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Schwarz maps associated with the triangle groups $(2,4,4)$ and $(2,3,6)$

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Abstract. We consider the Schwarz maps with monodromy groups isomorphic to the triangle groups $(2,4,4)$ and $(2,3,6)$ and their inverses. We apply our formulas to studies of mean iterations.

Key words: Schwarz map, theta function, mean iteration.

1. Introduction

The Gauss hypergeometric function $F(\alpha,\beta,\gamma;z)$ is defined by the series

$$F(\alpha,\beta,\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha,n)(\beta,n)}{(\gamma,n)(1,n)}z^n,$$

where z is the main variable in the unit disk $\mathbb{D}=\{z\in\mathbb{C}\mid |z|<1\}$, α,β,γ are parameters with $\gamma\neq 0,-1,-2,\dots$, and $(\alpha,n)=\alpha(\alpha+1)\cdots(\alpha+n-1)$. This function admits an integral representation

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_1^{\infty}t^{\beta-\gamma}(t-z)^{-\beta}(t-1)^{\gamma-\alpha-1}dt, \tag{1.1}$$

and satisfies the hypergeometric differential equation

$$\mathcal{F}(\alpha,\beta,\gamma): z(1-z)f''(z)+\{\gamma-(\alpha+\beta+1)z\}f'(z)-\alpha\beta f(z)=0, \tag{1.2}$$

which has only singular points of regular type at $z=0,1,\infty$. The Schwarz map is defined by the continuation to $X=\mathbb{C}-\{0,1\}$ of the ratio of two linearly independent solutions to $\mathcal{F}(\alpha,\beta,\gamma)$ in a small simply connected domain in X. It is well known that the inverse of the Schwarz map is single valued if and only if each of

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$$r_0 = \frac{1}{|1-\gamma|}, \quad r_1 = \frac{1}{|\gamma-\alpha-\beta|}, \quad r_\infty = \frac{1}{|\alpha-\beta|}$$

belongs to $\{2,3,\dots,\infty\}$. In this case, the projective monodromy group of $\mathcal{F}(\alpha,\beta,\gamma)$ is isomorphic to the triangle group (r_0,r_1,r_∞) and the image of the Schwarz map is isomorphic to

$$\begin{cases} \text{the complex projective line } \mathbb{P}^1 & \text{if } 1/r_0+1/r_1+1/r_\infty>1, \\ \text{the complex plane } \mathbb{C} & \text{if } 1/r_0+1/r_1+1/r_\infty=1, \\ \text{the upper half space } \mathbb{H} & \text{if } 1/r_0+1/r_1+1/r_\infty<1. \end{cases}$$

There are only finite sets

$$\{r_0,r_1,r_\infty\}=\{2,2,\infty\},\ \{2,4,4\},\ \{2,3,6\},\ \{3,3,3\},$$

such that $1/r_0+1/r_1+1/r_\infty=1$. All of them appear in studies of mean iterations in [HKM] and [MO]. In particular, a limit formula of a mean iteration associated to $\{2,2,\infty\}$ is extended in [Ma1] to that of an iteration of three means of three terms. Moreover, it is shown in [G] as a geometrical background that this extended limit formula can be obtained from the twice formula of an elliptic curve and the Abel-Jacobi map for it.

In this paper, we consider the Schwarz maps for two sets of the parameters

$$(\alpha,\beta,\gamma)=\left(\frac{1}{4},0,\frac{1}{2}\right), \quad \left(\frac{1}{3},0,\frac{1}{2}\right)$$

to study geometrically limit formulas of mean iterations associated to $\{2,4,4\}$ and $\{2,3,6\}$. The monodromy groups of $\mathcal{F}(\alpha,\beta,\gamma)$ for these sets of parameters are reducible and isomorphic to the triangle groups $(2,4,4)$ and $(2,3,6)$, respectively. We give circuit matrices generating these groups in Corollary 1. The images of the Schwarz maps are the quotient of the complex torus $E_i=\mathbb{C}/(i\mathbb{Z}+\mathbb{Z})$ by the multiplicative group $\langle i\rangle=\{\pm1,\pm i\}$ for $(\alpha,\beta,\gamma)=(1/4,0,1/2)$, and that of $E_\zeta=\mathbb{C}/(\zeta\mathbb{Z}+\mathbb{Z})$ by $\langle\zeta\rangle=\{\pm1,\pm\zeta,\pm\zeta^2\}$ for $(\alpha,\beta,\gamma)=(1/3,0,1/2)$, where $i=\sqrt{-1}$ and $\zeta=(1+\sqrt{3}i)/2$. We consider elliptic curves

$$C_i: u^4=t^2(t-1), \quad C_\zeta: u^6=t^3(t-1),$$

and relate these Schwarz maps and the Abel-Jacobi maps

$$\jmath_i : C_i \to E_i, \quad \jmath_\zeta : C_\zeta \to E_\zeta$$

defined by incomplete elliptic integrals on C_i and on C_ζ. We express the inverses of these Schwarz maps in terms of the theta function $\vartheta_{a,b}(z,\tau)$ with characteristics a, b; see Theorem 1 and Theorem 3. We study the pull-back of the $(1+i)$-multiple on E_i and that of the $(1+\zeta)$-multiple on E_ζ under the corresponding Abel-Jacobi maps. We show that Theorem 2 yields the limit formula of the mean iteration in [HKM]:

$$\lim_{n\to\infty} \overbrace{m \circ \cdots \circ m}^{n}(a,b) = \frac{a}{F(1/4,1/2,5/4;1-b^2/a^2)^2}(1,1),$$

where $a > b > 0$ and

$$m : (a,b) \mapsto \left(\frac{a+b}{2}, \sqrt{\frac{a(a+b)}{2}} \right).$$

We have a similar result from the $(1+\zeta)$-multiple formula on the elliptic curve E_ζ in Theorem 4. We elucidate a geometric background of these limit formulas as multiplications on the complex tori E_i and E_ζ.

As by-products of our results, we evaluate some $\vartheta_{a,b}(0,\tau)$ for $\tau = i, \zeta$ in terms of the Gamma function in Corollaries 3, 6, and give relations between $\theta_{a,b}(z,\tau)$ for $\tau = i, \zeta$ and the hypergeometric function in Corollaries 5, 8.

2. The Schwarz map

2.1. Fundamental system of solutions to $\mathcal{F}(\alpha,\beta,\gamma)$

We define the Schwarz map as the ratio of solutions to $\mathcal{F}(\alpha,\beta,\gamma)$ given by the Euler type integral representations

$$f_1(x) = \int_1^x t^{\beta-\gamma}(t-x)^{-\beta}(t-1)^{\gamma-\alpha}\frac{dt}{t-1},$$
$$f_2(x) = \int_1^\infty t^{\beta-\gamma}(t-x)^{-\beta}(t-1)^{\gamma-\alpha}\frac{dt}{t-1},$$

where

$$0 < \mathrm{Re}(\alpha) < \mathrm{Re}(\gamma), \quad \mathrm{Re}(\beta) < 1.$$

For an element x in $U = \{x \in X \mid |x| < 1, |x-1| < 1\}$, they can be expressed by the hypergeometric series. By (1.1),

$$f_2(x) = B(\gamma - \alpha, \alpha) \cdot F(\alpha, \beta, \gamma; x),$$

where $B(*, *)$ denotes the beta function. By the variable change

$$s = \frac{x-1}{t-1}, \quad \text{i.e.} \quad t = \frac{s+x-1}{s}, \quad dt = -\frac{(x-1)ds}{s^2}$$

for the integral representation of $f_1(x)$ and (1.1), we have

$$f_1(x) = e^{\pi i(\gamma-\alpha)} B(\gamma-\alpha, 1-\beta)(1-x)^{\gamma-\alpha-\beta} \cdot F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-x),$$

where $\theta_1 = \arg x$ and $\theta_2 = \arg(1-x)$ belong to the open interval $(-\pi/2, \pi/2)$, and the arguments of t, $t-x$, $t-1$ on the open segments $(1, x)$ and $(1, \infty)$ belong to the intervals in Table 1. Here pay your attention to the argument of $t-1$ and the orientation of the path integral.

Table 1. Arguments of t, $t-x$ and $t-1$.

	$t \in (x, 1)$	$t \in (1, \infty)$
$\arg(t)$	$[\min(0, \theta_1), \max(0, \theta_1)]$	0
$\arg(t-x)$	θ_2	$[\min(0, \theta_2), \max(0, \theta_2)]$
$\arg(t-1)$	$\pi + \theta_2$	0

Remark 1 When $\beta = 0$, the solution $f_1(x)$ is expressed as

$$f_1(x) = \frac{e^{\pi i(\gamma-\alpha)}}{\gamma - \alpha} \cdot (1-x)^{\gamma-\alpha} \cdot F(\gamma - \alpha, \gamma, \gamma - \alpha + 1; 1 - x)$$

for $|x-1| < 1$, and the solution $f_2(x)$ reduces to a constant

$$B(\gamma - \alpha, \alpha) = \frac{\Gamma(\gamma - \alpha)\Gamma(\alpha)}{\Gamma(\gamma)}.$$

2.2. Monodromy representation of $\mathcal{F}(\alpha, \beta, \gamma)$

We take a base point $\dot{x}$ in U. Let $\mathcal{M}$ be the monodromy representation of $\mathcal{F}(\alpha, \beta, \gamma)$ with respect to the base point $\dot{x}$. It is the homomorphism from the fundamental group $\pi_1(X, \dot{x})$ to the general linear group of the local solution space to $\mathcal{F}(\alpha, \beta, \gamma)$ on U arising from the analytic continuation along a loop with terminal $\dot{x}$. We denote the image of $\ell \in \pi_1(X, \dot{x})$ by $\mathcal{M}_\ell$. Let ℓ_0 and ℓ_1 be a loop starting from $\dot{x}$ turning positively once around the point $x = 0$ and that around the point $x = 1$, respectively. Since $\pi_1(X, \dot{x})$ is generated by ℓ_0 and ℓ_1, $\mathcal{M}$ is determined by $\mathcal{M}_0 = \mathcal{M}_{\ell_0}$ and $\mathcal{M}_1 = \mathcal{M}_{\ell_1}$. By the basis ${}^t(f_1(x), f_2(x))$, the transformations $\mathcal{M}_0$ and $\mathcal{M}_1$ are represented by matrices M_0 and M_1. That is, the basis ${}^t(f_1(x), f_2(x))$ is transformed into

$$M_i \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

by the analytic continuation along the loop ℓ_i. They are expressed by the intersection matrix

$$H = \begin{pmatrix} \dfrac{\mathbf{e}(\gamma-\alpha) - \mathbf{e}(\beta)}{\mathbf{e}(\gamma-\alpha) - 1} & \dfrac{-\mathbf{e}(\gamma-\alpha)}{\mathbf{e}(\gamma-\alpha) - 1} \\ \dfrac{-\mathbf{e}(\beta)+1}{\mathbf{e}(\gamma-\alpha) - 1} & \dfrac{-\mathbf{e}(\gamma)+1}{(\mathbf{e}(\gamma-\alpha) - 1)(\mathbf{e}(\alpha)-1)} \end{pmatrix}$$

as in [Ma2], where $\mathbf{e}(\alpha) = \exp(2\pi i \alpha)$.

Proposition 1 *Suppose that*

$$\alpha, \quad \alpha - \gamma, \quad \beta - \gamma \notin \mathbb{Z}, \quad \beta \notin \mathbb{N} = \{1, 2, 3, \dots\}.$$

Then we have

$$M_0 = \lambda_0 I_2 - \frac{\lambda_0 - 1}{e_2 H\, e_2^*} H\, e_2^* e_2 = \begin{pmatrix} \mathbf{e}(-\gamma) & 1 - \mathbf{e}(-\alpha) \\ 0 & 1 \end{pmatrix},$$

$$M_1 = I_2 - \frac{1 - \lambda_1}{e_1 H\, e_1^*} H\, e_1^* e_1 = \begin{pmatrix} \mathbf{e}(\gamma-\alpha-\beta) & 0 \\ -1 + \mathbf{e}(-\beta) & 1 \end{pmatrix},$$

where $\lambda_0 = \mathbf{e}(-\gamma)$, $\lambda_1 = \mathbf{e}(\gamma - \alpha - \beta)$,

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = (1,0), \quad e_2 = (0,1), \quad e_1^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We use this proposition for $\beta \in \mathbb{Z} - \mathbb{N}$ with a base change

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 - \mathbf{e}(\alpha) \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}.$$

Corollary 1 *In this case, M_0 and M_1 are transformed into*

$$N_0 = \begin{pmatrix} \mathbf{e}(-\gamma) & -\mathbf{e}(-\alpha) \\ 0 & 1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} \mathbf{e}(\gamma - \alpha) & 0 \\ 0 & 1 \end{pmatrix},$$

respectively. When $(\alpha, \beta, \gamma) = (1/4, 0, 1/2)$, N_0, N_1, $(N_0 N_1)^{-1}$ are

$$\begin{pmatrix} -1 & i \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} i & 1 \\ 0 & 1 \end{pmatrix}.$$

The group generated by these matrices is isomorphic to the triangle group $(2,4,4)$, and to the semi-direct product $\langle i \rangle \ltimes \mathbb{Z}[i]$. When $(\alpha, \beta, \gamma) = (1/3, 0, 1/2)$, N_0, N_1, $(N_0 N_1)^{-1}$ are

$$\begin{pmatrix} -1 & \zeta \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \zeta^2 & 1 \\ 0 & 1 \end{pmatrix}, \quad \zeta = \frac{1 + \sqrt{3}i}{2}.$$

The group generated by these matrices is isomorphic to the triangle group $(2,3,6)$, and to the semi-direct product $\langle \zeta \rangle \ltimes \mathbb{Z}[\zeta]$.

3. Theta functions

3.1. Basic properties of $\vartheta_{a,b}$

The theta function with characteristics is defined by

$$\vartheta_{a,b}(z, \tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i (n+a)^2 \tau + 2\pi i (n+a)(z+b)),$$

where $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ are main variables, and a, b are rational parameters. For a fixed τ, we denote $\vartheta_{a,b}(z, \tau)$ by $\vartheta_{a,b}(z)$. In this subsection, we collect useful formulas for $\vartheta_{a,b}(z, \tau)$ in our study from [I] and [Mu].

It is easy to see that this function satisfies

$$\begin{aligned}
\vartheta_{a,b}(z,\tau) &= \mathbf{e}\left(\frac{a^2\tau}{2}+a(z+b)\right)\vartheta_{0,0}(z+a\tau+b,\tau),\\
\vartheta_{-a,-b}(z,\tau) &= \vartheta_{a,b}(-z,\tau),\\
\vartheta_{a,b}(z+p\tau+q,\tau) &= \mathbf{e}\left(aq-\frac{p^2\tau}{2}-pz-bp\right)\vartheta_{a,b}(z,\tau)\\
\vartheta_{a+p,b+q}(z,\tau) &= \mathbf{e}(aq)\vartheta_{a,b}(z,\tau),\\
\frac{\vartheta_{a,b}(z+c\tau+d,\tau)}{\vartheta_{a',b'}(z+c\tau+d,\tau)} &= \mathbf{e}(c(b'-b))\frac{\vartheta_{a+c,b+d}(z,\tau)}{\vartheta_{a'+c,b'+d}(z,\tau)},
\end{aligned}$$

where $p,q\in\mathbb{Z}$ and $a',b'\in\mathbb{Q}$.

It is known that $\vartheta_{a,b}(z)=0$ if and only if

$$\left(-a+p+\frac{1}{2}\right)\tau+\left(-b+q+\frac{1}{2}\right)\quad (p,q\in\mathbb{Z}),$$

and they are simple zeroes. If $(a_1,b_1),\dots,(a_r,b_r)$ and $(a'_1,b'_1),\dots,(a'_r,b'_r)$ satisfy

$$\sum_{i=1}^{r}(a_i,b_i)\equiv\sum_{i=1}^{r}(a'_i,b'_i) \bmod \mathbb{Z}^2$$

then the product

$$F(z)=\prod_{i=1}^{r}\frac{\vartheta_{a_i,b_i}(z)}{\vartheta_{a'_i,b'_i}(z)}$$

becomes an elliptic function with respect to the lattice $L_\tau=\mathbb{Z}\tau+\mathbb{Z}$, i.e., it is meromorphic on $\mathbb{C}$ and satisfies

$$F(z)=F(z+1)=F(z+\tau).$$

Fact 1 (Jacobi's derivative formula)

$$\left.\frac{\partial}{\partial z}\vartheta_{1/2,1/2}(z,\tau)\right|_{z=0}=-\pi\vartheta_{0,0}(0,\tau)\vartheta_{0,1/2}(0,\tau)\vartheta_{1/2,0}(0,\tau).$$

Fact 2 (Transformation formulas)

$$\vartheta_{a,b}(z,\tau+1)=\mathbf{e}\left(\frac{a(1-a)}{2}\right)\vartheta_{a,a+b-1/2}(z,\tau),$$

$$\vartheta_{a,b}\left(\frac{z}{\tau},\frac{-1}{\tau}\right)=\mathbf{e}(ab)\sqrt{\frac{\tau}{i}}\mathbf{e}\left(\frac{z^2}{2\tau}\right)\vartheta_{b,-a}(z,\tau),$$

where $\sqrt{\tau/i}$ is positive when τ is purely imaginary.

Fact 3 (Addition formulas, Jacobi's identity)

$$\begin{aligned}
&\vartheta_{0,0}(z_1+z_2)\vartheta_{0,0}(z_1-z_2)\vartheta_{0,0}(0)^2\\
&\quad=\vartheta_{0,0}(z_1)^2\vartheta_{0,0}(z_2)^2+\vartheta_{1/2,1/2}(z_1)^2\vartheta_{1/2,1/2}(z_2)^2\\
&\quad=\vartheta_{0,1/2}(z_1)^2\vartheta_{0,1/2}(z_2)^2+\vartheta_{1/2,0}(z_1)^2\vartheta_{1/2,0}(z_2)^2,\\
&\vartheta_{0,1/2}(z_1+z_2)\vartheta_{0,1/2}(z_1-z_2)\vartheta_{0,1/2}(0)^2\\
&\quad=\vartheta_{0,0}(z_1)^2\vartheta_{0,0}(z_2)^2-\vartheta_{1/2,0}(z_1)^2\vartheta_{1/2,0}(z_2)^2\\
&\quad=\vartheta_{0,1/2}(z_1)^2\vartheta_{0,1/2}(z_2)^2-\vartheta_{1/2,1/2}(z_1)^2\vartheta_{1/2,1/2}(z_2)^2,\\
&\vartheta_{1/2,0}(z_1+z_2)\vartheta_{1/2,0}(z_1-z_2)\vartheta_{1/2,0}(0)^2\\
&\quad=\vartheta_{0,0}(z_1)^2\vartheta_{0,0}(z_2)^2-\vartheta_{0,1/2}(z_1)^2\vartheta_{0,1/2}(z_2)^2\\
&\quad=\vartheta_{1/2,0}(z_1)^2\vartheta_{1/2,0}(z_2)^2-\vartheta_{1/2,1/2}(z_1)^2\vartheta_{1/2,1/2}(z_2)^2,\\
&\vartheta_{1/2,1/2}(z_1+z_2)\vartheta_{1/2,1/2}(z_1-z_2)\vartheta_{0,0}(0)^2\\
&\quad=\vartheta_{1/2,1/2}(z_1)^2\vartheta_{0,0}(z_2)^2-\vartheta_{0,0}(z_1)^2\vartheta_{1/2,1/2}(z_2)^2\\
&\quad=\vartheta_{0,1/2}(z_1)^2\vartheta_{1/2,0}(z_2)^2-\vartheta_{1/2,0}(z_1)^2\vartheta_{0,1/2}(z_2)^2,\\
&\vartheta_{0,0}(0)^4=\vartheta_{0,1/2}(0)^4+\vartheta_{1/2,0}(0)^4.
\end{aligned}$$

3.2. Formulas for $\tau=i$

In this subsection, we obtain several formulas for $\vartheta_{a,b}(z,i)$ in the case of $\tau=i$.

Lemma 1 *We have*

$$\vartheta_{a,b}(iz,i)=\mathbf{e}(ab)\exp(\pi z^2)\vartheta_{-b,a}(z,i),$$

$$\vartheta_{0,0}(iz,i)=\exp(\pi z^2)\vartheta_{0,0}(z,i),\quad \vartheta_{0,1/2}(iz,i)=\exp(\pi z^2)\vartheta_{1/2,0}(z,i),$$

$$\vartheta_{1/2,0}(iz,i)=\exp(\pi z^2)\vartheta_{0,1/2}(z,i),\quad \vartheta_{1/2,1/2}(iz,i)=i\exp(\pi z^2)\vartheta_{1/2,1/2}(iz,i),$$

$$\vartheta_{0,1/2}(0,i)=\vartheta_{1/2,0}(0,i)=\frac{\vartheta_{0,0}(0,i)}{\sqrt[4]{2}}.$$

Proof. For the i-multiple formulas, we have only to substitute $\tau=i$ into the second formula for $\vartheta_{-a,-b}$ in Fact 2. We have $\vartheta_{0,1/2}(0)=\vartheta_{1/2,0}(0)$ by substituting $z=0$ into the identity between $\vartheta_{0,1/2}(iz)$ and $\vartheta_{1/2,0}(z)$. By Jacobi's identity, we have $\vartheta_{0,0}(0)^4 = 2\vartheta_{0,1/2}(0)^4$. Note that $\vartheta_{0,0}(0)$ and $\vartheta_{0,1/2}(0)$ take positive real values. □

Lemma 2 *We have*

$$\vartheta_{0,0}((1+i)z,i)=\frac{\vartheta_{0,0}(0,i)\vartheta_{0,1/2}(z,i)\vartheta_{1/2,0}(z,i)}{\exp(\pi i(1+i)z^2)\vartheta_{0,1/2}(0,i)\vartheta_{1/2,0}(0,i)},$$

$$\vartheta_{1/2,1/2}((1+i)z,i)=\mathbf{e}\left(\frac{1}{8}\right)\frac{\vartheta_{0,0}(0,i)\vartheta_{0,0}(z,i)\vartheta_{1/2,1/2}(z,i)}{\exp(\pi i(1+i)z^2)\vartheta_{0,1/2}(0,i)\vartheta_{1/2,0}(0,i)},$$

$$\vartheta_{0,1/2}((1+i)z,i)\vartheta_{1/2,0}((1+i)z,i)=\frac{\vartheta_{0,0}(z,i)^4-\vartheta_{0,1/2}(z,i)^2\vartheta_{1/2,0}(z,i)^2}{\exp(2\pi i(1+i)z^2)\vartheta_{0,1/2}(0,i)\vartheta_{1/2,0}(0,i)}.$$

Proof. We set

$$\eta(z)=\exp(\pi i(1+i)z^2)\vartheta_{0,0}((1+i)z,i).$$

Since $\vartheta_{0,0}(z)$ has simple zero at $z=(i+1)/2$, the function $\eta(z)$ has simple zero at $z=1/2, i/2$. By using the quasi periodicity of $\vartheta_{0,0}(z)$, we can show that

$$\eta(z+1)=-\eta(z),\quad \eta(z+i)=-\exp(-2\pi i(i+2z))\eta(z).$$

Thus the function

$$\frac{\eta(z)}{\vartheta_{0,1/2}(z)\vartheta_{1/2,0}(z)}$$

is a holomorphic elliptic function with respect to the lattice L_i; it is a constant. We can determine this constant by putting $z=0$. The second formula is obtained by the substitution $z+1/2$ into z for the first formula. We show the third formula. By Fact 3 for $z_1 = z$ and $z_2 = iz$, we have

$$\vartheta_{0,1/2}(z+iz)\vartheta_{0,1/2}(z-iz)\vartheta_{0,1/2}(0)^2 = \vartheta_{0,0}(z)^2\vartheta_{0,0}(iz)^2 - \vartheta_{1/2,0}(z)^2\vartheta_{1/2,0}(iz)^2.$$

This identity together with Lemma 1 leads the third formula. □

3.3. Formulas for $\tau = \zeta$

In this subsection, we obtain several formulas for $\vartheta_{a,b}(z,\zeta)$ in the case of $\tau = \zeta = (1+\sqrt{3}i)/2$.

Lemma 3 *We have*

$$\vartheta_{a,b}(\omega z,\zeta) = \mathbf{e}\left(\frac{a^2}{2} + ab - \frac{1}{24}\right)\mathbf{e}\left(\frac{z^2}{2\zeta}\right)\vartheta_{-a-b-1/2,a}(z,\zeta),$$

$$\vartheta_{a,b}(\omega^2 z,\zeta) = \mathbf{e}\left(ab + \frac{b^2+b}{2} + \frac{1}{24}\right)\mathbf{e}\left(\frac{z^2}{2\omega}\right)\vartheta_{b,-a-b-1/2}(z,\zeta),$$

$$\vartheta_{0,0}(\omega z,\zeta) = \mathbf{e}\left(\frac{-1}{24}\right)\mathbf{e}\left(\frac{z^2}{2\zeta}\right)\vartheta_{1/2,0}(z,\zeta),$$

$$\vartheta_{0,0}(\omega^2 z,\zeta) = \mathbf{e}\left(\frac{1}{24}\right)\mathbf{e}\left(\frac{z^2}{2\omega}\right)\vartheta_{0,1/2}(z,\zeta),$$

$$\vartheta_{0,1/2}(\omega z,\zeta) = \mathbf{e}\left(\frac{-1}{24}\right)\mathbf{e}\left(\frac{z^2}{2\zeta}\right)\vartheta_{0,0}(z,\zeta),$$

$$\vartheta_{0,1/2}(\omega^2 z,\zeta) = \mathbf{e}\left(\frac{-1}{12}\right)\mathbf{e}\left(\frac{z^2}{2\omega}\right)\vartheta_{1/2,0}(z,\zeta),$$

$$\vartheta_{1/2,0}(\omega z,\zeta) = \mathbf{e}\left(\frac{1}{12}\right)\mathbf{e}\left(\frac{z^2}{2\zeta}\right)\vartheta_{0,1/2}(z,\zeta),$$

$$\vartheta_{1/2,0}(\omega^2 z,\zeta) = \mathbf{e}\left(\frac{1}{24}\right)\mathbf{e}\left(\frac{z^2}{2\omega}\right)\vartheta_{0,0}(z,\zeta),$$

$$\vartheta_{1/2,1/2}(\omega z,\zeta) = \omega\mathbf{e}\left(\frac{z^2}{2\zeta}\right)\vartheta_{1/2,1/2}(z,\zeta),$$

$$\vartheta_{1/2,1/2}(\omega^2 z,\zeta) = \omega^2\mathbf{e}\left(\frac{z^2}{2\omega}\right)\vartheta_{1/2,1/2}(z,\zeta),$$

where $\omega = \zeta^2 = (-1+\sqrt{3}i)/2$.

Proof. Fact 2 yields that

$$\begin{aligned}\vartheta_{a,b}\left(\frac{z}{\zeta},\frac{-1}{\zeta}\right) &= \mathbf{e}(ab)\mathbf{e}\left(\frac{-1}{24}\right)\mathbf{e}\left(\frac{z^2}{2\zeta}\right)\vartheta_{b,-a}(z,\zeta),\\ &= \vartheta_{a,b}(-\omega z,\zeta-1) = \mathbf{e}\left(\frac{a(a-1)}{2}\right)\vartheta_{a,-a+b+1/2}(-\omega z,\zeta)\\ &= \mathbf{e}\left(\frac{a(a-1)}{2}\right)\vartheta_{-a,a-b-1/2}(\omega z,\zeta).\end{aligned}$$

By rewriting $(a',b') = (-a,a-b-1/2)$ i.e., $(a,b) = (-a',-a'-b'-1/2)$ for the identity

$$\mathbf{e}(ab)\mathbf{e}\left(\frac{-1}{24}\right)\mathbf{e}\left(\frac{z^2}{2\zeta}\right)\vartheta_{b,-a}(z,\zeta) = \mathbf{e}\left(\frac{a(a-1)}{2}\right)\vartheta_{-a,a-b-1/2}(\omega z,\zeta),$$

we have the first formula. To get the second formula, substitute $z = \omega^2 z$ into the first formula. These formulas yield the others. □

Lemma 4 *For $\tau = \zeta$, we have*

$$\vartheta_{0,0}((1+\zeta)z) = \frac{\mathbf{e}(1/8)\mathbf{e}((\omega^2+\omega/2)z^2)}{\vartheta_{0,0}(0)^2}\vartheta_{1/2,0}(z)\{\vartheta_{0,0}(z)^2 - i\vartheta_{0,1/2}(z)^2\},$$

$$\vartheta_{0,1/2}((1+\zeta)z) = \frac{\mathbf{e}(1/8)\mathbf{e}((\omega^2+\omega/2)z^2)}{\vartheta_{0,1/2}(0)^2}\vartheta_{0,0}(z)\{\vartheta_{0,1/2}(z)^2 - \vartheta_{1/2,0}(z)^2\},$$

$$\vartheta_{1/2,0}((1+\zeta)z) = \frac{\mathbf{e}((\omega^2+\omega/2)z^2)}{\vartheta_{1/2,0}(0)^2}\vartheta_{0,1/2}(z)\{\vartheta_{0,0}(z)^2 + i\vartheta_{1/2,0}(z)^2\},$$

$$\vartheta_{1/2,1/2}((1+\zeta)z) = \frac{\mathbf{e}((\omega^2+\omega/2)z^2)}{\vartheta_{0,0}(0)^2}\vartheta_{1/2,1/2}(z)\{\vartheta_{0,0}(z)^2 + i\vartheta_{0,1/2}(z)^2\}.$$

Proof. We apply addition formulas in Fact 3 to $z_1 = z$ and $z_2 = \zeta z$, and use Lemma 3. For example, we have

$$\begin{aligned}&\vartheta_{0,0}((1+\zeta)z)\vartheta_{0,0}((1-\zeta)z)\vartheta_{0,0}(0)^2\\ &\quad = \vartheta_{0,1/2}(z)^2\vartheta_{0,1/2}(\zeta z)^2 + \vartheta_{1/2,0}(z)^2\vartheta_{1/2,0}(\zeta z)^2,\end{aligned}$$

$$\vartheta_{0,0}((1-\zeta)z) = \vartheta_{0,0}(-\omega z) = \vartheta_{0,0}(\omega z) = \mathbf{e}\left(\frac{-1}{24}\right)\mathbf{e}\left(\frac{z^2}{2\zeta}\right)\vartheta_{1/2,0}(z),$$

$$\vartheta_{0,1/2}(\zeta z)^2 = \vartheta_{0,1/2}(-\omega^2 z)^2 = \vartheta_{0,1/2}(\omega^2 z)^2 = \mathbf{e}\left(\frac{-1}{6}\right)\mathbf{e}\left(\frac{z^2}{\omega}\right)\vartheta_{1/2,0}(z)^2,$$

$$\vartheta_{1/2,0}(\zeta z)^2 = \vartheta_{1/2,0}(-\omega^2 z)^2 = \vartheta_{1/2,0}(\omega^2 z)^2 = \mathbf{e}\left(\frac{1}{12}\right)\mathbf{e}\left(\frac{z^2}{\omega}\right)\vartheta_{0,0}(z)^2,$$

which yield the first formula. □

Lemma 5 *Some theta constants $\vartheta_{a,b}(0,\zeta)$ are related as follows*:

$$\vartheta_{0,1/2}(0,\zeta)=\mathbf{e}\left(\frac{-1}{24}\right)\vartheta_{0,0}(0,\zeta), \qquad \vartheta_{1/2,0}(0,\zeta)=\mathbf{e}\left(\frac{1}{24}\right)\vartheta_{0,0}(0,\zeta),$$

$$\vartheta_{5/6,1/3}(0,\zeta)=\mathbf{e}\left(\frac{-1}{8}\right)\vartheta_{1/3,1/3}(0,\zeta), \quad \vartheta_{1/3,5/6}(0,\zeta)=\mathbf{e}\left(\frac{-17}{24}\right)\vartheta_{1/3,1/3}(0,\zeta).$$

$$\vartheta_{1/3,1/3}(0,\zeta)=\mathbf{e}\left(\frac{1}{18}\right)\frac{1}{\sqrt[3]{2}}\vartheta_{0,0}(0,\zeta), \quad \vartheta_{1/6,1/6}(0,\zeta)=\mathbf{e}\left(\frac{1}{72}\right)\frac{\sqrt[4]{3}}{\sqrt[3]{2}}\vartheta_{0,0}(0,\zeta).$$

Proof. By substituting $z=0$ and $z=(\zeta+1)/3$ into formulas in Lemma 3, we have the formulas in the first and second lines in this lemma. We show the formulas in the third line. Substitute $z=(\zeta+1)/3$ and $z=(\zeta+1)/6$ into the first formula in Lemma 4. Then we have

$$\begin{aligned}\vartheta_{0,0}(\zeta) &= \frac{\mathbf{e}(1/8)\mathbf{e}((\omega^2+\omega/2)(\zeta+1)^2/9)}{\vartheta_{0,0}(0)^2} \\ &\quad \times \vartheta_{1/2,0}\left(\frac{\zeta+1}{3}\right)\left\{\vartheta_{0,0}\left(\frac{\zeta+1}{3}\right)^2 - i\vartheta_{0,1/2}\left(\frac{\zeta+1}{3}\right)^2\right\}, \\ \vartheta_{0,0}\left(\frac{\zeta}{2}\right) &= \frac{\mathbf{e}(1/8)\mathbf{e}((\omega^2+\omega/2)(\zeta+1)^2/36)}{\vartheta_{0,0}(0)^2} \\ &\quad \times \vartheta_{1/2,0}\left(\frac{\zeta+1}{6}\right)\left\{\vartheta_{0,0}\left(\frac{\zeta+1}{6}\right)^2 - i\vartheta_{0,1/2}\left(\frac{\zeta+1}{6}\right)^2\right\}.\end{aligned}$$

By using shown formulas in this lemma, we can transform these identities into

$$\vartheta_{0,0}(0,\zeta)^3 = \frac{2}{\zeta}\vartheta_{1/3,1/3}(0,\zeta)^3,$$
$$\vartheta_{0,0}(0,\zeta)^3 = \vartheta_{1/3,1/3}(0,\zeta)(\vartheta_{1/6,1/6}(0,\zeta)^2 - \zeta\vartheta_{1/3,1/3}(0,\zeta)^2).$$

Note that the last identity is equivalent to

$$\vartheta_{1/6,1/6}(0,\zeta)^2 = \frac{\vartheta_{0,0}(0,\zeta)^3 + \zeta\vartheta_{1/3,1/3}(0,\zeta)^3}{\vartheta_{1/3,1/3}(0,\zeta)} = \frac{\zeta+1}{2}\cdot\frac{\vartheta_{0,0}(0,\zeta)^3}{\vartheta_{1/3,1/3}(0,\zeta)}.$$

By numerical computations, we can see that the identity

$$\vartheta_{1/3,1/3}(0,\zeta) = \mathbf{e}\left(\frac{1}{18}\right)\frac{1}{\sqrt[3]{2}}\vartheta_{0,0}(0,\zeta)$$

holds. This identity yields that

$$\vartheta_{1/6,1/6}(0,\zeta)^2 = \mathbf{e}\left(\frac{1}{36}\right)\frac{\sqrt{3}}{\sqrt[3]{4}}\vartheta_{0,0}(0,\zeta)^2.$$

By numerical computations, we can select a square root of $\mathbf{e}(1/36)$ so that identity between $\vartheta_{1/6,1/6}(0,\zeta)$ and $\vartheta_{0,0}(0,\zeta)$ holds. □

4. The Schwarz map for $(\alpha,\beta,\gamma) = (1/4, 0, 1/2)$

We study the Schwarz map for $(\alpha,\beta,\gamma) = (1/4, 0, 1/2)$ and its inverse by using an elliptic curve with i-action and $\vartheta_{a,b}(z,i)$.

4.1. Abel-Jacobi map for C_i

Let C_i be an algebraic curve in $\mathbb{P}^2$ defined by

$$C_i : s_2^4 = s_0 s_1^2(s_1 - s_0).$$

By affine coordinates $(t,u) = (s_1/s_0, s_2/s_0)$, C_i is expressed by

$$u^4 = t^2(t-1).$$

Note that the point $(t,u) = (0,0)$ in C_i is a node. We use the same symbol C_i for a non-singular model of C_i. By a projection pr from the non-singular model C_i to the complex projective line $\mathbb{P}^1$ arising from

$$C_i \ni (t,u) \mapsto t \in \mathbb{C},$$

we regard C_i as a branched covering $\mathbb{P}^1$ with a covering transformation ρ_i arising from a map

$$\rho_i : C_i \ni (t,u) \mapsto (t, iu) \in C_i.$$

The branch points of pr are $t = 0, 1, \infty$. Each preimage of $pr^{-1}(1)$ and $pr^{-1}(\infty)$ consists of a point; $P_1 = pr^{-1}(1)$ and $P_\infty = pr^{-1}(\infty)$ are expressed as $(t,u) = (1,0)$ and $[s_0, s_1, s_2] = [0,1,0]$, respectively. On the other hand, the preimage $pr^{-1}(0)$ consists of two points, which are denoted by $P_{0,1}$ and $P_{0,2}$. The point $P_{0,1}$ corresponds to

$$\lim_{\substack{x\to 0\\ x\in(0,1)}} \big(x, \sqrt[4]{x^2(x-1)}\big), \quad \arg x^2(x-1) = \pi$$

for x in the open interval $(0,1)$, and $P_{0,2}$ is given by $\rho_i(P_{0,1})$. By the Hurwitz formula, C_i is an elliptic curve.

Let $I_{1\infty}$ be an oriented path in C_i given by

$$\big(x, \sqrt[4]{x^2(x-1)}\big) \in C_i, \quad x \in [1, \infty],$$

where $\sqrt[4]{x^2(x-1)}$ takes positive real values for $x \in [1,\infty)$ and the interval $[1,\infty]$ is naturally oriented. We define a cycle B by $I_{1\infty} - \rho_i \cdot I_{1\infty}$ and a cycle A by $\rho_i \cdot B$. Since

$$B \cdot A = 1,$$

A and B form a basis of $H_1(C_i, \mathbb{Z})$.

The space of holomorphic 1-forms on C_i is one dimensional and it is spanned by a form expressed by

$$\varphi = \frac{udt}{t(t-1)} = \frac{dt}{\sqrt[4]{t^2(t-1)^3}}.$$

The period integral $\int_B \varphi$ is evaluated as

$$(1-i)\int_1^\infty \frac{dt}{\sqrt[4]{t^2(t-1)^3}} = (1-i)B\Big(\frac{1}{4}, \frac{1}{4}\Big).$$

On the other hand, we have

$$\int_A \varphi = \int_{\rho_i(B)} \varphi = \int_B \rho_i^*(\varphi) = i \int_B \varphi.$$

We normalize φ to φ_1 as

$$\varphi_1 = \frac{1}{(1-i)B(1/4,1/4)}\varphi.$$

Then we have

$$\int_B \varphi_1 = 1, \quad \int_A \varphi_1 = i$$

and the Abel-Jacobi map

$$\jmath_i : C_i \ni P = \left(x, \sqrt[4]{x^2(x-1)}\right) \mapsto z = \int_{P_1}^{P} \varphi_1 \in E_i = \mathbb{C}/L_i,$$

where $L_i = \mathbb{Z}i + \mathbb{Z} \subset \mathbb{C}$. The map $\jmath_i$ is an isomorphism between C_i and E_i.

Proposition 2 *The Abel-Jacobi map* $\jmath_i$ *sends points* P_1, P_∞, $P_{0,1}$ *and* $P_{0,2}$ *to*

$$\jmath_i(P_1) = 0, \quad \jmath_i(P_\infty) = \frac{i+1}{2}, \quad \jmath_i(P_{0,1}) = \frac{i}{2}, \quad \jmath_i(P_{0,2}) = \frac{1}{2}$$

as elements of E_i.

Proof. It is clear that $\jmath_i(P_1) = 0$ and $\jmath_i(P_\infty) = (i+1)/2$. We have

$$\begin{aligned}\jmath_i(P_{0,1}) &= \frac{1}{(1-i)B(1/4,1/4)} \int_1^0 \exp(\pi i/4) \frac{\sqrt[4]{s^2(1-s)}ds}{s(s-1)} \\ &= \frac{i}{\sqrt{2}} \cdot \frac{\Gamma(1/2)^2\Gamma(1/4)}{\Gamma(1/4)^2\Gamma(3/4)} = \frac{i}{\sqrt{2}} \cdot \frac{\pi}{\pi/\sin(\pi/4)} = \frac{i}{2}.\end{aligned}$$

Since $P_{0,2} = \rho_i(P_{0,1})$, $\jmath_i(P_{0,2})$ is equal to $i\jmath_i(P_{0,1}) = -1/2 \equiv 1/2 \bmod L_i$. □

We consider the relation between the Abel-Jacobi map $\jmath_i$ and the Schwarz map

$$x \mapsto \frac{f_1(x)}{(1-i)f_2(x)} = \frac{2\sqrt{2}i}{B(1/4,1/4)}\sqrt[4]{1-x}F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},1-x\right) \tag{4.1}$$

for $\mathcal{F}(1/4,0,1/2)$. By Corollary 1, its monodromy group is generated by the three transformations

$$N_0 : z \mapsto -z+i, \quad N_1 : z \mapsto iz, \quad (N_0N_1)^{-1} : z \mapsto iz+1,$$

and this group is isomorphic to the semi-direct product $\langle i\rangle \ltimes \mathbb{Z}[i]$. Note that the information of a branch of $\sqrt[4]{x^2(x-1)}$ is lost in the Schwarz map. Thus we can regard the Schwarz map as the Abel-Jacobi map $\jmath_i$ modulo the actions of ρ_i and i; that is

$$C_i/\langle\rho_i\rangle \ni x \mapsto \int_1^x \varphi_1 \in E_i/\langle i\rangle,$$

where $\langle\rho_i\rangle$ and $\langle i\rangle$ are the groups generated by ρ_i and i, respectively.

4.2. The inverse of $\jmath_i$

In this subsection, we express the inverse of the Abel-Jacobi map $\jmath_i$ in terms of $\vartheta_{a,b}(z,\tau)$. We fix the variable τ to i and denote $\vartheta_{a,b}(z,i)$ by $\vartheta_{a,b}(z)$ in short. Since the pull-backs ${\jmath_i^{-1}}^*(t)$ and ${\jmath_i^{-1}}^*(u)$ are elliptic functions with respect to the lattice L_i, they can be expressed as

$${\jmath_i^{-1}}^*(t) = \theta_t(z), \quad {\jmath_i^{-1}}^*(u) = \theta_u(z)$$

in terms of $\vartheta_{a,b}(z)$. It turns out that the map

$$E_i \ni z \mapsto (\theta_t(z), \theta_u(z)) \in C_i$$

is the inverse of $\jmath_i$.

Theorem 1 *The inverse of $\jmath_i : C_i \ni (t,u) \mapsto z \in E_i$ is given by*

$$t = 2\frac{\vartheta_{0,1/2}(z,i)^2\vartheta_{1/2,0}(z,i)^2}{\vartheta_{0,0}(z,i)^4} = 1 - \frac{\vartheta_{1/2,1/2}(z,i)^4}{\vartheta_{0,0}(z,i)^4},$$

$$u = -(1-i)\frac{\vartheta_{0,1/2}(z,i)\vartheta_{1/2,0}(z,i)\vartheta_{1/2,1/2}(z,i)}{\vartheta_{0,0}(z,i)^3}.$$

The holomorphic 1-form $\varphi = udt/t(t-1)$ on C_i corresponds to

$$2(1-i)\pi\vartheta_{0,0}(0,i)^2 dz = (1-i)B\left(\frac{1}{4},\frac{1}{4}\right)dz$$

by the Abel-Jacobi map $\jmath_i$.

Proof. We regard the coordinate t of C_i as a meromorphic function on C_i. Its divisor is

$$2P_{0,1} + 2P_{0,2} - 4P_\infty.$$

We construct an elliptic function for L_i with zero of order 2 at $z = i/2, 1/2$ and pole of order 4 at $z = (i+1)/2$. Since

$$2\cdot\left(0,\frac{1}{2}\right) + 2\cdot\left(\frac{1}{2},0\right) \equiv 4\cdot\left(\frac{1}{2},\frac{1}{2}\right) \bmod \mathbb{Z}^2,$$

the function

$$\frac{\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2}{\vartheta_{0,0}(z)^4}$$

becomes an elliptic function for L_i. Moreover, it has zero of order 2 at $z = i/2, 1/2$, and pole of order 4 at $z = (i+1)/2$, since $\vartheta_{a,b}(z) = 0$ if and only if $z \equiv (-a+1/2)i + (-b+1/2) \mod \mathbb{Z}^2$. Thus the pull-back $F(P)$ of this function under the map $\jmath_i$ is a constant multiple of t by Proposition 2. Let us determine this constant. Lemma 1 yields that

$$\frac{\vartheta_{0,1/2}(0)^2\vartheta_{1/2,0}(0)^2}{\vartheta_{0,0}(0)^4} = \frac{\vartheta_{0,1/2}(0)^4}{\vartheta_{0,0}(0)^4} = \frac{1}{2}.$$

Thus $2F(P)$ is equal to t.

Similarly we regard $t-1$ as a meromorphic function on C_i whose divisor is

$$4P_1 - 4P_\infty.$$

The function

$$\frac{\vartheta_{1/2,1/2}(z)^4}{\vartheta_{0,0}(z)^4}$$

becomes an elliptic function for L_i with zero of order 4 at $z=0$ and pole of order 4 at $z=(i+1)/2$. The pull-back of this function under the map $\jmath_i$ is a constant multiple of $t-1$. By substituting $P_{0,1}$ into this pull-back, we can determine the constant. We have

$$t-1=-\frac{\vartheta_{1/2,1/2}(z)^4}{\vartheta_{0,0}(z)^4}.$$

By regarding the coordinate u of C_i as a meromorphic function on C_i, we see that its divisor is

$$P_{0,1}+P_{0,2}+P_1-3P_\infty.$$

Thus it is the pull-back of

$$c\cdot\frac{\vartheta_{0,1/2}(z)\vartheta_{1/2,0}(z)\vartheta_{1/2,1/2}(z)}{\vartheta_{0,0}(z)^3}$$

under $\jmath_i$, where c is a constant. Let us determine c. By $u^4=t^2(t-1)$, we have

$$c^4\cdot\frac{\vartheta_{0,1/2}(z)^4\vartheta_{1/2,0}(z)^4\vartheta_{1/2,1/2}(z)^4}{\vartheta_{0,0}(z)^{12}}=\frac{4\vartheta_{0,1/2}(z)^4\vartheta_{1/2,0}(z)^4}{\vartheta_{0,0}(z)^8}\cdot\frac{-\vartheta_{1/2,1/2}(z)^4}{\vartheta_{0,0}(z)^4},$$

which yields that $c^4=-4$, i.e., $c=i^k\cdot(1+i)$ for some $k\in\{0,1,2,3\}$.

By the expressions t, $t-1$ and u in terms of $\vartheta_{a,b}(z)$, it turns out that the holomorphic 1-from $\varphi=udt/t(t-1)$ corresponds to

$$i^k(1+i)\cdot\frac{\vartheta_{0,1/2}(z)\vartheta_{1/2,0}(z)\vartheta_{1/2,1/2}(z)}{\vartheta_{0,0}(z)^3}\cdot\frac{\vartheta_{0,0}(z)^4}{2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2}\cdot\frac{-\vartheta_{0,0}(z)^4}{\vartheta_{1/2,1/2}(z)^4}$$

$$\cdot\frac{4\{\vartheta_{0,0}(z)^3\vartheta_{0,0}(z)'\vartheta_{1/2,1/2}(z)^4-\vartheta_{1/2,1/2}(z)^3\vartheta_{1/2,1/2}(z)'\vartheta_{0,0}(z)^4\}}{\vartheta_{0,0}(z)^8}dz$$

$$=-2i^k(1+i)\cdot\frac{\{\vartheta_{0,0}(z)'\vartheta_{1/2,1/2}(z)-\vartheta_{1/2,1/2}(z)'\vartheta_{0,0}(z)\}}{\vartheta_{0,1/2}(z)\vartheta_{1/2,0}(z)}dz,$$

which should be a constant multiple of dz. By putting $z=0$ and using Fact 1, we have

$$\varphi=-2i^k(1+i)\pi\vartheta_{0,0}(0)^2\jmath_i^*(dz).$$

Since $\vartheta_{0,0}(0)^2$ and

$$B\left(\frac{1}{4}, \frac{1}{4}\right) = \int_1^\infty \varphi = \int_{\jmath_i(P_1)}^{\jmath_i(P_\infty)} -2i^k(1+i)\pi\vartheta_{0,0}(0)^2 dz$$
$$= -2i^k(1+i)\pi\vartheta_{0,0}(0)^2 \cdot \frac{1+i}{2}$$

are positive real numbers, k is equal to 1. Hence we have the expressions of u and φ. □

Corollary 2 *Let $z \in E_i$ be the image of $(t, u) \in C_i$ under the Abel-Jacobi map $\jmath_i$. Then we have*

$$i\frac{u^2}{t} = \frac{\vartheta_{1/2,1/2}(z)^2}{\vartheta_{0,0}(z)^2}, \quad 1 + i\frac{u^2}{t} = \sqrt{2}\frac{\vartheta_{0,1/2}(z)^2}{\vartheta_{0,0}(z)^2}, \quad 1 - i\frac{u^2}{t} = \sqrt{2}\frac{\vartheta_{1/2,0}(z)^2}{\vartheta_{0,0}(z)^2}.$$

Moreover, $\vartheta_{a,b}(z)$'s satisfy relations

$$\sqrt{2}\vartheta_{0,1/2}(z)^2 = \vartheta_{0,0}(z)^2 + \vartheta_{1/2,1/2}(z)^2,$$
$$\sqrt{2}\vartheta_{1/2,0}(z)^2 = \vartheta_{0,0}(z)^2 - \vartheta_{1/2,1/2}(z)^2.$$

Proof. The first identity is a direct consequence of Theorem 1. The right hand side of the second identity is an elliptic function with respect to L_i. It has zero of order 2 at $\jmath_i(P_{0,1})$ and pole of order 2 at $\jmath_i(P_\infty)$. Since $P_{0,1}$ corresponds to the limit as $t \to 0$ given by the branch of u with $\arg(u) = \pi/4$ on the interval $(0, 1)$, $\lim_{t\to 0} i(u^2/t) = -1$. By comparing the zero and pole of both functions, $1 + i(u^2/t)$ is a constant multiple of the pull-back of $\vartheta_{0,1/2}(z)^2/\vartheta_{0,0}(z)^2$ under $\jmath_i$. We can determine this constant by the substitution $z = 0$. The third identity is obtained by the action of ρ_i on the second identity. By eliminating $i(u^2/t)$ from these identities, we have the relations among $\vartheta_{a,b}(z)$'s □

Corollary 3 *We have*

$$\vartheta_{0,0}(0, i) = \frac{\Gamma(1/4)}{\sqrt[4]{4\pi^3}} = \frac{\sqrt[4]{\pi}}{\Gamma(3/4)},$$

$$\vartheta_{0,1/2}(0, i) = \vartheta_{1/2,0}(0, i) = \frac{\Gamma(1/4)}{\sqrt[4]{(2\pi)^3}} = \frac{\sqrt[4]{\pi}}{\sqrt[4]{2}\Gamma(3/4)}.$$

Proof. By Theorem 1, we have

$$2\pi\vartheta_{0,0}(0)^2 = B\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{\Gamma(1/4)^2}{\sqrt{\pi}}.$$

Note that $\vartheta_{0,0}(0)$ and $\Gamma(1/4)$ are positive. To show the rest, use the inversion formula for the Γ-function and Lemma 1. □

Corollary 4 *The inverse of the Schwarz map* (4.1) *for* $\mathcal{F}(1/4, 0, 1/2)$ *is given by*

$$x = 2\frac{\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2}{\vartheta_{0,0}(z)^4} = 1 - \frac{\vartheta_{1/2,1/2}(z)^4}{\vartheta_{0,0}(z)^4}.$$

Proof. It is clear by Theorem 1. We can check this map is invariant under the action of $\langle i \rangle$ by Lemma 1. □

Corollary 5 *For any point* z *around* 0, *we have*

$$-\frac{2\sqrt{2\pi}}{\Gamma(1/4)^2} \cdot \frac{\vartheta_{1/2,1/2}(z,i)}{\vartheta_{0,0}(z,i)} \cdot F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; \frac{\vartheta_{1/2,1/2}(z,i)^4}{\vartheta_{0,0}(z,i)^4}\right) = z.$$

Proof. By Corollary 4, we have

$$\frac{2\sqrt{2\pi}}{\Gamma(1/4)^2} \cdot \frac{\vartheta_{1/2,1/2}(z,i)}{\vartheta_{0,0}(z,i)} \cdot F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; \frac{\vartheta_{1/2,1/2}(z,i)^4}{\vartheta_{0,0}(z,i)^4}\right) \equiv z$$

modulo the monodromy group of $\mathcal{F}(1/4, 0, 1/2)$. Since the both sides of the above become 0 for $z = 0$, their difference is represented as the group $\langle i \rangle$. Consider the limit of the both sides as $z \to i/2$ along the imaginary axis. Use

$$\frac{\vartheta_{1/2,1/2}(i/2,i)}{\vartheta_{0,0}(i/2,i)} = \mathbf{e}\left(\frac{1}{2} \cdot \frac{-1}{2}\right) \cdot \frac{\vartheta_{0,1/2}(0,i)}{\vartheta_{1/2,0}(0,i)} = -i,$$

and the Gauss-Kummer formula

$$F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}$$

for $\mathrm{Re}(\gamma - \alpha - \beta) > 0$. □

4.3. $(1+i)$-multiplication

Theorem 2 *Let $z \in E_i$ be the image of $(t,u) \in C_i$ under the Abel-Jacobi map $\jmath_i$. Then we have*

$$\jmath_i^{-1}((1+i)z) = \left(\left(\frac{t-2}{t}\right)^2, (1+i)\frac{u(2-t)}{t^2}\right). \tag{4.2}$$

Proof. We set

$$(t',u') = \jmath_i^{-1}((1+i)z).$$

By Theorem 1, we have

$$t' = 2\frac{\vartheta_{0,1/2}((1+i)z)^2\vartheta_{1/2,0}((1+i)z)^2}{\vartheta_{0,0}((1+i)z)^4},$$

$$u' = -(1-i)\frac{\vartheta_{0,1/2}((1+i)z)\vartheta_{1/2,0}((1+i)z)\vartheta_{1/2,1/2}((1+i)z)}{\vartheta_{0,0}((1+i)z)^3}.$$

We transform them as

$$t' = 2\frac{\vartheta_{0,1/2}^4(0)\vartheta_{1/2,0}^4(0)}{\vartheta_{0,0}^4(0)\vartheta_{0,1/2}(z)^4\vartheta_{1/2,0}(z)^4} \cdot \frac{(\vartheta_{0,0}(z)^4 - \vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2)^2}{\vartheta_{0,1/2}(0)^2\vartheta_{1/2,0}(0)^2}$$

$$= 2\frac{\vartheta_{0,1/2}(0)^2\vartheta_{1/2,0}(0)^2}{\vartheta_{0,0}(0)^4} \cdot \frac{(\vartheta_{0,0}(z)^4 - \vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2)^2}{\vartheta_{0,1/2}(z)^4\vartheta_{1/2,0}(z)^4} = \left(\frac{2}{t}-1\right)^2,$$

$$u' = -(1-i)\cdot\frac{\mathbf{e}(1/8)\vartheta_{0,0}(0)\vartheta_{0,0}(z)\vartheta_{1/2,1/2}(z)}{\vartheta_{0,1/2}(0)\vartheta_{1/2,0}(0)}$$

$$\cdot\frac{\vartheta_{0,0}(z)^4 - \vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2}{\vartheta_{0,1/2}(0)\vartheta_{1/2,0}(0)}$$

$$\cdot\frac{\vartheta_{0,1/2}(0)^3\vartheta_{1/2,0}(0)^3}{\vartheta_{0,0}(0)^3}\frac{1}{\vartheta_{0,1/2}(z)^3\vartheta_{1/2,0}(z)^3}$$

$$= -\sqrt{2}\cdot\frac{\vartheta_{0,1/2}(0)\vartheta_{1/2,0}(0)}{\vartheta_{0,0}(0)^2}$$

$$\cdot\frac{\vartheta_{0,0}(z)\vartheta_{1/2,1/2}(z)(\vartheta_{0,0}(z)^4 - \vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2)}{\vartheta_{0,1/2}(z)^3\vartheta_{1/2,0}(z)^3}$$

$$
\begin{aligned}
&= -\frac{\vartheta_{0,0}(z)^8\vartheta_{1/2,1/2}(z)\vartheta_{0,1/2}(z)\vartheta_{1/2,0}(z)}{\vartheta_{0,1/2}(z)^4\vartheta_{1/2,0}(z)^4\vartheta_{0,0}(z)^3} \\
&\quad + \frac{\vartheta_{0,0}(z)^4\vartheta_{1/2,1/2}(z)\vartheta_{0,1/2}(z)\vartheta_{1/2,0}(z)}{\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2\vartheta_{0,0}(z)^3} \\
&= \frac{4}{(1-i)}\frac{u}{t^2} - \frac{2}{(1-i)}\frac{u}{t} = (1+i)\frac{u(2-t)}{t^2},
\end{aligned}
$$

by Lemma 2 and Theorem 1. □

5. The Schwarz map for $(\alpha, \beta, \gamma) = (1/3, 0, 1/2)$

In this section, we study the Schwarz map for $(\alpha, \beta, \gamma) = (1/3, 0, 1/2)$ and its inverse by using an elliptic curve with ζ-action and $\vartheta_{a,b}(z, \zeta)$, where $\zeta = (1 + \sqrt{3}i)/2$.

5.1. The Abel-Jacobi map for C_ζ

Let C_ζ be an algebraic curve in $\mathbb{P}^2$ defined by

$$C_\zeta : s_2^6 = s_0^2 s_1^3 (s_1 - s_0).$$

By affine coordinates $(t, u) = (s_1/s_0, s_2/s_0)$, C_ζ is expressed as

$$u^6 = t^3(t-1).$$

Note that $(t, u) = (0, 0)$ is a triple node and $[s_0, s_1, s_2] = [0, 1, 0]$ is a node. We use the same symbol C_ζ for a non-singular model of C_ζ. We regard C_ζ as a cyclic 6-fold covering of the t-space with covering transformation

$$\rho_\zeta : (t, u) \mapsto (t, \zeta u), \quad \zeta = \frac{1 + \sqrt{-3}}{2}.$$

The branching information of this covering is as in Table 2. Here we set some points in the non-singular model C_ζ as follows:

$$P_{0,1} = \lim_{\substack{t \to 0 \\ t \in (0,1)}} (t, t^{1/2}(t-1)^{1/6}), \quad P_{0,2} = \rho_\zeta(P_{0,1}), \quad P_{0,3} = \rho_\zeta^2(P_{0,1}),$$

$$P_{\infty,1} = \lim_{\substack{t \to \infty \\ t \in (1,\infty)}} (t, t^{1/2}(t-1)^{1/6}), \quad P_{\infty,2} = \rho_\zeta(P_{\infty,1}),$$

Table 2. Branching information.

ramification point	$P_{0,1}$	$P_{0,2}$	$P_{0,3}$	$P_1=(1,0)$	$P_{\infty,1}$	$P_{\infty,2}$
projected point	0	0	0	1	∞	∞
ramification index	2	2	2	6	3	3

where $\arg(t)=\arg(t-1)=0$ on the open interval $I_\infty=(1,\infty)$ and $\arg(t)=0$, $\arg(t-1)=\pi$ on the open interval $I_0=(0,1)$. By the Hurwitz formula, C_ζ is an elliptic curve.

We can regard t and u as meromorphic functions on C_ζ. We give some meromorphic functions on C_ζ and their zero and pole divisors as in Table 3. Pay your attention to the last three meromorphic functions for the setting of branch of u. Note that

$$\left(1+\frac{t}{u^2}\right)\left(1+\frac{\zeta^2 t}{u^2}\right)\left(1+\frac{\zeta^4 t}{u^2}\right)=1+\frac{1}{t-1}.$$

The preimage of I_∞ under the natural projection consists of six copies $\rho_\zeta^i\cdot I_\infty$ $(i=0,1,\dots,5)$. Since the terminal points of $\rho_\zeta^2\cdot I_\infty$ coincide with that of I_∞, the formal difference $B=\rho_\zeta^0\cdot I_\infty-\rho_\zeta^2\cdot I_\infty=(1-\rho_\zeta^2)\cdot I_\infty$ is a cycle of C_ζ. Let A be the cycle $\rho_\zeta\cdot B$. Then the intersection number $B\cdot A$ of the cycles B and A is 1. Thus the cycles A and B form a basis of the first homology group $H_1(C_\zeta,\mathbb{Z})$ of C_ζ.

A non-zero holomorphic 1-form ψ on C_ζ is given by

$$\psi=\frac{t^2dt}{u^5}=\frac{udt}{t(t-1)}=\frac{t^{1/2}(t-1)^{1/6}dt}{t(t-1)}.$$

It is easy to see that

$$\rho_\zeta^*(\psi)=\zeta\psi.$$

Note that

$$\int_{I_\infty}\psi=\int_1^\infty t^{1/2-1}(t-1)^{1/6-1}dt=\int_0^1 s^{1/3-1}(1-s)^{1/6-1}ds=B\left(\frac{1}{3},\frac{1}{6}\right),$$

$$\int_A\psi=\zeta(1-\zeta^2)B\left(\frac{1}{3},\frac{1}{6}\right),\quad \int_B\psi=(1-\zeta^2)B\left(\frac{1}{3},\frac{1}{6}\right).$$

Table 3. Meromorphic functions on C_ζ.

functions	zero divisor	pole divisor
t	$2P_{0,1}+2P_{0,2}+2P_{0,3}$	$3P_{\infty,1}+3P_{\infty,2}$
$t-1$	$6P_1$	$3P_{\infty,1}+3P_{\infty,2}$
$1+\dfrac{1}{t-1}$	$2P_{0,1}+2P_{0,2}+2P_{0,3}$	$6P_1$
u	$P_{0,1}+P_{0,2}+P_{0,3}+P_1$	$2P_{\infty,1}+2P_{\infty,2}$
$\dfrac{u^2}{t}\left(=\sqrt[3]{t-1}\right)$	$2P_1$	$P_{\infty,1}+P_{\infty,2}$
$\dfrac{u^3}{t}\left(=\sqrt{t(t-1)}\right)$	$P_{0,1}+P_{0,2}+P_{0,3}+3P_1$	$3P_{\infty,1}+3P_{\infty,2}$
$\dfrac{u^3}{t(t-1)}\left(=\sqrt{\dfrac{t}{t-1}}\right)$	$P_{0,1}+P_{0,2}+P_{0,3}$	$3P_1$
$1+\dfrac{\zeta^4 t}{u^2}\left(=1+\dfrac{\zeta^4}{\sqrt[3]{t-1}}\right)$	$2P_{0,1}$	$2P_1$
$1+\dfrac{t}{u^2}\left(=1+\dfrac{1}{\sqrt[3]{t-1}}\right)$	$2P_{0,2}$	$2P_1$
$1+\dfrac{\zeta^2 t}{u^2}\left(=1+\dfrac{\zeta^2}{\sqrt[3]{t-1}}\right)$	$2P_{0,3}$	$2P_1$

We normalize ψ to ψ_1 as

$$\psi_1 = \frac{1}{(1-\zeta^2)B(1/3,1/6)}\psi,$$

then we have

$$\int_A \psi_1 = \zeta, \quad \int_B \psi_1 = 1.$$

The Abel-Jacobi map is defined by

$$\jmath_\zeta : C_\zeta \ni P \mapsto \int_{P_1}^{P} \psi_1 \in E_\zeta = \mathbb{C}/L_\zeta,$$

where $L_\zeta = \mathbb{Z}\zeta + \mathbb{Z} \subset \mathbb{C}$. The map $\jmath_\zeta$ is an isomorphism between C_ζ and E_ζ.

Proposition 3 *We have*

$$\jmath_\zeta(P_1) = 0, \quad \jmath_\zeta(P_{\infty,1}) = \frac{\zeta+1}{3}, \quad \jmath_\zeta(P_{\infty,2}) = \frac{2\zeta+2}{3},$$
$$\jmath_\zeta(P_{0,1}) = \frac{\zeta}{2}, \quad \jmath_\zeta(P_{0,2}) = \frac{\zeta+1}{2}, \quad \jmath_\zeta(P_{0,3}) = \frac{1}{2}$$

as elements of E_ζ.

Proof. It is obvious that $\jmath_\zeta(P_1) = 0$. It is easy to see that

$$\jmath_\zeta(P_{\infty,1}) = \int_{I_\infty} \psi_1 = \frac{1}{1-\zeta^2} = \frac{\zeta+1}{3},$$
$$\jmath_\zeta(P_{\infty,2}) = \int_{\rho_\zeta \cdot I_\infty} \psi_1 = \int_{I_\infty} \rho_\zeta^*(\psi_1) = \zeta \jmath_\zeta(P_{\infty,1}) = \frac{\zeta^2+\zeta}{3} \equiv \frac{2\zeta+2}{3} \bmod L_\zeta.$$

Note that

$$\jmath_\zeta(P_{0,1}) = \int_{I_0} \psi_1 = \frac{1}{(1-\zeta^2)B(1/3,1/6)} \int_1^0 t^{1/2}(t-1)^{1/6} \frac{dt}{t(t-1)},$$
$$\int_1^0 t^{1/2}(t-1)^{1/6} \frac{dt}{t(t-1)} = \mathbf{e}\left(\frac{1}{12}\right) \int_0^1 t^{1/2}(1-t)^{1/6} \frac{dt}{t(1-t)}$$
$$= \mathbf{e}\left(\frac{1}{12}\right) B\left(\frac{1}{2}, \frac{1}{6}\right).$$

Thus we have

$$\jmath_\zeta(P_{0,1}) = \frac{\mathbf{e}(1/12)}{1-\zeta^2} \cdot \frac{B(1/2,1/6)}{B(1/3,1/6)} = \frac{(\zeta+1)\mathbf{e}(1/12)}{3} \cdot \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(2/3)\Gamma(1/3)}$$
$$= \frac{\sqrt{3}\mathbf{e}(1/6)}{3} \cdot \frac{\sqrt{3}}{2} = \frac{\zeta}{2}.$$

The rests are obtained as

$$\jmath_\zeta(P_{0,2}) = \zeta \jmath_\zeta(P_{0,1}) \equiv \frac{\zeta+1}{2} \bmod L_\zeta, \quad \jmath_\zeta(P_{0,3}) = \zeta^2 \jmath_\zeta(P_{0,1}) \equiv \frac{1}{2} \bmod L_\zeta,$$

since $P_{0,2} = \rho_\zeta \cdot P_{0,1}$ and $P_{0,3} = \rho_\zeta^2 \cdot P_{0,1}$. □

We consider the relation between the Abel-Jacobi map $\jmath_\zeta$ and the Schwarz map

$$x \mapsto \frac{f_1(x)}{(1-\zeta^2)f_2(x)} = \frac{2\sqrt{3}\zeta}{B(1/3,1/6)}\sqrt[6]{1-x}F\left(\frac{1}{6},\frac{1}{2},\frac{7}{6};1-x\right) \tag{5.1}$$

for $\mathcal{F}(1/3,0,1/2)$. By Corollary 1, its monodromy group is generated by the three transformations

$$N_0 : z \mapsto -z+\zeta, \quad N_1 : z \mapsto \zeta z, \quad (N_0N_1)^{-1} : z \mapsto \zeta^2 z + 1,$$

and this group is isomorphic to the semi-direct product $\langle\zeta\rangle \ltimes \mathbb{Z}[\zeta]$. Note that the information of a branch of $u = \sqrt[6]{x^3(x-1)}$ is lost in the Schwarz map. Thus we can regard the Schwarz map as the Abel-Jacobi map $\jmath_\zeta$ modulo the actions of ρ_ζ and ζ; that is

$$C_\zeta/\langle\rho_\zeta\rangle \ni x \mapsto \int_1^x \psi_1 \in E_\zeta/\langle\zeta\rangle.$$

5.2. The inverse of $\jmath_\zeta$

We express the inverse of the Abel-Jacobi map $\jmath_\zeta$. We regard the coordinates t and u as meromorphic functions on C_ζ. The pull-backs $\jmath_\zeta^{-1^*}(t)$ and $\jmath_\zeta^{-1^*}(u)$ are elliptic functions with respect to the lattice L_ζ, they can be expressed as

$$\jmath_\zeta^{-1^*}(t) = \theta_t(z), \quad \jmath_\zeta^{-1^*}(u) = \theta_u(z)$$

in terms of theta functions with characteristics. It turns out that the map

$$E_\zeta \ni z \mapsto (\theta_t(z), \theta_u(z)) \in C_\zeta$$

is the inverse of $\jmath_\zeta$.

Lemma 6 *Let z be the image of $(t,u) \in C_\zeta$ under the Abel-Jacobi map. Then we have*

$$1+\frac{t}{u^2} = \sqrt{3}i\frac{\vartheta_{0,0}(z,\zeta)^2}{\vartheta_{1/2,1/2}(z,\zeta)^2}, \quad 1+\frac{\zeta^2 t}{u^2} = -\sqrt{3}\frac{\vartheta_{1/2,0}(z,\zeta)^2}{\vartheta_{1/2,1/2}(z,\zeta)^2},$$

$$1+\frac{\zeta^4 t}{u^2}=\sqrt{3}\frac{\vartheta_{0,1/2}(z,\zeta)^2}{\vartheta_{1/2,1/2}(z,\zeta)^2}.$$

Proof. By Table 3, we have

$$1+\frac{t}{u^2}=c\cdot\frac{\vartheta_{0,0}(z)^2}{\vartheta_{1/2,1/2}(z)^2},$$

where c is a constant. We substitute $P_{0,1}$ into the above, we have

$$1-\omega=c\cdot\frac{\vartheta_{0,0}(\zeta/2)^2}{\vartheta_{1/2,1/2}(\zeta/2)^2}=c\cdot\bigg(-\frac{\vartheta_{1/2,0}(0)^2}{\vartheta_{0,1/2}(0)^2}\bigg)=-c\cdot\mathbf{e}\bigg(\frac{1}{6}\bigg),$$

which yields $c=\sqrt{3}i$. The rests can be shown similarly. □

Lemma 7 *The functions* $\vartheta_{0,1/2}(z,\zeta)^2$ *and* $\vartheta_{1/2,0}(z,\zeta)^2$ *are expressed as linear combinations of* $\vartheta_{0,0}(z,\zeta)^2$ *and* $\vartheta_{1/2,1/2}(z,\zeta)^2$*:*

$$\vartheta_{0,1/2}(z,\zeta)^2=\mathbf{e}\bigg(\frac{-1}{12}\bigg)\big(\vartheta_{0,0}(z,\zeta)^2-\omega^2\vartheta_{1/2,1/2}(z,\zeta)^2\big),$$
$$\vartheta_{1/2,0}(z,\zeta)^2=\mathbf{e}\bigg(\frac{1}{12}\bigg)\big(\vartheta_{0,0}(z,\zeta)^2+\omega\vartheta_{1/2,1/2}(z,\zeta)^2\big).$$

Proof. By Lemma 6, we have

$$-\sqrt{3}\frac{\vartheta_{1/2,0}(z)^2}{\vartheta_{1/2,1/2}(z)^2}-1=\omega\frac{t}{u^2}=\omega\bigg(\sqrt{3}i\frac{\vartheta_{0,0}(z)^2}{\vartheta_{1/2,1/2}(z)^2}-1\bigg),$$
$$\sqrt{3}\frac{\vartheta_{0,1/2}(z)^2}{\vartheta_{1/2,1/2}(z)^2}-1=\omega^2\frac{t}{u^2}=\omega^2\bigg(\sqrt{3}i\frac{\vartheta_{0,0}(z)^2}{\vartheta_{1/2,1/2}(z)^2}-1\bigg),$$

which yield this lemma. □

Lemma 8 *Let* z *be the image of* $(t,u)\in C_\zeta$ *under the Abel-Jacobi map. Then we have*

$$\frac{u^3}{t(t-1)}=\mathbf{e}\bigg(\frac{-1}{8}\bigg)\sqrt[4]{27}\frac{\vartheta_{0,0}(z,\zeta)\vartheta_{0,1/2}(z,\zeta)\vartheta_{1/2,0}(z,\zeta)}{\vartheta_{1/2,1/2}(z,\zeta)^3}.$$

Proof. By Table 3, we have

$$\frac{u^3}{t(t-1)} = c' \frac{\vartheta_{0,0}(z)\vartheta_{0,1/2}(z)\vartheta_{1/2,0}(z)}{\vartheta_{1/2,1/2}(z)^3},$$

where c' is a constant. We consider the limit as $t \to \infty$ with $t \in (1, \infty)$, $u \in (0, \infty)$. The left hand side of the above converges to 1. On the other hand, the right hand side of the above converges to

$$\begin{aligned} &c' \frac{\vartheta_{0,0}((\zeta+1)/3)\vartheta_{0,1/2}((\zeta+1)/3)\vartheta_{1/2,0}((\zeta+1)/3)}{\vartheta_{1/2,1/2}((\zeta+1)/3)^3} \\ &= c'\mathbf{e}\left(\frac{1}{3}\cdot\left(\frac{1}{2}+\frac{1}{2}\right)\right) \frac{\vartheta_{1/3,1/3}(0)\vartheta_{1/3,5/6}(0)\vartheta_{5/6,1/3}(0)}{\vartheta_{5/6,5/6}(0)^3} \\ &= c'\mathbf{e}\left(\frac{1}{3}-\frac{1}{8}-\frac{17}{24}+\frac{3}{6}\right) \frac{\vartheta_{1/3,1/3}(0)^3}{\vartheta_{1/6,1/6}(0)^3} = c' \frac{\vartheta_{1/3,1/3}(0)^3}{\vartheta_{1/6,1/6}(0)^3} = c'\mathbf{e}\left(\frac{1}{8}\right)\frac{1}{\sqrt[4]{27}} \end{aligned}$$

by Lemma 5. Hence we have $c' = \mathbf{e}(-1/8)\sqrt[4]{27}$. □

Theorem 3 *The inverse of $\jmath_\zeta : C_\zeta \ni (t,u) \mapsto z \in E_\zeta$ is given by*

$$\begin{aligned} t &= \frac{-3\sqrt{3}i\vartheta_{0,0}(z,\zeta)^2\vartheta_{0,1/2}(z,\zeta)^2\vartheta_{1/2,0}(z,\zeta)^2}{\left(\sqrt{3}i\vartheta_{0,0}(z,\zeta)^2 - \vartheta_{1/2,1/2}(z,\zeta)^2\right)^3}, \\ u &= \mathbf{e}\left(\frac{-1}{8}\right)\sqrt[4]{27}\frac{\vartheta_{0,0}(z,\zeta)\vartheta_{0,1/2}(z,\zeta)\vartheta_{1/2,0}(z,\zeta)\vartheta_{1/2,1/2}(z,\zeta)}{\left(\sqrt{3}i\vartheta_{0,0}(z,\zeta)^2 - \vartheta_{1/2,1/2}(z,\zeta)^2\right)^2}. \end{aligned}$$

Proof. Note that

$$\left(1+\frac{t}{u^2}\right)\left(1+\frac{\zeta^2 t}{u^2}\right)\left(1+\frac{\zeta^4 t}{u^2}\right) = 1 + \frac{t^3}{u^6} = 1 + \frac{1}{t-1}.$$

By Lemma 6, we have

$$1 + \frac{1}{t-1} = -3\sqrt{3}i\frac{\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2}{\vartheta_{1/2,1/2}(z)^6},$$

which yields

$$t = \frac{3\sqrt{3}i\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2}{3\sqrt{3}i\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2 + \vartheta_{1/2,1/2}(z)^6}.$$

Rewrite $\vartheta_{0,1/2}(z)^2$ and $\vartheta_{1/2,0}(z)^2$ in the denominator of this expression by $\vartheta_{0,0}(z)^2$ and $\vartheta_{1/2,1/2}(z)^2$ by Lemma 7. Then it can be factorized as

$$\begin{aligned}&3\sqrt{3}i\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2+\vartheta_{1/2,1/2}(z)^6\\&=-\big(\sqrt{3}i\vartheta_{0,0}(z)^2-\vartheta_{1/2,1/2}(z)^2\big)^3.\end{aligned}$$

Hence we have the expression of t.

By Lemmas 6 and 8, the functions $1+t/u^2$ and $u^3/t(t-1)$ are expressed in terms $\vartheta_{a,b}(z,\zeta)$. We have

$$\begin{aligned}u&=\frac{u^3}{t(t-1)}\cdot\left(\left(1+\frac{t}{u^2}\right)-1\right)\cdot(t-1)\\&=\mathbf{e}\left(\frac{-1}{8}\right)\sqrt[4]{27}\frac{\vartheta_{0,0}(z)\vartheta_{0,1/2}(z)\vartheta_{1/2,0}(z)}{\vartheta_{1/2,1/2}(z)^3}\cdot\frac{\sqrt{3}i\vartheta_{0,0}(z)^2-\vartheta_{1/2,1/2}(z)^2}{\vartheta_{1/2,1/2}(z)^2}\\&\quad\cdot\frac{-\vartheta_{1/2,1/2}(z)^6}{3\sqrt{3}i\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2+\vartheta_{1/2,1/2}(z)^6}\\&=\mathbf{e}\left(\frac{-1}{8}\right)\sqrt[4]{27}\\&\quad\cdot\frac{\vartheta_{0,0}(z)\vartheta_{0,1/2}(z)\vartheta_{1/2,0}(z)\vartheta_{1/2,1/2}(z)(\sqrt{3}i\vartheta_{0,0}(z)^2-\vartheta_{1/2,1/2}(z)^2)}{-3\sqrt{3}i\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2-\vartheta_{1/2,1/2}(z)^6}.\end{aligned}$$

Note that the denominator of the last term is $(\sqrt{3}i\vartheta_{0,0}(z)^2-\vartheta_{1/2,1/2}(z)^2)^3$. Hence we have the expression of u. □

Corollary 6 *The pull back of the holomorphic* 1*-from* $\psi=udt/t(t-1)$ *under the map* $\jmath_\zeta^{-1}$ *is*

$$\mathbf{e}\left(\frac{-1}{8}\right)2\pi\sqrt[4]{27}\vartheta_{0,0}(0,\zeta)^2dz.$$

The theta constant $\vartheta_{0,0}(0,\zeta)$ *is evaluated as*

$$\vartheta_{0,0}(0,\zeta)=\mathbf{e}\left(\frac{1}{48}\right)\frac{\sqrt[8]{3}}{\sqrt[3]{4}\pi}\Gamma\left(\frac{1}{3}\right)^{3/2}.$$

The other theta constants $\vartheta_{a,b}(0,\zeta)$ *are*

$$\vartheta_{0,1/2}(0,\zeta)=\mathbf{e}\Big(\frac{-1}{48}\Big)\frac{\sqrt[8]{3}}{\sqrt[3]{4}\pi}\varGamma\Big(\frac{1}{3}\Big)^{3/2},$$

$$\vartheta_{1/2,0}(0,\zeta)=\mathbf{e}\Big(\frac{1}{16}\Big)\frac{\sqrt[8]{3}}{\sqrt[3]{4}\pi}\varGamma\Big(\frac{1}{3}\Big)^{3/2},$$

$$\vartheta_{1/3,1/3}(0,\zeta)=\mathbf{e}\Big(\frac{11}{144}\Big)\frac{\sqrt[8]{3}}{2\pi}\varGamma\Big(\frac{1}{3}\Big)^{3/2},$$

$$\vartheta_{1/6,1/6}(0,\zeta)=\mathbf{e}\Big(\frac{5}{144}\Big)\frac{\sqrt[8]{27}}{2\pi}\varGamma\Big(\frac{1}{3}\Big)^{3/2}.$$

$$\vartheta_{\frac{5}{6},1/3}(0,\zeta)=\mathbf{e}\Big(\frac{-7}{144}\Big)\frac{\sqrt[8]{3}}{2\pi}\varGamma\Big(\frac{1}{3}\Big)^{3/2},$$

$$\vartheta_{1/3,\frac{5}{6}}(0,\zeta)=\mathbf{e}\Big(\frac{53}{144}\Big)\frac{\sqrt[8]{3}}{2\pi}\varGamma\Big(\frac{1}{3}\Big)^{3/2}.$$

Proof. Recall that

$$\frac{t}{t-1}=-3\sqrt{3}i\frac{\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2}{\vartheta_{1/2,1/2}(z)^6}.$$

Thus we have

$$\begin{aligned}\frac{dt}{t^2}&=d\Big(1-\frac{1}{t}\Big)=d\Big(\frac{t-1}{t}\Big)\\&=\frac{dz}{-3\sqrt{3}i}\bigg[\frac{6\vartheta_{1/2,1/2}(z)^5\vartheta_{1/2,1/2}(z)'\cdot\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2}{\vartheta_{0,0}(z)^4\vartheta_{0,1/2}(z)^4\vartheta_{1/2,0}(z)^4}\\&\qquad-\frac{\vartheta_{1/2,1/2}(z)^6\cdot(\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2)'}{\vartheta_{0,0}(z)^4\vartheta_{0,1/2}(z)^4\vartheta_{1/2,0}(z)^4}\bigg],\end{aligned}$$

where $f(z)'=df(z)/dz$. Since $\psi=u\cdot t/(t-1)\cdot dt/t^2$, the pull-back of ψ under the map $\jmath_\zeta^{-1}$ is $\mathbf{e}(-1/8)\sqrt[4]{27}$ times

$$\frac{6\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2\vartheta_{1/2,1/2}(z)'-\vartheta_{1/2,1/2}(z)(\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2)'}{(\sqrt{3}i\vartheta_{0,0}(z)^2-\vartheta_{1/2,1/2}(z)^2)^2\vartheta_{0,0}(z)\vartheta_{0,1/2}(z)\vartheta_{1/2,0}(z)}dz.$$

It should be a constant times dz. We determine this constant by substituting $z=0$ into the above. By Fact 1 and Lemma 5, we have

$$\jmath_\zeta^{-1*}(\psi)=\mathbf{e}\left(\frac{-1}{8}\right)2\pi\sqrt[4]{27}\vartheta_{0,0}(0,\zeta)^2dz.$$

Note that

$$\begin{aligned}B\left(\frac{1}{3},\frac{1}{6}\right)&=\int_1^\infty\psi=\int_{\jmath_\zeta(P_1)}^{\jmath_\zeta(P_{\infty,1})}\jmath_\zeta^{-1*}(\psi)\\&=\mathbf{e}\left(\frac{-1}{8}\right)2\pi\sqrt[4]{27}\vartheta_{0,0}(0,\zeta)^2\cdot\left(\frac{\zeta+1}{3}-0\right)\end{aligned}$$

by Proposition 3. The well-known formula

$$\Gamma\left(\frac{1}{6}\right)=\frac{1}{\sqrt[3]{2}}\frac{\sqrt{3}}{\sqrt{\pi}}\Gamma\left(\frac{1}{3}\right)^2$$

yields that

$$B\left(\frac{1}{3},\frac{1}{6}\right)=\frac{\Gamma(1/3)\Gamma(1/6)}{\Gamma(1/2)}=\frac{\sqrt{3}}{\sqrt[3]{2}\pi}\Gamma\left(\frac{1}{3}\right)^3.$$

Hence we evaluate the theta constant as

$$\vartheta_{0,0}(0,\zeta)^2=\mathbf{e}\left(\frac{1}{24}\right)\frac{\sqrt[4]{3}}{\sqrt[3]{16\pi^2}}\Gamma\left(\frac{1}{3}\right)^3.$$

We can determine the sign of $\vartheta_{0,0}(0,\zeta)$ by a numerical computation. The rests can be obtained by Lemma 5. □

Corollary 7 *The inverse of the Schwarz map* (5.1) *for* $\mathcal{F}(1/3,0,1/2)$ *is given by*

$$x=\frac{-3\sqrt{3}i\vartheta_{0,0}(z,\zeta)^2\vartheta_{0,1/2}(z,\zeta)^2\vartheta_{1/2,0}(z,\zeta)^2}{(\sqrt{3}i\vartheta_{0,0}(z,\zeta)^2-\vartheta_{1/2,1/2}(z,\zeta)^2)^3}.$$

Proof. It is clear by Theorem 3. We can check this map is invariant under the action of $\langle\zeta\rangle$ by Lemma 3. □

Corollary 8 *For any point z around 0, we have*

$$\frac{\sqrt[3]{16}\pi\zeta^2}{\Gamma(1/3)^3}\cdot\frac{1}{\sqrt{1-\sqrt{3}i\dfrac{\vartheta_{0,0}(z,\zeta)^2}{\vartheta_{1/2,1/2}(z,\zeta)^2}}}$$
$$\cdot F\left(\frac{1}{6},\frac{1}{2},\frac{7}{6};\frac{\vartheta_{1/2,1/2}(z,\zeta)^6}{(\vartheta_{1/2,1/2}(z,\zeta)^2-\sqrt{3}i\vartheta_{0,0}(z,\zeta)^2)^3}\right)=z,$$

where the branch of the square root is selected as $\sqrt{\zeta^2}=\zeta$ for $z=\zeta/2$.

Proof. Let z be the image of the Schwarz map (5.1). We have seen in Proof of Theorem 3 that

$$\begin{aligned}\frac{1}{1-x}&=\frac{3\sqrt{3}i\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2+\vartheta_{1/2,1/2}(z)^6}{\vartheta_{1/2,1/2}(z)^6}\\&=\frac{(\vartheta_{1/2,1/2}(z)^2-\sqrt{3}i\vartheta_{0,0}(z)^2)^3}{\vartheta_{1/2,1/2}(z)^6}.\end{aligned}$$

Thus we have the desired identity modulo the monodromy group of $\mathcal{F}(1/3,0,1/2)$. Since the both sides of the above become 0 for $z=0$, their difference is represented as the group $\langle\zeta\rangle$. Consider the limit of the both sides as $z\to\zeta/2$ along the segment connecting 0 and $\zeta/2$. Since $1/(1-x)$ converges to 1 by this limit, it turns out that x converges to 0. Use

$$1-\sqrt{3}i\frac{\vartheta_{0,0}(\zeta/2,\zeta)^2}{\vartheta_{1/2,1/2}(\zeta/2,\zeta)^2}=1+\sqrt{3}i\frac{\vartheta_{1/2,0}(0,\zeta)^2}{\vartheta_{0,1/2}(0,\zeta)^2}=1+\sqrt{3}i\zeta=\zeta^2$$

and the Gauss-Kummer formula. □

5.3. $(1+\zeta)$-multiplication

Theorem 4 *Let $z\in E_\zeta$ be the image of $(t,u)\in C_\zeta$ under the Abel-Jacobi map $\jmath_\zeta$. Then we have*

$$\jmath_\zeta^{-1}((1+\zeta)z)=\left(\frac{t(9-8t)^2}{(4t-3)^3},\mathbf{e}\left(\frac{1}{12}\right)\sqrt{3}u\frac{9-8t}{(4t-3)^2}\right).\qquad(5.2)$$

Proof. We set $(t', u') = \jmath_{\zeta}^{-1}((1+\zeta)z)$. Then t' is given by the substitution z to $(z+1)z$ into the expression of t in Theorem 3. Rewrite $\vartheta_{a,b}((1+\zeta)z)$ in terms of $\vartheta_{a,b}(z)$ by Lemma 4. Its numerator $N(t')$ is

$$\begin{aligned} N(t') = {} & 3\sqrt{3}i\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\vartheta_{1/2,0}(z)^2 \\ & \times (\vartheta_{0,0}(z)^2 - i\vartheta_{0,1/2}(z)^2)^2(\vartheta_{0,0}(z)^2 + i\vartheta_{1/2,0}(z)^2)^2 \\ & \times (\vartheta_{0,1/2}(z)^2 - \vartheta_{1/2,0}(z)^2)^2, \end{aligned}$$

and its denominator $D(t')$ is

$$\begin{aligned} D(t') = \Big\{ & -\big(\sqrt{3}\vartheta_{1/2,0}(z)^2 + \vartheta_{1/2,1/2}(z)^2\big)(\vartheta_{0,0}(z)^4 - \vartheta_{0,1/2}(z)^4) \\ & + 2i\big(\sqrt{3}\vartheta_{1/2,0}(z)^2 - \vartheta_{1/2,1/2}(z)^2\big)\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\Big\}^3. \end{aligned}$$

In this computation, the theta constants $\vartheta_{0,0}(0)$, $\vartheta_{0,1/2}(0)$, $\vartheta_{1/2,0}(0)$ are canceled by Lemma 5. Divide them by $\vartheta_{1/2,1/2}(z)^{18}$ and rewrite

$$\frac{\vartheta_{0,0}(z)^2}{\vartheta_{1/2,1/2}(z)^2} = \frac{1+t/u^2}{\sqrt{3}i}, \quad \frac{\vartheta_{1/2,0}(z)^2}{\vartheta_{1/2,1/2}(z)^2} = \frac{1+\omega t/u^2}{-\sqrt{3}}, \\ \frac{\vartheta_{0,1/2}(z)^2}{\vartheta_{1/2,1/2}(z)^2} = \frac{1+\omega^2 t/u^2}{\sqrt{3}}.$$

Then we have

$$t' = \left(\frac{-(t^3+u^6)(t^3-8u^6)^2}{27u^{18}}\right) \Big/ \left(\frac{-(t^3+4u^6)^3}{27u^{18}}\right) = \frac{t(9-8t)^2}{(4t-3)^3},$$

where we use the relation $u^6 = t^3(t-1)$.

By the same way, we can express u' in terms of $\vartheta_{a,b}(z)$'s, whose numerator $N(u')$ and denominator $D(u')$ are

$$\begin{aligned} N(u') = {} & \mathbf{e}\left(\frac{1}{8}\right)\sqrt[4]{27}\vartheta_{0,0}(z)\vartheta_{0,1/2}(z)\vartheta_{1/2,0}(z)\vartheta_{1/2,1/2}(z) \\ & \cdot \big(\vartheta_{0,1/2}(z)^2 - \vartheta_{1/2,0}(z)^2\big)\big(\vartheta_{0,0}(z)^4 + \vartheta_{0,1/2}(z)^4\big) \\ & \cdot \big(\vartheta_{0,0}(z)^2 + i\vartheta_{1/2,0}(z)^2\big), \end{aligned}$$

$$D(u') = \big\{\big(\sqrt{3}\vartheta_{1/2,0}(z)^2 + \vartheta_{1/2,1/2}(z)^2\big)\big(\vartheta_{0,0}(z)^4 - \vartheta_{0,1/2}(z)^4\big) \\ - 2i\big(\sqrt{3}\vartheta_{1/2,0}(z)^2 - \vartheta_{1/2,1/2}(z)^2\big)\vartheta_{0,0}(z)^2\vartheta_{0,1/2}(z)^2\big\}^2.$$

Divide them by $\vartheta_{1/2,1/2}(z)^8(\sqrt{3}i\vartheta_{0,0}(z)^2 - \vartheta_{1/2,1/2}(z)^2)^2$. We factor out u from the numerator as

$$\frac{N(u')}{\vartheta_{1/2,1/2}(z)^8(\sqrt{3}i\vartheta_{0,0}(z)^2 - \vartheta_{1/2,1/2}(z)^2)^2} \\ = iu\frac{\big(\vartheta_{0,1/2}(z)^2 - \vartheta_{1/2,0}(z)^2\big)\big(\vartheta_{0,0}(z)^4 + \vartheta_{0,1/2}(z)^4\big)\big(\vartheta_{0,0}(z)^2 + i\vartheta_{1/2,0}(z)^2\big)}{\vartheta_{1/2,1/2}(z)^8}.$$

Since the rest terms are expressed in terms of $\vartheta_{a,b}(z)^2$, we can compute them quite similarly to the case of t'. Hence we have

$$u' = iu\bigg(\frac{(3i-\sqrt{3})(-t^3+8u^6)t}{18u^8}\bigg)\bigg/\bigg(\frac{(t^3+4u^6)^2}{9t^2u^8}\bigg) = \frac{3+\sqrt{3}i}{2}\cdot u\cdot\frac{(9-8t)}{(4t-3)^2}.$$

It is easy to see that (t', u') satisfies $u'^6 = t'^3(t'-1)$. □

6. Limits of mean iterations

6.1. Limit formula by $F(1/4, 1/2, 5/4; x)$

Theorem 2 is interpreted as follows.

Theorem 5 *Let $P_x = (x, \sqrt[4]{x^2(x-1)})$ be a point of the curve C. We set*

$$P_{x'} = \left(\frac{(2-x)^2}{x^2}, \frac{(1+i)(2-x)\sqrt[4]{x^2(x-1)}}{x^2}\right) \in C.$$

Then we have

$$\int_{P_1}^{P_{x'}} \varphi \equiv (1+i)\int_{P_1}^{P_x} \varphi \mod \langle i\rangle \ltimes \mathbb{Z}[i].$$

Corollary 9 *The following identity holds around $x = 1$:*

$$\frac{1}{\sqrt{x}}F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, 1 - \frac{(2-x)^2}{x^2}\right) = F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, 1-x\right).$$

Proof. Theorem 5 implies that

$$\int_1^{(2-x)^2/x^2} \frac{\sqrt[4]{t^2(t-1)}dt}{t(t-1)} \equiv (1+i)\int_1^x \frac{\sqrt[4]{t^2(t-1)}dt}{t(t-1)} \quad \bmod \langle i \rangle \ltimes \mathbb{Z}[i].$$

Note that

$$\int_1^x \frac{\sqrt[4]{t^2(t-1)}dt}{t(t-1)} = 2\sqrt{2}(1+i)\sqrt[4]{1-x}F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},1-x\right),$$

$$\sqrt[4]{1-\frac{(2-x)^2}{x^2}} = \sqrt[4]{\frac{4x-4}{x^2}} = (1+i)\frac{\sqrt[4]{1-x}}{\sqrt{x}},$$

for $0 < x < 1$ and $\arg(1-(2-x)^2/x^2) = \pi$. We can cancel the factor $\sqrt[4]{1-x}$ and determine the action of $\langle i \rangle \ltimes \mathbb{Z}[i]$ by the substitution $x = 1$. Thus we have the desired identity. □

Let $a = a_1$ and $b = b_1$ be positive real numbers. We define a pair $\{a_n, b_n\}_{n\in\mathbb{N}}$ of sequences by the recursive relations

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{\frac{a_n(a_n+b_n)}{2}}. \tag{6.1}$$

Corollary 10 (A formula in Theorem 2 in [HKM]) *We have*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{a}{F(1/4,1/2,5/4;1-b^2/a^2)^2}.$$

Proof. We can show that the sequences $\{a_n\}$ and $\{b_n\}$ converge and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ by Lemma 1 in [HKM]. Substitute $x = 2a/(a+b)$ into the identity between hypergeometric series in Corollary 9. Since

$$\frac{2-x}{x} = \frac{b}{a},$$

we have

$$\frac{\sqrt{a+b}}{\sqrt{2a}}F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4};1-\frac{b^2}{a^2}\right) = F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4};1-\frac{2a}{a+b}\right),$$

$$\begin{aligned}
&\frac{a}{F(1/4,1/2,5/4;1-b^2/a^2)^2}\\
&\quad=\frac{(a+b)/2}{F\big(1/4,1/2,5/4;1-\big(\sqrt{a(a+b)/2}/((a+b)/2)\big)^2\big)^2}\\
&\quad=\frac{a_2}{F(1/4,1/2,5/4;1-b_2^2/a_2^2)^2}=\cdots=\frac{a_n}{F(1/4,1/2,5/4;1-b_n^2/a_n^2)^2}.
\end{aligned}$$

The last term is equal to $\lim_{n\to\infty} a_n$ since $\lim_{n\to\infty}(b_n^2/a_n^2)=1$ and $F(1/4,1/2,5/4;0)=1$. □

Hence we see that the $(1+i)$-multiple formula (4.2) in Theorem 2 implies this limit formula for the sequences defined by the mean iteration (6.1).

6.2. Limit formula by $F(1/6,1/2,7/6;x)$

Theorem 4 is interpreted as follows.

Theorem 6 *Let $P_x=(x,\sqrt[6]{x^3(x-1)})$ be a point of the curve C_ζ. We set*

$$P_{x'}=\left(\frac{x(9-8x)^2}{(4x-3)^3},\mathbf{e}\left(\frac{1}{12}\right)\sqrt{3}\sqrt[6]{x^3(x-1)}\frac{9-8x}{(4x-3)^2}\right)\in C_\zeta.$$

Then we have

$$\int_{P_1}^{P_{x'}}\psi\equiv(1+\zeta)\int_{P_1}^{P_x}\psi\qquad \mathrm{mod}\ \langle\zeta\rangle\ltimes\mathbb{Z}[\omega].$$

Corollary 11 *The following identity holds around $x=1$:*

$$F\left(\frac16,\frac12,\frac76;1-x\right)=\frac{1}{\sqrt{4x-3}}F\left(\frac16,\frac12,\frac76;1-\frac{x(9-8x)^2}{(4x-3)^3}\right),$$

where $\sqrt{4x-3}=1$ for $x=1$.

Proof. Theorem 6 implies that

$$\int_1^{x'}\frac{\sqrt[6]{t^3(t-1)}dt}{t(t-1)}\equiv(1+\zeta)\int_1^{x}\frac{\sqrt[6]{t^3(t-1)}dt}{t(t-1)}\qquad \mathrm{mod}\ \langle\zeta\rangle\ltimes\mathbb{Z}[\omega],$$

for $x'=x(9-8x)^2/(4x-3)^3$. By this relation, there exists $k\in\mathbb{N}$ such that

$$\zeta^k \frac{\sqrt[6]{27(x-1)}}{\sqrt{4x-3}} F\left(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - \frac{x(9-8x)^2}{(4x-3)^3}\right)$$
$$= (1+\zeta)\sqrt[6]{1-x} F\left(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1-x\right).$$

We cancel $(1+\zeta)\sqrt[6]{1-x}$ and $\sqrt[6]{27(x-1)}$, and choose $k=0$ so that the identity holds for $x=1$. □

By Corollary 11, we define two means as follows. We solve the cubic equation

$$\frac{x(9-8x)^2}{(4x-3)^3} = \frac{b^2}{a^2}$$

of the variable x, where we assume $0 < a < b$. A real solution x_0 of this equation is

$$\frac{3}{8}\left[\frac{\sqrt[3]{a^2}}{\sqrt{b^2-a^2}}\left(\sqrt[3]{b+\sqrt{b^2-a^2}} - \sqrt[3]{b-\sqrt{b^2-a^2}}\right) + 2\right].$$

We set

$$\eta_1 = b + \sqrt{b^2-a^2}, \quad \eta_2 = b - \sqrt{b^2-a^2}. \tag{6.2}$$

Note that

$$\eta_1\eta_2 = a^2, \quad \frac{\eta_1+\eta_2}{2} = b, \quad \frac{\eta_1-\eta_2}{2} = \sqrt{b^2-a^2}.$$

We express x_0 and $4x_0 - 3$ in terms of η_1 and η_2 as

$$x_0 = \frac{3}{8}\left[\frac{\sqrt[3]{\eta_1}\sqrt[3]{\eta_2}}{(\eta_1-\eta_2)/2}\left(\sqrt[3]{\eta_1} - \sqrt[3]{\eta_2}\right) + 2\right] = \frac{3}{4}\left[\frac{\sqrt[3]{\eta_1}\sqrt[3]{\eta_2}}{\sqrt[3]{\eta_1^2} + \sqrt[3]{\eta_1}\sqrt[3]{\eta_2} + \sqrt[3]{\eta_2^2}} + 1\right]$$
$$= \frac{3}{4}\frac{(\sqrt[3]{\eta_1} + \sqrt[3]{\eta_2})^2}{\sqrt[3]{\eta_1^2} + \sqrt[3]{\eta_1}\sqrt[3]{\eta_2} + \sqrt[3]{\eta_2^2}},$$
$$4x_0 - 3 = \frac{3}{2}\left[\frac{\sqrt[3]{\eta_1}\sqrt[3]{\eta_2}}{(\eta_1-\eta_2)/2}\left(\sqrt[3]{\eta_1} - \sqrt[3]{\eta_2}\right) + 2\right] - 3 = \frac{3\sqrt[3]{\eta_1}\sqrt[3]{\eta_2}}{\sqrt[3]{\eta_1^2} + \sqrt[3]{\eta_1}\sqrt[3]{\eta_2} + \sqrt[3]{\eta_2^2}}.$$

Thus the identity in Corollary 11 is transformed into

$$F\left(\frac{1}{6},\frac{1}{2},\frac{7}{6};1-\left(\frac{\sqrt[3]{\eta_1}+\sqrt[3]{\eta_2}}{2}\right)^2\Big/\left(\sqrt{\frac{\eta_1^{2/3}+\eta_1^{1/3}\eta_2^{1/3}+\eta_2^{2/3}}{3}}\right)^2\right)$$
$$=\frac{1}{\sqrt[3]{a}}\sqrt{\frac{\eta_1^{2/3}+\eta_1^{1/3}\eta_2^{1/3}+\eta_2^{2/3}}{3}}F\left(\frac{1}{6},\frac{1}{2},\frac{7}{6};1-\frac{b^2}{a^2}\right).$$

This formula is equivalent to

$$\frac{a}{F(1/6,1/2,7/6;1-b^2/a^2)} = \frac{m_1(a,b)}{F(1/6,1/2,7/6;1-m_2(a,b)^2/m_1(a,b)^2)} \tag{6.3}$$

if we define two means m_1 and m_2 of positive real numbers a and b by

$$m_1(a,b)=\frac{a^{2/3}\sqrt{\eta_1^{2/3}+\eta_1^{1/3}\eta_2^{1/3}+\eta_2^{2/3}}}{\sqrt{3}},\quad m_2(a,b)=\frac{a^{2/3}(\eta_1^{1/3}+\eta_2^{1/3})}{2},$$

where η_1 and η_2 are given in (6.2) with conditions

$$-\frac{\pi}{6}<\arg(\eta_i^{1/3})<\frac{\pi}{6},\quad \eta_1^{1/3}\eta_2^{1/3}=a^{2/3}.$$

Let $a_1=a$ and $b_1=b$ be positive real numbers. We give a pair of sequences $\{a_n,b_n\}_{n\in\mathbb{N}}$ with initial terms $a_1=a$, $b_1=b$ by the recursive relations

$$a_{n+1}=m_1(a_n,b_n),\quad b_{n+1}=m_2(a_n,b_n). \tag{6.4}$$

Corollary 12 (A formula in Theorem 3 in [HKM]) *We have*

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=\frac{a}{F(1/6,1/2,7/6;1-b^2/a^2)}.$$

Proof. It is shown in §5 of [HKM] that the sequences $\{a_n\}$ and $\{b_n\}$ converge and satisfy $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n$. By (6.3), we have

$$\begin{aligned}\frac{a}{F(1/6,1/2,7/6;1-b^2/a^2)} &= \frac{a_2}{F(1/6,1/2,7/6;1-b_2^2/a_2^2)}\\ &= \frac{a_3}{F(1/6,1/2,7/6;1-b_3^2/a_3^2)} = \cdots\\ &= \frac{a_n}{F(1/6,1/2,7/6;1-b_n^2/a_n^2)} = \cdots = \lim_{n\to\infty} a_n,\end{aligned}$$

since $\lim_{n\to\infty}(b_n^2/a_n^2) = 1$ and $F(1/6,1/2,7/6;0) = 1$. □

Hence we see that the $(1+\zeta)$-multiple formula (5.2) in Theorem 4 implies this limit formula for the sequences defined by the mean iteration (6.4).

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