$$\nu \int_{\Omega} \tilde{\Delta} u_i \Delta u_i dV = \nu \int_{\Omega} (\tilde{\Delta} u_i)^2 dV$$

since $\Delta u_i - \tilde{\Delta} u_i$ is a gradient orthogonal to $\tilde{\Delta} u_i$. Finally, the nonlinear term yields

$$\begin{aligned} &\left| \int_{\Omega} \tilde{\Delta} u_i u_k u_{i,k} dV \right| \\ &\leq \|\tilde{\Delta} u\|_2 \|u\|_6 \|\nabla u\|_3 \\ &\leq C \|\tilde{\Delta} u\|_2 \|\nabla u\|_2 \|\nabla u\|_6^{1/2} \|\nabla u\|_2^{1/2} \\ &\leq C \|\tilde{\Delta} u\|_2^{3/2} \|\nabla u\|_2^{3/2} \\ &\leq \frac{\nu}{2} \|\tilde{\Delta} u\|_2^2 + C_\nu \|\nabla u\|_2^6 \end{aligned}$$

where we have used the inequalities of Hölder, Sobolev and Young as well as the Lemma of Section 2.

THEOREM 1. *For a solution u of the Navier Stokes equations in a domain Ω in three space dimensions the estimate*

$$\frac{d}{dt} \|\nabla u\|_2^2 + \nu \|\tilde{\Delta} u\|_2^2 \leq K \|\nabla u\|_2^6$$

holds.

An estimate of this form was obtained by Leray [7] in the case $\Omega = R^3$, and by Foias, Guillopé and Temam [4] for the case of a periodic parallelepiped in R^3. Heywood [7] gives somewhat similar estimates but with an additional term $C\|\nabla u\|_2^4$ on the right. While this fourth order term is of no consequence near singularities when $\|\nabla u\|_2$ becomes large, it is possible to derive improved asymptotic estimates as $t\to\infty$ from the estimate of Theorem 1. The estimate of Theorem 1 is given by Shinbrot [9, p. 201] in essentially the same form for bounded domains.

4. The path length estimates

The motion of a particle or element of fluid at $x_i(t)$ at time t is given by

$$\frac{dx_i}{dt}=u_i(x_j, t)$$

while its speed is given by

$$\frac{ds}{dt}=\left|u(x, t)\right|,$$

where s denotes arc length travelled. For a motion commencing at $t=0$ we have

$$\begin{aligned} s&=\int_0^T\left|u(x, t)\right|dt\\ &\leq\int_0^T||u||_\infty dt, \end{aligned}$$

because $||u||_\infty\equiv\text{ess.}\max_x|u(x, t)|$.

To estimate the maximum norm we shall use an inequality of Adams and Fournier [1, Theorem 4, p. 718], namely

$$||u||_\infty\leq K||u||_{m,p}^\theta||u||_q^{1-\theta}$$

where $q\geq1$, $p>1$, $mp-p<n<mp$, $\theta=np/(np+(mp-n)q)$. With the numerical choices $n=3$, $m=1$, $p=6$ and $q=6/\varepsilon$ large these conditions hold with $\theta=\varepsilon/1+\varepsilon$. Here also

$$\begin{aligned} ||u||_{m,p}=||u||_{1,6}&=\left\{||\nabla u||_6^6+||u||_6^6\right\}^{1/6}\\ &\leq||\nabla u||_6+||u||_6\\ &\leq K||\Delta u||_2+K||\nabla u||_2, \end{aligned}$$

by the Lemma and Sobolev's inequality.

For large q we write $||u||_q\leq c||\nabla u||_p$ with $p=3q/(3+q)=6/(2+\varepsilon)$. By Hölder's inequality and the Lemma

$$\begin{aligned} ||\nabla u||_p&\leq||\nabla u||_6^{1-\alpha}||\nabla u||_2^\alpha\\ &\leq C||\Delta u||_2^{1-\alpha}||\nabla u||_2^\alpha, \end{aligned}$$

where $\alpha=(1+\varepsilon)/2$. Hence, finally,

$$\begin{aligned}||u||_\infty &\le K(\varepsilon)\left\{||\varDelta u||_2+||\nabla u||_2\right\}^{\frac{\varepsilon}{1+\varepsilon}}||u||_q^{\frac{1}{1+\varepsilon}}\\ &\le K_1(\varepsilon)\left\{||\varDelta u||_2^{\frac{\varepsilon}{1+\varepsilon}}+||\nabla u||_2^{\frac{\varepsilon}{1+\varepsilon}}\right\}||\nabla u||_p^{\frac{1}{1+\varepsilon}}\\ &\le K_1(\varepsilon)\left\{||\varDelta u||_2^{\frac{\varepsilon}{1+\varepsilon}}+||\nabla u||_2^{\frac{\varepsilon}{1+\varepsilon}}\right\}||\varDelta u||_2^{\frac{1-\varepsilon}{2(1+\varepsilon)}}||\nabla u||_2^{1/2}\\ &\le K_1(\varepsilon)\left\{||\varDelta u||_2^{1/2}||\nabla u||_2^{1/2}+||\varDelta u||_2^{\frac{1-\varepsilon}{2(1+\varepsilon)}}||\nabla u||_2^{\frac{1+3\varepsilon}{2(1+\varepsilon)}}\right\}.\end{aligned}$$

To estimate the integral of $||u||_\infty$ we modify a method of Foias, Guillopé and Temam [4]. Let $f(t)$ be a smooth positive nonincreasing function for $t>0$. From Theorem 1 we have

$$\frac{d}{dt}\left(f(t)+||\nabla u||_2^2\right)+\nu||\varDelta u||_2^2\le K||\nabla u||_2^2\left(f(t)+||\nabla u||_2^2\right)^2$$

whence on division by the last factors on the right we find

$$-\frac{d}{dt}\left(f(t)+||\nabla u||_2^2\right)^{-1}+\frac{\nu||\varDelta u||_2^2}{(f(t)+||\nabla u||_2^2)^2}\le K||\nabla u||_2^2.$$

Now integrate over $(0, T)$ to obtain

$$\frac{1}{f(0)+||\nabla u_0||_2^2}+\nu\int_0^T\frac{||\varDelta u||_2^2dt}{(f(t)+||\nabla u||_2^2)^2}\le K\int_0^\infty||\nabla u||_2^2dt+\frac{1}{f(T)+||\nabla u(T)||_2^2}.$$

Noting that the integral on the right converges we may obtain estimates for the integral on the left.

Now

$$\begin{aligned}\int_0^T||\varDelta u||_2^{2/3}dt &=\int_0^T\left(f(t)+||\nabla u||_2^2\right)^{2/3}\frac{||\varDelta u||_2^{2/3}dt}{(f(t)+||\nabla u||_2^2)^{2/3}}\\ &\le\left[\int_0^T\left(f(t)+||\nabla u||_2^2\right)dt\right]^{2/3}\left[\int_0^T\frac{||\varDelta u||_2^2dt}{(f(t)+||\nabla u||_2^2)^2}\right]^{1/3}\end{aligned}$$

by Hölder's inequality. Hence

$$\int_0^T||\varDelta u||_2^{2/3}dt\le\left[\int_0^T f(t)dt+A\right]^{2/3}\cdot\left[KA+\frac{1}{f(T)+||\nabla u||_2^2}\right]^{1/3}.$$

The integral on the left is bounded for any finite T and so is bounded during any possible epoch of turbulence. Referring to Lemma 2.1 we now obtain

LEMMA 4.1. $\varDelta u\in L^{2/3}(0, T;\ L^2(\Omega))$

and

$$\nabla u \in L^{2/3}\Big(0, T;\ L^6(\Omega)\Big).$$

It will be shown below that as $T \to \infty$ we have $\|\nabla u\|_2^2 = \sigma(T^{-1})$. Hence the choice $f(t) = (1+T)^{-1}$ is convenient and this yields

$$\int_0^T \|\tilde{\Delta} u\|_2^{2/3} dt \leq \Big(A + \log(1+T)\Big)^{2/3} (KA + 1 + T)^{1/3}$$
$$\leq BT^{1/3} \log(1+T)^{2/3} \text{ as } T \to \infty.$$

Now

$$s \leq \int_0^T \|u\|_\infty dt$$
$$\leq K(\varepsilon) \int_0^T \Big\{ \|\tilde{\Delta} u\|_2^{1/2} \|\nabla u\|_2^{1/2} + \|\tilde{\Delta} u\|_2^{\frac{1}{2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)} \|\nabla u\|_2^{\frac{1+3\varepsilon}{2(1+\varepsilon)}} \Big\} dt.$$

The first term is bounded by

$$\leq K(\varepsilon) \Big[\int_0^T \|\tilde{\Delta} u\|_2^{2/3} dt \Big]^{3/4} \Big[\int_0^T \|\nabla u\|_2^2 dt \Big]^{1/4}$$
$$\leq K(\varepsilon)\, B_1 T^{1/4} \log(1+T)^{1/2} \text{ as } T \to \infty.$$

The second term may be estimated by

$$K(\varepsilon) \Big[\int_0^T \|\tilde{\Delta} u\|_2^{2/3} dt \Big]^{\frac{3}{4}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)} \Big[\int_0^T \|\nabla u\|_2^2 dt \Big]^{\frac{1+3\varepsilon}{4(1+\varepsilon)}} \Big[\int_0^T dt \Big]^{\frac{\varepsilon}{1+\varepsilon}}$$
$$\leq K(\varepsilon)\, B_2 T^{\frac{1}{4}\left(\frac{1+3\varepsilon}{1+\varepsilon}\right)} (\log T)^{\frac{1}{2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)}$$
$$\leq K(\varepsilon)\, B_3 T^{\frac{1}{4}+\frac{\varepsilon}{2}}, \text{ as } T \to \infty.$$

Hence we have

THEOREM 2. *As* $T \to \infty$, $s \leq K_1(\varepsilon)\, T^{\frac{1}{4}+\frac{\varepsilon}{2}}$, $\varepsilon > 0$.

This result extends and completes Theorem 1 of [3] for which the abbreviated proof there given is valid for R^3, for a periodic rectangular domain or for a closed three dimensional manifold, with a slight improvement in the order of magnitude as $T \to \infty$.

We now prove the further

LEMMA 4.2. *For t sufficiently large, the norm* $\|\nabla u\|_2$ *is decreasing in t, and*

$$\|\nabla u\|_2 = \sigma(t^{-\frac{1}{2}}) \text{ as } t \to \infty.$$

PROOF. We have $(\nabla u, \nabla u) = -(u, \Delta u)$
$$= -(u, \tilde{\Delta} u)$$

since u is orthogonal to gradients, being solenoidal and vanishing on the boundary. Hence

$$||\nabla u||_2^2 \leq ||u||_2 ||\tilde{\Delta} u||_2 .$$

Dividing the main estimate by $||\nabla u||_2^4$, we find

$$-\frac{d}{dt}\frac{1}{||\nabla u||_2^2}+\frac{\nu}{||u||_2^2}\leq -\frac{d}{dt}\frac{1}{||\nabla u||_2^2}+\frac{\nu||\tilde{\Delta} u||_2^2}{||\nabla u||_2^4}$$
$$\leq K||\nabla u||_2^2 .$$

Hence $\dfrac{d}{dt}\dfrac{1}{||\nabla u||_2^2}\geq \dfrac{\nu}{||u||_2^2}-K||\nabla u||_2^2$.

Since $||u||_2$ is decreasing with t, the first term on the right is increasing, while the second term being integrable over $(0, \infty)$ must take arbitrarily small values. Hence for some t_0 the right side is positive, and it follows that the time rate of change of $||\nabla u||_2^2$ is negative there. Consequently the right side remains positive and it is easily seen that $||\nabla u||_2^2$ is decreasing for $t>t_0$. Thus for $T>2t_0$ we have

$$\frac{1}{2}T||\nabla u||_2^2(T)\leq \int_{\frac{1}{2}T}^{T}||\nabla u||_2^2 dt\leq \varepsilon(T)$$

so that $||\nabla u||_2^2(T)\leq \varepsilon(T)/T$ as $T\to\infty$. This completes the proof of the lemma. See also [9, p. 203].

THEOREM 3. *If* $||u||_2^2\leq \dfrac{t}{c}||\nabla u||_2^2$, *for* $t>t_1$, *where* $4\nu c>1$, *then all path lengths are essentially uniformly bounded for all times and in the limit* $t\to\infty$.

PROOF. From the energy integral

$$\frac{d}{dt}||u||_2^2=-2\nu||\nabla u||_2^2$$

we deduce

$$\frac{d}{dt}||u||_2^2\leq -\frac{2c\nu}{t}||u||_2^2$$

and hence

$$||u||_2\leq Ct^{-c\nu} .$$

Thus $||u||_2\in L^p(t_0, \infty)$ for $t_0>0$, with $pc\nu>1$.

The two following lemmas are now required.

LEMMA 4.3. *If* $||u||_2 \in L^p(t_0, \infty)$ *and* t_0 *is sufficiently large, then* $||\nabla u||_2 \in L^q(t_0, \infty)$ *where* $\frac{1}{p}+\frac{1}{2}-\varepsilon=\frac{1}{q}$, $\varepsilon>0$.

PROOF OF LEMMA. From the energy integral we obtain, dividing by $||u||_2^{2-\varepsilon}$,

$$\frac{1}{\varepsilon}\frac{d}{dt}||u||_2^{\varepsilon}+\nu\frac{||\nabla u||_2^2}{||u||_2^{2-\varepsilon}}=0\,.$$

After integration we find

$$\frac{1}{\varepsilon}||u||_2^{\varepsilon}(t)+\nu\int_{t_0}^{t}\frac{||\nabla u||_2^2}{||u||_2^{2-\varepsilon}}dt'=\frac{1}{\varepsilon}||u||_2^{\varepsilon}(0)\,.$$

Hence as $t\to\infty$ the integral on the left is convergent for fixed $\varepsilon>0$. Now

$$\int_{t_0}^{\infty}||\nabla u||_2^q dt=\int_{t_0}^{\infty}||\nabla u||_2^q\frac{||u||_2^{q(1-\frac{\varepsilon}{2})}}{||u||_2^{q(1-\frac{\varepsilon}{2})}}dt$$

$$\leq\left[\int_{t_0}^{\infty}||u||_2^p dt\right]^{\frac{q(-1\frac{\varepsilon}{2})}{p}}\left[\int_{t_0}^{\infty}\frac{||\nabla u||_2^2 dt}{||u||_2^{2-\varepsilon}}\right]^{q/2}<\infty$$

by Hölder's inequality, where

$$\frac{1}{p}-\frac{\varepsilon}{2p}+\frac{1}{2}=\frac{1}{q}\,.$$

This gives the conclusion of the Lemma, with a slight change of notation for ε.

LEMMA 4.4. *If* $||\nabla u||_2 \in L^q(t_0, \infty)$ *where* t_0 *is sufficiently large, then* $||\Delta u||_2 \in L^r(t_0, \infty)$ *with* $\frac{1}{r}=\frac{1}{q}+\frac{1}{2}-\varepsilon$.

PROOF OF LEMMA. From the main estimate we find, on division by $||\nabla u||_2^{2-\varepsilon}$,

$$\frac{1}{\varepsilon}\frac{d}{dt}||\nabla u||_2^{\varepsilon}+\nu\frac{||\Delta u||_2^2}{||\nabla u||_2^{2-\varepsilon}}\leq K||\nabla u||_2^{4+\varepsilon}\,.$$

Noting that the term on the right side is integrable for $t_0<t<\infty$, since $||\nabla u||_2^2$ is integrable and $||\nabla u||_2$ is shown to be a decreasing function of t by Lemma 4.2, we find

$$\frac{1}{\varepsilon}||\nabla u||_2^{\varepsilon}+\nu\int_{t_0}^{t}\frac{||\Delta u||_2^2 d\tau}{||\nabla u||_2^{2-\varepsilon}}\leq K\int_{t_0}^{\infty}||\nabla u||_2^{4+\varepsilon}d\tau+\frac{1}{\varepsilon}||\nabla u||_2^{\varepsilon}\,.$$

Hence the integral on the left converges as $t\to\infty$. Again

$$\int_{t_0}^{\infty} ||\tilde{\Delta} u||_2^r dt = \int_{t_0}^{\infty} ||\tilde{\Delta} u||_2^r \frac{||\nabla u||_2^{r\left(1-\frac{\varepsilon}{2}\right)}}{||\nabla u||_2^{r\left(1-\frac{\varepsilon}{2}\right)}} dt$$

$$\leq \left[\int_{t_0}^{\infty} ||\nabla u||_2^q dt\right]^{r\left(1-\frac{\varepsilon}{2}\right)/q} \left[\int_{t_0}^{\infty} \frac{||\tilde{\Delta} u||_2^2 dt}{||\nabla u||_2^{2-\varepsilon}}\right]^{r/2}$$

by Hölder's inequality, where

$$\frac{1}{q} - \frac{\varepsilon}{2q} + \frac{1}{2} = \frac{1}{r}.$$

With a small change of notation for ε, this is the result stated in the Lemma.

To complete the proof of the Theorem, we now observe that if inclusion for $||u||_2$ holds in $L^p(t_0, \infty)$ with $p<4$, we can by the lemmas secure convergence of $||\nabla u||_2$ in $L^q(t_0, \infty)$ for some $q<\frac{4}{3}$ and for $||\tilde{\Delta} u||_2$ in $L^r(t_0, \infty)$ for some $r<\frac{4}{5}$. Reference to the path length calculations of Section 4 shows this suffices to establish $||u||_\infty \in L^1(t_0, \infty)$. Explicitly, we have, with $\eta=8\varepsilon/5$, $\xi=8\varepsilon/3$,

$$\int_{t_0}^{\infty} ||u||_\infty dt \leq K(\varepsilon) \int_{t_0}^{\infty} \left[||\tilde{\Delta} u||_2^{1/2} ||\nabla u||_2^{1/2} + ||\tilde{\Delta} u||_2^{1/2-\varepsilon} ||\nabla u||_2^{1/2+\varepsilon}\right] dt$$

$$\leq K(\varepsilon) \left\{\left[\int_{t_0}^{\infty} ||\tilde{\Delta} u||_2^{4/5} dt\right]^{5/8} \left[\int_{t_0}^{\infty} ||\nabla u||_2^{4/3} dt\right]^{3/8}\right.$$

$$\left. + \left[\int_{t_0}^{\infty} ||\tilde{\Delta} u||_2^{4/5-\eta} dt\right]^{5/8} \left[\int_{t_0}^{\infty} ||\nabla u||_2^{4/3+\xi} dt\right]^{3/8}\right\} < \infty$$

under the conditions established. When $4\nu c>1$, a range of positive values for $\varepsilon, \eta, \xi, \cdots$ is possible and this suffices for the estimates. This completes the proof of the theorem.

THEOREM 4. *If the domain Ω satisfies a Poincaré inequality $||u||_2 \leq C||\nabla u||_2$ or equivalently has a positive lowest eigenvalue, then the path lengths are essentially uniformly bounded as in Theorem* 3.

This conclusion is immediate, since the hypothesis of the Theorem is satisfied for t sufficiently large.

Support of this research by the Natural Sciences and Engineering Research Council grant A 3004 is gratefully acknowledged.

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Department of Mathematics
University of Toronto
Toronto, Canada