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Certain bilinear operators on Morrey spaces

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Abstract. In this paper, we consider that $T(f,g)$ is a bilinear operator satisfying

$$|T(f,g)(x)| \preceq \int_{\mathbb{R}^n} \frac{|f(x-ty)g(x-y)|}{|y|^n} dy$$

for x such that $0 \notin \text{supp}\ (f(x-t\cdot)) \cap \text{supp}\ (g(x+\cdot))$. We obtain the boundedness of $T(f,g)$ on the Morrey spaces with the assumption of the boundedness of the operator $T(f,g)$ on the Lebesgues spaces. As applications, we yield that many well known bilinear operators, as well as the first Calderón commutator, are bounded from the Morrey spaces $L^{q,\lambda_1} \times L^{r,\lambda_2}$ to $L^{p,\lambda}$, where $\lambda/p = \lambda_1/q + \lambda_2/r$.

Key words: Multilinear operators, bilinear Hilbert transform, the first Calderón commutator, Morrey spaces.

1. Introduction

Let T be a multilinear operator from the product of Schwartz space $S(\mathbb{R}^n) \times \cdots \times S(\mathbb{R}^n)$ into the space of tempered distribution $S'(\mathbb{R}^n)$, which commutes with simultaneous translations. The operator T can be formally written in the integral form of

$$T(f_1,\ldots,f_m)(x) = \int_{\mathbb{R}^{nm}} K(y_1,\ldots,y_m) \prod_{j=1}^{m} f_j(x-y_j) dy_1 \ldots dy_m,$$

where $x,\ y_j \in \mathbb{R}^n$, $j = 1,2,\ldots,m$, and K is a distribution kernel. This operator has received extensive study in the last two decades, see [4], [8], [9] and [10]. For instance, if one chooses $n = 1$, $k = 2$ and $K(y_1,y_2) = {y_1}^{-1}\delta(y_2+y_1)$ with the Dirac delta function δ, then one obtains the famous bilinear Hilbert transform

$$H(f_1,f_2)(x) = p.v. \int_{\mathbb{R}} \frac{f_1(x-y_1)f_2(x+y_1)}{y_1} dy_1.$$

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Connected to the operator $H(f_1, f_2)$ is the famous Calderón conjecture that says $H(f_1, f_2)$ is a bounded operator from $L^\infty \times L^2 \to L^2$, see [11]. The conjecture was solved on a more general setting by Lacey and Thiele in their celebrated theorem published in 1997.

Theorem A (Lacey and Thiele [13]) *Let* $1 < q, r \le \infty$*, and* $2/3 < p < \infty$*. Then*

$$\|H(f_1, f_2)\|_{L^p(\mathbb{R})} \preceq \|f_1\|_{L^q(\mathbb{R})}\|f_2\|_{L^r(\mathbb{R})},$$

provided $1/p = 1/q + 1/r$*.*

We note that the original proof of Theorem A is on $f_1,\ f_2 \in S(\mathbb{R})$. Then it is naturally extended to all $f_1 \in L^q(\mathbb{R})$ and $f_2 \in L^r(\mathbb{R})$. Thus H is a bounded operator from $L^q(\mathbb{R}) \times L^r(\mathbb{R})$ to $L^p(\mathbb{R})$.

Besides the Lebesgue space L^p, the Morrey space $L^{p,\lambda}$ is a function space raised from studying some well-posed problem in partial differential equations (see [15]). Let f be a locally integrable function. For $0 < \lambda < 1$ and $1 \le p < \infty$, we define the norm $\|f\|_{L^{p,\lambda}}$ by

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_B \left\{ \frac{1}{|B|^\lambda} \int_B |f(x)|^p dx \right\}^{1/p},$$

where the supreme runs over all balls B in $\mathbb{R}^n$, and $|B|$ denotes the volume of B. The Morrey space $L^{p,\lambda}$ is the linear space consists of all locally integrable functions f for which

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty.$$

Also, for convention, we denote

$$\|f\|_{L^{\infty,\lambda}(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)}$$

for all $0 < \lambda < 1$.

The Morrey spaces have also recently received a lot of attentions. The reader can see [1], [2], [5], [14], [16] and [17], among numerous references. Based on this observation, we naturally hope to establish the Lacey-Thiele theorem on the Morrey space. Recall that the proof of Theorem A is very difficult and it is completed with a very elegant method of time-frequency analysis. Clearly, we do not expect using this difficult method, since the structure of the Morrey space seems more complicated than the Lebesgue space if we invoke the time-frequency analysis. Thus, in this paper we apply some "transference" method to transfer the Lacey-Thiele theorem from the Lebesgue space to the Morrey space. Our result can be stated in a more general setting in the following theorem.

Theorem 1.1 *Let* $1 \le p < \infty$, $1 < q$, $r \le \infty$, $0 < \lambda_i < 1$ *for* $i = 1, 2$, *and* $T(f, g)$ *be a bilinear operator satisfying*

$$|T(f,g)(x)| \preceq \int_{\mathbb{R}^n} \frac{|f(x-ty)g(x-y)|}{|y|^n} dy \tag{1}$$

for x *such that* $0 \notin \operatorname{supp}\,(f(x-t\cdot)) \,\cap\, \operatorname{supp}\,(g(x+\cdot))$. *If* $1/p = 1/q + 1/r$, *and*

$$\|T(f,g)\|_{L^p} \le C\|f\|_{L^q}\|g\|_{L^r},$$

then there exist two positive constants C_1 *and* C_2 *independent of* $0 < |t| \le 1$ *such that*

$$\|T(f,g)\|_{L^{p,\lambda}} \le (C_1 C + C_2 |t|^{(1-\lambda_2)n/r})\|f\|_{L^{q,\lambda_1}}\|g\|_{L^{r,\lambda_2}},$$

where $\lambda/p = \lambda_1/q + \lambda_2/r$.

As applications, we obtain the boundedness from the Morrey spaces $L^{q,\lambda_1} \times L^{r,\lambda_2}$ to $L^{p,\lambda}$ for the bilinear Hilbert transform $H(f_1, f_2)$, the bilinear singular integral

$$T_\Omega(f,g)(x) = p.v. \int_{\mathbb{R}^n} \frac{f(x-y)g(x+y)}{|y|^n} \Omega(y') dy,$$

and the first Calderón commutator, where $\lambda/p = \lambda_1/q + \lambda_2/r$. Also, we will discuss the boundedness on the Morrey spaces for the bilinear oscillatory singular integral

$$\mathcal{T}_P(f,g)(x) = p.v. \int_{\mathbb{R}} \frac{e^{iP(x,y)} f(x-y)g(x+y)}{y} dy.$$

Throughout this article, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. The symbol $A \preceq B$ means that there exists a constant $C > 0$ independent of all essential variables such that $A \leq CB$. We also use the notation $A \simeq B$ if $A \preceq B$ and $B \preceq A$.

2. Proof of Theorem 1.1

To prove Theorem1.1, we need some preliminary work. For any ball $B_\delta = B(x_0, \delta)$, there is a $k_0 \in \mathbb{Z}$ such that $2^{k_0} < \delta \leq 2^{k_0+1}$. Let $B = B(x_0, 2^{k_0+4})$.

For the sake of simplicity in the notation, we assume $x_0 = 0$ since the general case can be achieved similarly by shifting an x_0 unit. We let

$$A_0 = B = B(0, 2^{k_0+4}),$$

$$A_i = \{x : 2^{k_0+3+i} < |x| \leq 2^{k_0+4+i}\},\ i = 1, 2, \ldots$$

and

$$B_i = \{x : |x| \leq 2^{k_0+4+i}\},\ i = 1, 2, \ldots$$

Now for fixed t with $0 < |t| \leq 1$, there is a nonnegative integer k_1 such that

$$2^{k_1} \leq \frac{1}{|t|} < 2^{k_1+1}.$$

We have three lemmas in the following.

Lemma 2.1 *For $x \in B_\delta$ and $0 < |t| \leq 1$, if $x - ty \in A_j$, then*

$$2^{k_0+k_1+j+2} < |y| < 2^{k_0+k_1+j+6}.$$

Proof. For $x \in B_\delta$, $2^{k_0} < \delta \leq 2^{k_0+1}$, and $x - ty \in A_j$, we have

$$\begin{aligned} |y| &\leq \frac{1}{|t|}(|x - ty| + |x|) < 2^{k_1+1}(2^{k_0+4+j} + 2^{k_0+1}) \\ &= 2^{k_0+k_1+2}(2^{3+j} + 1) < 2^{k_0+k_1+j+6}, \end{aligned}$$

and

$$|y| \geq \frac{1}{|t|}(|x-ty|-|x|) > 2^{k_1}(2^{k_0+3+j}-2^{k_0+1})$$
$$> 2^{k_0+k_1+1}(2^{2+j}-2^j) > 2^{k_0+k_1+j+2}. \qquad \square$$

Lemma 2.2 *For $x \in B_\delta$ and $0 < |t| \leq 1$, if*

$$j > 10 \text{ and } |j+k_1-i| > 4,$$

then

$$\{y : x-ty \in A_j\} \cap \{y : x-y \in A_i\} = \emptyset.$$

Proof. We will use a contradiction argument to prove the following two cases:

Case 1: $i > j+k_1+4$; Case 2: $i < j+k_1-4$.

In Case 1, if $x-ty \in A_j$, by Lemma 2.1, we have

$$|y| < 2^{k_0+k_1+j+6}.$$

On the other hand, if $x-y \in A_i$, we have

$$|y| \geq |x-y|-|x| \geq 2^{k_0+i+3}-2^{k_0+1} > 2^{k_0+1}(2^{k_1+j+6}-1) > 2^{k_0+k_1+j+6}.$$

This leads to a contradiction.

In Case 2, for $x-ty \in A_j$, by Lemma 2.1, we have

$$|y| > 2^{k_0+k_1+j+2}.$$

On the other hand, if $x-y \in A_i$, we have

$$|y| \leq |x-y|+|x| \leq 2^{k_0+i+4}+2^{k_0+1} < 2^{k_0+1}(2^{j+k_1-1}+1) \leq 2^{k_0+k_1+j+1}.$$

Again, it leads to a contradiction. The lemma is proved. $\square$

Using the same idea as in the proof above, we have

Lemma 2.3 *For $x \in B_\delta$ and $0 < |t| \leq 1$, if*

$$i > 10 \text{ and } |j+k_1-i| > 4,$$

then

$$\{y : x - ty \in A_j\} \cap \{y : x - y \in A_i\} = \emptyset.$$

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. For any $f \in L^{q,\lambda_1}$ and $g \in L^{r,\lambda_2}$, we write

$$f = \sum_{j=0}^{\infty} f_j \chi_{A_j}, \ \ g = \sum_{i=0}^{\infty} g_i \chi_{A_i},$$

where χ_E is the characteristic function of a set E. Therefore,

$$\|T(f,g)\|_{L^{p,\lambda}} \le \Big\| \sum_{i \ge j \ge 0} T(f\chi_{A_j}, g\chi_{A_i}) \Big\|_{L^{p,\lambda}} + \Big\| \sum_{j \ge i \ge 0} T(f\chi_{A_j}, g\chi_{A_i}) \Big\|_{L^{p,\lambda}}.$$

To prove the theorem, we first show that

$$\Big\| \sum_{j \ge i \ge 0} T(f\chi_{A_j}, g\chi_{A_i}) \Big\|_{L^{p,\lambda}} \le (C_1 C + C_2 |t|^{(1-\lambda_2)n/r}) \|f\|_{L^{q,\lambda_1}} \|g\|_{L^{r,\lambda_2}}.$$

By the assumption and the Minkowski integral inequality, it is easy to check

$$\begin{aligned}
&\left\{ \frac{1}{|B_\delta|^\lambda} \int_{B_\delta} \Big| \sum_{10 \ge j \ge i \ge 0} T(f\chi_{A_j}, g\chi_{A_i})(x) \Big|^p dx \right\}^{1/p} \\
&\le \sum_{10 \ge j \ge i \ge 0} \left\{ \frac{1}{|B_\delta|^\lambda} \int_{B_\delta} |T(f\chi_{A_j}, g\chi_{A_i})(x)|^p dx \right\}^{1/p} \\
&\le C \sum_{10 \ge j \ge i \ge 0} \frac{1}{|B_\delta|^{\lambda/p}} \left\{ \int_{B_j} |f(x)|^q dx \right\}^{1/q} \left\{ \int_{B_i} |g(x)|^r dx \right\}^{1/r} \\
&\le CC_1 \|f\|_{L^{q,\lambda_1}} \|g\|_{L^{r,\lambda_2}},
\end{aligned}$$

where the last inequality holds because that there are only less than 11! terms in the summation and in each term $|B_j| \simeq |B_i| \simeq |B_\delta|$.

We now need to show that the following estimate holds:

$$\Big\| \sum_{j>10, j\ge i\ge 0} T(f\chi_{A_j}, g\chi_{A_i})\Big\|_{L^{p,\lambda}} \le C_2|t|^{(1-\lambda_2)n/r}\|f\|_{L^{q,\lambda_1}}\|g\|_{L^{r,\lambda_2}}.$$

By Lemma 2.2, we know that for $x\in B_\delta$,

$$\begin{aligned}
&\sum_{j\ge i\ge 0, j>10} (f\chi_{A_j})(x-ty)(g\chi_{A_i})(x-y)\\
&\quad = \sum_{j>10, |j+k_1-i|\le 4} (f\chi_{A_j})(x-ty)(g\chi_{A_i})(x-y)\\
&\quad = \sum_{j>10, |l-j|\le 4} (f\chi_{A_j})(x-ty)(g\chi_{A_{l+k_1}})(x-y).
\end{aligned}$$

By the Minkowski inequality, we have

$$\begin{aligned}
&\Big\| \sum_{j>10, j\ge i\ge 0} T(f\chi_{A_j}, g\chi_{A_i})\Big\|_{L^{p,\lambda}} \le \sum_{j>10, j\ge i\ge 0} \|T(f\chi_{A_j}, g\chi_{A_i})\|_{L^{p,\lambda}}\\
&\quad = \sum_{j>10}\sum_{|j+k_1-i|\le 4} \|T(f\chi_{A_j}, g\chi_{A_i})\|_{L^{p,\lambda}}\\
&\quad = \sum_{j>10}\sum_{|l-j|\le 4} \|T(f\chi_{A_j}, g\chi_{A_{l+k_1}})\|_{L^{p,\lambda}}.
\end{aligned}$$

The relationship $|l-j|\le 4$ means that $l\simeq j$ for all $j>10$. So the above double series $\sum_{j>10}\sum_{|l-j|\le 4}\|T(f\chi_{A_j}, g\chi_{A_{l+k_1}})\|_{L^{p,\lambda}}$ essentially is a single series $\sum_{j>10}\|T(f\chi_{A_j}, g\chi_{A_{j+k_1}})\|_{L^{p,\lambda}}$. Thus it suffices to show

$$\sum_{j>10}\|T(f\chi_{A_j}, g\chi_{A_{j+k_1}})\|_{L^{p,\lambda}} \le C_2|t|^{(1-\lambda_2)n/r}\|f\|_{L^{q,\lambda_1}}\|g\|_{L^{r,\lambda_2}}.$$

Here by the assumption and Lemma 2.1, we have that for $x\in B_\delta$,

$$\begin{aligned}
&|T(f\chi_{A_j}, g\chi_{A_{j+k_1}})(x)|\\
&\quad \preceq \int_{\mathbb{R}^n} \frac{|(f\chi_{A_j})(x-ty)(g\chi_{A_{j+k_1}})(x-y)|}{|y|^n}dy\\
&\quad \preceq \frac{1}{2^{(k_0+k_1+j)n}}\int_{B_{k_1+j+2}} |(f\chi_{A_j})(x-ty)(g\chi_{A_{j+k_1}})(x-y)|dy.
\end{aligned}$$

Using Hölder's inequality, noting that for $x \in B_\delta$, $g\chi_{A_{j+k_1}}(x-y)$ has support in A_{j+k_1}, and changing the variables, the previous term is controlled by

$$\begin{aligned}
&\preceq \frac{1}{2^{(k_0+k_1+j)n/p}} \frac{1}{|t|^{n/q}} \|f\chi_{A_j}\|_{L^q} \|g\chi_{A_{j+k_1}}\|_{L^r} \\
&\preceq \frac{1}{2^{(k_0+k_1+j)n/p}} 2^{k_1 n/q} 2^{n(k_0+j)\lambda_1/q} 2^{n(k_0+k_1+j)\lambda_2/r} \|f\|_{L^{q,\lambda_1}} \|g\|_{L^{r,\lambda_2}} \\
&= 2^{n(\lambda-1)(k_0+j)/p} 2^{n(\lambda_2-1)k_1/r} \|f\|_{L^{q,\lambda_1}} \|g\|_{L^{r,\lambda_2}}.
\end{aligned}$$

The previous estimate leads to for $j \geq i \geq 0,\ j > 10$

$$\begin{aligned}
&\left\{ \frac{1}{|B_\delta|^\lambda} \int_{B_\delta} |T(f\chi_{A_j}, g\chi_{A_i})(x)|^p dx \right\}^{1/p} \\
&\quad \preceq 2^{n(\lambda-1)(k_0+j)/p} 2^{n(\lambda_2-1)k_1/r} |B_\delta|^{(1-\lambda)/p} \|f\|_{L^{q,\lambda_1}} \|g\|_{L^{r,\lambda_2}} \\
&\quad \preceq 2^{n(\lambda-1)(k_0+j)/p} 2^{n(\lambda_2-1)k_1/r} 2^{k_0 n(1-\lambda)/p} \|f\|_{L^{q,\lambda_1}} \|g\|_{L^{r,\lambda_2}} \\
&\quad = 2^{n(\lambda-1)j/p} 2^{n(\lambda_2-1)k_1/r} \|f\|_{L^{q,\lambda_1}} \|g\|_{L^{r,\lambda_2}}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\left\| \sum_{j>10, j\geq i\geq 0} T(f\chi_{A_j}, g\chi_{A_i}) \right\|_{L^{p,\lambda}} \leq \sum_{j>10} \|T(f\chi_{A_j}, g\chi_{A_j})\|_{L^{p,\lambda}} \\
&\quad \leq C_2 \sum_{j>10} 2^{n(\lambda-1)j/p} 2^{n(\lambda_2-1)k_1/r} \|f\|_{L^{q,\lambda_1}} \|g\|_{L^{r,\lambda_2}} \\
&\quad \leq C_2 |t|^{(1-\lambda_2)n/r} \|f\|_{L^{q,\lambda_1}} \|g\|_{L^{r,\lambda_2}}.
\end{aligned}$$

The proof of the estimate on the integral

$$\left\| \sum_{i>j\geq 0} T(f\chi_{A_j}, g\chi_{A_i}) \right\|_{L^{p,\lambda}}$$

is similar to that of the above one. The main change is to use Lemma 2.3 in place of Lemma 2.2. This completes the proof. $\square$

3. Some applications

As applications, we can easily obtain the boundedness on the Morrey spaces for several well known bilinear operators. These operators include the bilinear Hilbert transform, the bilinear oscillatory Hilbert transform, the bilinear singular integral with rough kernel and the first Calderón commutator.

Our first result is about the bilinear Hilbert transform.

Corollary 3.1 *Let $\lambda,\ \lambda_1,\ \lambda_2$ be as in Theorem* 1.1 *and let* $1/q + 1/r = 1/p,\ 1 < q,\ r \leq \infty,\ 1 \leq p < \infty$. *Then*

$$\|H(f, g)\|_{L^{p,\lambda}} \preceq \|f\|_{L^{q,\lambda_1}} \|g\|_{L^{r,\lambda_2}},$$

if and only if

$$\lambda/p = \lambda_1/q + \lambda_2/r.$$

Proof. The sufficient part follows from Theorem 1.1 and Theorem A.

To prove the necessary part, let

$$f_\varepsilon(x) = f(\varepsilon x),\ g_\varepsilon(x) = g(\varepsilon x).$$

By changing the variable, we have

$$\|H(f_\varepsilon, g_\varepsilon)\|_{L^{p,\lambda}(\mathbb{R})} = \varepsilon^{(\lambda-1)/p} \|H(f, g)\|_{L^{p,\lambda}(\mathbb{R})},$$
$$\|f_\varepsilon\|_{L^{q,\lambda_1}(\mathbb{R})} = \varepsilon^{(\lambda_1-1)/q} \|f\|_{L^{q,\lambda_1}(\mathbb{R})},$$

and

$$\|g_\varepsilon\|_{L^{r,\lambda_2}(\mathbb{R})} = \varepsilon^{(\lambda_2-1)/r} \|g\|_{L^{r,\lambda_2}(\mathbb{R})}.$$

Therefore,

$$\|H(f, g)\|_{L^{p,\lambda}(\mathbb{R})} \preceq \|f\|_{L^{q,\lambda_1}(\mathbb{R})} \|g\|_{L^{r,\lambda_2}(\mathbb{R})},$$

for all $f,\ g$ if and only if

$$\|H(f_\varepsilon, g_\varepsilon)\|_{L^{p,\lambda}(\mathbb{R})} \preceq \|f_\varepsilon\|_{L^{q,\lambda_1}(\mathbb{R})} \|g_\varepsilon\|_{L^{r,\lambda_2}(\mathbb{R})},$$

if and only if

$$\|H(f_\varepsilon, g_\varepsilon)\|_{L^{p,\lambda}(\mathbb{R})} \preceq \varepsilon^{\lambda_1/q+\lambda_2/r-\lambda/p}\|f\|_{L^{q,\lambda_1}(\mathbb{R})}\|g\|_{L^{r,\lambda_2}(\mathbb{R})}.$$

Since ε is arbitrary, we must require

$$\lambda_1/q + \lambda_2/r - \lambda/p = 0.$$

The corollary is proved. □

Next, we consider the bilinear oscillatory Hilbert transform

$$\mathcal{T}_P(f,g)(x) = p.v. \int_{\mathbb{R}} \frac{e^{iP(x,y)} f(x-y)g(x+y)}{y} dy,$$

where P is a real-valved polynomial. In [3], the authors proved that the operator $\mathcal{T}_P$ is a bounded operator from $L^q(\mathbb{R}) \times L^r(\mathbb{R})$ to $L^p(\mathbb{R})$ with $1/r + 1/q = 1/p \le 1$. Also they showed that the operator bound

$$\|\mathcal{T}_P\|_{L^q(\mathbb{R})\times L^r(\mathbb{R}) \ \to \ L^p(\mathbb{R})}$$

is independent of the coefficients of P. By this result and Theorem 1.1, it is easy to obtain the following result.

Corollary 3.2 *Let* λ, λ_1, λ_2 *be as in Theorem 1.1 and let* $1/q+1/r = 1/p$, $1 < q, r \le \infty$, $1 \le p < \infty$. *If*

$$\lambda/p = \lambda_1/q + \lambda_2/r,$$

then we have

$$\|\mathcal{T}_P(f,g)\|_{L^{p,\lambda}} \preceq \|f\|_{L^{q,\lambda_1}}\|g\|_{L^{r,\lambda_2}}.$$

Also the operator norm

$$\|\mathcal{T}_P\|_{L^{q,\lambda_1}\times L^{r,\lambda_2}\to L^{p,\lambda}}$$

is independent of the coefficients of the polynomial P.

In the high dimensional case, we consider the bilinear singular integral with rough kernel

$$T_\Omega(f,g)(x) = p.v. \int_{\mathbb{R}^n} \frac{f(x-y)g(x+y)}{|y|^n}\Omega(y')dy,$$

and its maximal operator

$$T_\Omega^*(f,g)(x) = \sup_{\varepsilon>0} |T_{\Omega,\varepsilon}(f,g)(x)|,$$

where $T_{\Omega,\varepsilon}$ is the truncated bilinear operator defined by

$$T_{\Omega,\varepsilon}(f,g)(x) = \int_{|y|>\varepsilon} \frac{f(x-y)g(x+y)}{|y|^n} \Omega(y')dy, \qquad \text{for } \varepsilon > 0.$$

where $\Omega(y')$ is a function defined on the unit sphere S^{n-1} in the Euclidean space $\mathbb{R}^n$, and $y' = y/|y|$ for any $y \neq 0$.

Corollary 3.3 *Let* λ, λ_1, λ_2 *be as in Theorem* 1.1 *and let* $1/q+1/r = 1/p$, $1 < q$, $r \leq \infty$, $1 \leq p < \infty$. *If* $\Omega \in L^\infty(S^{n-1})$ *is an odd function, then*

$$\|T_\Omega(f,g)\|_{L^{p,\lambda}} \preceq \|f\|_{L^{q,\lambda_1}} \|g\|_{L^{r,\lambda_2}},$$

if and only if

$$\lambda/p = \lambda_1/q + \lambda_2/r.$$

Proof. By Theorem 1.1, we need to show

$$\|T_\Omega(f,g)\|_{L^p(\mathbb{R}^n)} \preceq \|f\|_{L^q(\mathbb{R}^n)} \|g\|_{L^r(\mathbb{R}^n)}.$$

The proof for the above inequality can follow a standard rotation method by Calderón and Zygmund. We state its proof for completeness. By the spherical coordinates and changing variables,

$$\begin{aligned} T_\Omega(f,g)(x) &= p.v. \int_{S^n} \left\{ \int_0^\infty \frac{f(x-ty')g(x+ty')}{t} dt \right\} \Omega(y') d\sigma(y') \\ &= p.v. \int_{S^n} \left\{ \int_{-\infty}^0 \frac{f(x-ty')g(x+ty')}{t} dt \right\} \Omega(y') d\sigma(y'). \end{aligned}$$

This means that

$$2T_\Omega(f,g)(x) = p.v. \int_{S^n} \left\{ \int_{-\infty}^\infty \frac{f(x-ty')g(x+ty')}{t} dt \right\} \Omega(y') d\sigma(y').$$

Thus by the Minkowski integral inequality,

$$\|T_\Omega(f,g)\|_{L^p(\mathbb{R}^n)}$$
$$\preceq p.v. \int_{S^n} \left\{ \left\| \int_{-\infty}^{\infty} \frac{|f(\cdot - ty')g(\cdot + ty')|}{t} dt \right\|_{L^p(\mathbb{R}^n)} \right\} d\sigma(y').$$

Now for y' fixed, take the rotation O such that $O(y') = \mathbf{1} = (1, 0, 0, \ldots, 0)$. Denote function $f_{O^{-1}}$ by $f_{O^{-1}}(x) = f(O^{-1}x)$. Then

$$\int_{S^n} \left\{ \left\| \int_{-\infty}^{\infty} \frac{|f(\cdot - ty')g(\cdot + ty')|}{t} dt \right\|_{L^p(\mathbb{R}^n)} \right\} d\sigma(y')$$
$$= \int_{S^n} \left\{ \int_{\mathbb{R}^n} \left| \int_{-\infty}^{\infty} \frac{|f_{O^{-1}}(Ox - t\mathbf{1})g_{O^{-1}}(Ox + t\mathbf{1})|}{t} dt \right|^p dx \right\}^{1/p} d\sigma(y')$$
$$= \int_{S^n} \left\{ \int_{\mathbb{R}^n} \left| \int_{-\infty}^{\infty} \frac{|f_{O^{-1}}(x - t\mathbf{1})g_{O^{-1}}(x + t\mathbf{1})|}{t} dt \right|^p dx \right\}^{1/p} d\sigma(y')$$
$$= \int_{S^n} \left\{ \int_{\mathbb{R}^n} \left| \int_{-\infty}^{\infty} \frac{|f_{O^{-1}}(x_1 - t, \overline{x})g_{O^{-1}}(x_1 + t, \overline{x})|}{t} dt \right|^p dx \right\}^{1/p} d\sigma(y'),$$

where $\overline{x} = (x_2, x_3, \ldots, x_n)$.

Using the well known result of the bilinear Hilbert transform

$$\left\{ \int_{\mathbb{R}^n} \left| \int_{-\infty}^{\infty} \frac{|f_{O^{-1}}(x_1 - t, \overline{x})g_{O^{-1}}(x_1 + t, \overline{x})|}{t} dt \right|^p dx \right\}^{1/p}$$
$$\preceq \left\{ \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} |f_{O^{-1}}(x_1, \overline{x})|^q dx_1 \right)^{p/q} \left(\int_{\mathbb{R}} |g_{O^{-1}}(x_1, \overline{x})|^r dx_1 \right)^{p/r} d\overline{x} \right\}^{1/p}.$$

Using Hölder's inequality and iterating the integral, we obtain

$$\left\{ \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} |f_{O^{-1}}(x_1, \overline{x})|^q dx_1 \right)^{p/q} \left(\int_{\mathbb{R}} |g_{O^{-1}}(x_1, \overline{x})|^r dx_1 \right)^{p/r} d\overline{x} \right\}^{1/p}$$
$$\preceq \|f_{O^{-1}}\|_{L^q} \|g_{O^{-1}}\|_{L^r} = \|f\|_{L^q} \|g\|_{L^r}.$$

This shows

$$\|T_\Omega(f,g)\|_{L^p(\mathbb{R}^n)} \preceq \|f\|_{L^q(\mathbb{R}^n)} \|g\|_{L^r(\mathbb{R}^n)}.$$

We prove the corollary. $\square$

It is well known that for the maximal bilinear Hilbert transform

$$\mathcal{H}^*(f,g)(x) = \sup_{\varepsilon>0}\left|\int_{|t|>\varepsilon}\frac{f(x-t)g(x+t)}{t}dt\right|,$$

Lacey in [12] obtained the following remarkable result.

Lemma 3.1 *Let* $1 < q,\ r \leq \infty$ *and* $1/p = 1/q + 1/r$. *If* $2/3 < p < \infty$, *then*

$$\|\mathcal{H}^*(f,g)\|_{L^p(\mathbb{R})} \preceq \|f\|_{L^q(\mathbb{R})}\|g\|_{L^r(\mathbb{R})}.$$

The same arguments as in proof of Corollary 3.3 and Lemma 3.1 apply to yield

Corollary 3.4 *Let* λ, λ_1, λ_2 *be as in Theorem* 1.1 *and let* $1/q+1/r = 1/p$, $1 < q,\ r \leq \infty$, $1 \leq p < \infty$. *If* $\Omega \in L^\infty(S^{n-1})$ *is an odd function, then*

$$\|T^*_\Omega(f,g)\|_{L^{p,\lambda}} \preceq \|f\|_{L^{q,\lambda_1}}\|g\|_{L^{r,\lambda_2}},$$

if and only if

$$\lambda/p = \lambda_1/q + \lambda_2/r.$$

The first Calderón commutator is defined by

$$C_\varphi f(x) = p.v.\int_{\mathbb{R}}\frac{\varphi(x)-\varphi(y)}{(x-y)^2}f(y)dy,$$

where, for a Lipschitz function φ, we may write,

$$\frac{\varphi(x)-\varphi(y)}{x-y} = \frac{1}{x-y}\int_x^y \varphi'(t)dt = \int_0^1 \varphi'((1-t)x+ty)dt.$$

Thus,

$$\begin{aligned} C_\varphi f(x) &= p.v.\int_{\mathbb{R}}\frac{\varphi(x)-\varphi(y)}{(x-y)^2}f(y)dy \\ &= p.v.\int_{\mathbb{R}}\frac{f(y)}{x-y}f(y)\int_0^1 \varphi'((1-t)x+ty)dtdy \end{aligned}$$

$$= p.v. \int_0^1 \int_{\mathbb{R}} f(x-y)\varphi'(x-ty)\frac{dy}{y}dt$$
$$= \int_0^1 H_t(f,\varphi')(x)dt.$$

A celebrated result due to Grafakos and Li [7] showed that the following

Lemma 3.2 *For $2 < q,\ r < \infty$ and $1 < p = qr/(q+r) < 2$, then there is a constant $C = C(q,r)$ such that for all $f,\ g$ on $\mathbb{R}$,*

$$\sup_{t\in\mathbb{R}} \|H_t(f,g)\|_{L^p(\mathbb{R})} \le C\|f\|_{L^q(\mathbb{R})}\|g\|_{L^r(\mathbb{R})}.$$

We have the following result for the operator C_φ.

Corollary 3.5 *Let $\lambda,\ \lambda_1,\ \lambda_2$ be as in Theorem 1.1 and let $p,\ q$ and r be as in Lemma 3.2. If φ is a Lipschitz function, then we have*

$$\|C_\varphi(f)\|_{L^{p,\lambda}} \preceq \|f\|_{L^{q,\lambda_1}}\|\varphi'\|_{L^{r,\lambda_2}},$$

where

$$\lambda/p = \lambda_1/q + \lambda_2/r.$$

Proof. By Theorem 1.1, it is easy to check that

$$\begin{aligned}
\|C_\varphi(f)\|_{L^{p,\lambda}} &\le \int_0^1 \|H_t(f,\varphi')\|_{L^{p,\lambda}}dt \\
&\le \int_0^1 (C_1C + C_2t^{(1-\lambda_1)/r})\|f\|_{L^{q,\lambda_1}}\|\varphi'\|_{L^{r,\lambda_2}}dt \\
&\preceq \left(1 + \int_0^1 t^{(1-\lambda_1)/r}dt\right)\|f\|_{L^{q,\lambda_1}}\|\varphi'\|_{L^{r,\lambda_2}} \\
&\preceq \|f\|_{L^{q,\lambda_1}}\|\varphi'\|_{L^{r,\lambda_2}}.
\end{aligned}$$

The corollary is proved. □

4. Final Remarks

In this section, we give two remarks. First, our method in the proof of Theorem 1.1 may be still valid if the condition (1) is replaced by some similar size conditions. For example, the bilinear oscillatory integral along the parabola studied by Fan and Li in [6],

$$\mathcal{T}_\beta(f,g)(x) = \int_{-1}^{1} f(x-y)g(x-y^2)e^{i|y|^{-\beta}}\frac{dy}{|y|}.$$

They proved that if $\beta > 1$, then

$$\|\mathcal{T}_\beta(f,g)\|_{L^2(\mathbb{R})} \preceq \|f\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}.$$

Using this result and following the same method in the proof for Theorem 1.1, we also easily obtain the following

Corollary 4.1 *Let $\beta > 1$. Then for $0 < \lambda < 1$, we have*

$$\|\mathcal{T}_\beta(f,g)\|_{L^{2,\lambda}(\mathbb{R})} \preceq \|f\|_{L^{\infty,\lambda}(\mathbb{R})}\|g\|_{L^{2,\lambda}(\mathbb{R})}.$$

We leave the proof of this corollary to the reader.

Secondly, our method works for the multilinear operator T satisfying certain integral size condition

$$|\mathcal{T}(f_1,f_2,\ldots,f_m)(x)| \preceq \int_{\mathbb{R}^n}\prod_{j=1}^{m}|f_j(x-\theta_j y)|\;|y|^{-n}dy$$

for x such that $0 \notin \cap_{j=1}^{m}$ supp $(f_j(x-\theta_j\cdot))$, where $\theta_j \neq 0$, $j = 1,2,\ldots,m$, are fixed real numbers. In fact, without loss of generality, we assume

$$|\theta_1| \geq |\theta_2| \geq \cdots \geq |\theta_m| > 0.$$

Then

$$\int_{\mathbb{R}^n}\prod_{j=1}^{m}|f_j(x-\theta_j y)|\,|y|^{-n}dy = \int_{\mathbb{R}^n}\prod_{j=2}^{m}|f_j(x-\eta_j y)|\,|f_1(x-y)|\,|y|^{-n}dy,$$

where $\eta_i = \theta_i/\theta_1$. Thus, we may assume that all η_j satisfying $|\eta_j| \leq 1$. Now pick some suitable nonnegative integers $k_1,k_2,\ldots,k_{m-1}$ satisfying

$$\frac{1}{|\eta_j|} \simeq 2^{k_{j-1}} \text{ for } j = 2,3,\ldots,m.$$

We can follow the same proof of Theorem 1.1 to show a multilinear version of Theorem 1.1.

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