Hokkaido Mathematical Journal Vol. 47 (2018) p. 1–15

A remark on modified Morrey spaces on metric measure spaces

Yoshihiro Sawano, Tetsu Shimomura and Hitoshi Tanaka

(Received February 28, 2015; Revised August 9, 2015)

Abstract. Morrey norms, which are originally endowed with two parameters, are considered on general metric measure spaces. Volberg, Nazarov and Treil showed that the modified Hardy-Littlewood maximal operator is bounded on Legesgue spaces. The modified Hardy-Littlewood maximal operator is known to be bounded on Morrey spaces on Euclidean spaces, if we introduce the third parameter instead of adopting a natural extension of Morrey spaces. When it comes to geometrically doubling, as long as an auxiliary parameter is introduced suitably, the Morrey norm does not depend on the third parameter and this norm extends naturally the original Morrey norm. If the underlying space has a rich geometric structure, there is still no need to introduce auxiliary parameters. However, an example shows that this is not the case in general metric measure spaces. In this paper, we present an example showing that Morrey spaces depend on the auxiliary parameters.

Key words: Morrey space, non-doubling, metric measure spaces.

1. Introduction

In this paper, we present an example showing that the Morrey space $\mathcal{M}^p_q(k,\mu)$ depends on the auxiliary parameter k which we shall define by (1.2) below. Our construction will supplement [2, Theorem 7], [14, Proposition 1.1] and [16, p. 314]. Based on the metric measure space (X,d,μ) obtained in [13, Section 2], we shall show that the conclusion of [2, Theorem 7] fails when we do not assume the metric geometrically doubling condition on X; see Definition 1.10 for the definition of geometrically doubling metric spaces.

The analysis on a metric measure space (X,d,μ) developed dramatically due to the work of Nazarov, Treil and Volberg [11]. One of the main contributions is the following:

2010 Mathematics Subject Classification : Primary 42B35; Secondary 42B25.

The first author is supported by Grant-in-Aid for Young Scientists (B) (No. 21740104), the Japan Society for the Promotion of Science.

The second author is supported by Grant-in-Aid for Scientific Research (C) (No. 24540174), the Japan Society for the Promotion of Science.

The third author is supported by the FMSP program at Graduate School of Mathematical Sciences, the University of Tokyo, and Grant-in-Aid for Scientific Research (C) (No. 23540187), the Japan Society for the Promotion of Science.

Proposition 1.1 ([11, Lemma 3.1]) *Let (X, d, μ) be a metric measure space. Define the modified maximal operator $\tilde{M}_k$, $k > 0$, by*

$$\tilde{M}_k f(x) := \sup_{r>0} \frac{1}{\mu(B(x, kr))} \int_{B(x,r)} |f(y)| \, d\mu(y).$$

Then, $\mu(\{x \in X : \tilde{M}_3 f(x) > \lambda\}) \leq (1/\lambda)\|f\|_{L^1(\mu)}$ for all μ-measurable functions f and $\lambda > 0$.

The proof of Proposition 1.1 depends on the Vitali covering theorem:

Lemma 1.2 *Fix some $R > 0$. Let $E \subset X$ be any finite set, and let $\{B(x, r_x)\}_{x \in E}$ be a family of balls of radii $0 < r_x < R$. Then there exists a countable subfamily $\{B(x_j, r_{x_j})\}_{j=1}^{\infty}$ of disjoint balls such that $E \subset \bigcup_j B(x_j, 3r_{x_j})$.*

Terasawa noticed that $k = 3$ is superfluous; he pointed out that $k = 2$ suffices.

Proposition 1.3 ([17, Theorem 2.4]) *Let (X, d, μ) be a metric measure space such that, for any $x \in X$, $\mu(B(x, \cdot)) : (0, \infty) \to [0, \infty)$ is a continuous function. Assume that μ is a Radon measure. Then $\mu(\{x \in X : \tilde{M}_2 f(x) > \lambda\}) \leq (1/\lambda)\|f\|_{L^1(\mu)}$ for all μ-measurable functions f and $\lambda > 0$.*

After he proved Proposition 1.3, the first author proved the following:

Proposition 1.4 ([12, Theorem 1.2]) *Let (X, d, μ) be a separable metric measure space. Assume that μ is a Radon measure. Then $\mu(\{x \in X : \tilde{M}_2 f(x) > \lambda\}) \leq (1/\lambda)\|f\|_{L^1(\mu)}$ for all μ-measurable functions f and $\lambda > 0$.*

The proofs of Propositions 1.3 and 1.4 essentially rely on the following parameterized covering lemma:

Lemma 1.5 ([12, Lemma 2.1]) *Fix some $R > 0$ and $\delta > 0$. Let $E \subset X$ be any compact set and let $\{B(x, r_x)\}_{x \in E}$ be a finite family of balls of radii $0 < r_x < R$. Then there exists a countable subfamily $\{B(x_j, r_{x_j})\}_{j=1}^{\infty}$ of disjoint balls such that*

$$\bigcup_{x \in E} B(x, \delta r_x) \subset \bigcup_j B(x_j, (2+\delta) r_{x_j}).$$

When $\delta = 1$, the lemma above is exactly the Vitali covering lemma.

Moreover, we have the following (see [12, Remark 2.9] for (1.1)):

Proposition 1.6 ([12, Section 2]) *There exists a compact metric measure space (X, d, μ) such that*

$$\sup\left\{\mu(\{\tilde{M}_2 f > 1\}) : \|f\|_{L^1(\mu)} = 1\right\} = 1 \tag{1.1}$$

and, as long as $0 < k < 2$,

$$\sup\left\{\mu(\{\tilde{M}_k f > 1\}) : \|f\|_{L^1(\mu)} = 1\right\} = \infty.$$

Motivated by the above fact, we investigate Morrey spaces, which can be used to understand the behaviour of the Hardy–Littlewood maximal function.

Morrey spaces play an important role in harmonic analysis. Let $1 \le q \le p < \infty$ and f a μ-measurable function. On a metric measure space (X, d, μ) define the modified Morrey space $\mathcal{M}^p_q(k, \mu)$ of order $k > 0$ by the norm;

$$\|f\|_{\mathcal{M}^p_q(k,\mu)} := \sup_{B \in \mathcal{B}} \mu(kB)^{1/p-1/q} \left(\int_B |f(y)|^q \, d\mu(y) \right)^{1/q}, \tag{1.2}$$

where $\mathcal{B}$ denotes the set of all balls: $\mathcal{B} = \{B(x, r) : x \in X, r > 0\}$. In [16], the following is proved:

Proposition 1.7 ([16, Theorem 2.2]) *Let $1 < q \le p < \infty$. Then there exists $C > 0$ such that*

$$\|\tilde{M}_2 f\|_{\mathcal{M}^p_q(6,\mu)} \le C \|f\|_{\mathcal{M}^p_q(2,\mu)}.$$

When (X, d) is the Euclidean space $\mathbb{R}^d$, more can be said:

Proposition 1.8 ([14, Proposition 1.1]) *Let $1 < q \le p < \infty$. Then, for any $k > 1$,*

$$\mathcal{M}^p_q(2, \mu) = \mathcal{M}^p_q(k, \mu)$$

when (X, d) is the Euclidean space $\mathbb{R}^d$.

The conclusion remains valid for any geometrically doubling metric measure space; see [8, Proposition 5.4 and Theorem 5.6]. We refer to [20] for further results on Morrey spaces on quasi-metric measure spaces, where the authors developed a theory of the Hajłasz gradient.

Originally, Proposition 1.8 is proved with balls in the definition replaced by cubes. Here, "by a cube", we mean a compact cube in $\mathbb{R}^d$ whose edges are parallel to coordinate axes. Denote by kQ the k-times expansion of Q for $k > 1$. The proof of Proposition 1.8 depends on the following covering lemma for cubes:

Lemma 1.9 *Let Q be a cube and $k > 1$. Bisect Q into 2^d subcubes and let R be one of them. Then*

$$(2k-1)R \subset kQ.$$

We also have an example showing that $\mathcal{M}^p_q(1,\mu)$ and $\mathcal{M}^p_q(2,\mu)$ are not isomorphic, where μ is a Radon measure on the Euclidean space $\mathbb{R}^d$ (see [15, Example 2.3]).

A similar assertion can be obtained for a geometrically doubling metric measure space (X,d,μ).

Definition 1.10 A metric space is geometrically doubling, if there exists a constant N such that, for any $x \in X$ and $R > 0$, $B(x,2R)$ is covered by N balls of radius R.

With these results in mind, we expect that $\mathcal{M}^p_q(6,\mu)$ and $\mathcal{M}^p_q(2,\mu)$ are the same function spaces even in the general metric measure spaces. However, $\mathcal{M}^p_q(6,\mu)$ and $\mathcal{M}^p_q(2,\mu)$ can be different when (X,d,μ) is not a metric geometrically doubling measure space. In Section 2 we shall show;

Theorem 1.11 *Let $1 \le q < p < \infty$. Then, it can happen that $\mathcal{M}^p_q(2,\mu)$ is a proper subset of $\mathcal{M}^p_q(6,\mu)$.*

When $p = q$, then the Morrey space $\mathcal{M}^p_q(k,\mu)$ is the Lebesgue space $L^p(\mu)$ and the parameter k does not come into play in the definition. So, we are mainly concerned with the case $1 \le q < p < \infty$.

Theorem 1.11 is a counterpart of the Morrey norm defined by

$$\|f\|_{L^{q,\lambda}(k,\mu)} := \sup_{Q:cubes(or\ balls),\mu(Q)>0} \ell(Q)^{\lambda}\mu(k\,Q)^{-1/q}\left(\int_Q |f(y)|^q\,d\mu(y)\right)^{1/q} \tag{1.3}$$

on $\mathbb{R}^d$ or (X, d, μ), where $\ell(Q)$ denotes the side length or diameter of Q, $1 \le q < \infty$ and $\lambda > 0$. This Morrey norm is defined in [9] in which the Poincaré inequality is obtained (see [9, Section 4]). With this definition in mind, let us summarize the results on dependence on the parameter k in the definition of the Morrey space $\|f\|_{\mathcal{M}_q^p(k,\mu)}$ given by (1.2) and the Morrey space $\|f\|_{L^{q,\lambda}(k,\mu)}$ given by (1.3).

Morrey spaces	results
$\mathcal{M}_q^p(k, \mu)$ on $\mathbb{R}^d$ in [14]	positive [14]
$\mathcal{M}_q^p(k, \mu)$ on (X, d, μ) in [16]	negative; see Theorem 1.11 in this paper
$L^{q,\lambda}(k, \mu)$ on $\mathbb{R}^d$ in [9]	positive [9]
$L^{q,\lambda}(k, \mu)$ on (X, d, μ) in [9]	negative [13]

In [9, Remark 2.1], on $\mathbb{R}^d$, an example is presented showing that the norms $\| \cdot \|_{L^{q,\lambda}(k,\mu)}$ are equivalent for $k > 1$ but that the norms $\| \cdot \|_{L^{q,\lambda}(1,\mu)}$ and $\| \cdot \|_{L^{q,\lambda}(2,\mu)}$ are not equivalent. Also, a counterpart of Theorem 1.11 for this Morrey norm on general metric measure spaces is obtained in [13, Section 2]; the norms $\| \cdot \|_{L^{q,\lambda}(2,\mu)}$ and $\| \cdot \|_{L^{q,\lambda}(4,\mu)}$ are not equivalent.

Coifman and Weiss pointed out any metric measure space (X, d, μ) is geometrically doubling if μ is a doubling measure [3, p. 4]. Despite this fact, the notion of the geometrically doubling spaces itself dates back to around early 80's. At that time the geometrically doubling spaces were called homogeneous metric spaces. Assouad's embedding theorem asserts that, a metric space (X, d) is geometrically doubling if and only if there exist $D \in \mathbb{N}$, $\varepsilon \in (0, 1)$, $C > 0$ and $f : X \to \mathbb{R}^D$ such that $C^{-1}d(x, y)^\varepsilon \le |f(x) - f(y)| \le Cd(x, y)^\varepsilon$; see Assouad's 1983 paper [1]. Recently more and more people pay attention to geometrically doubling spaces after the advent of Hytönen's 2010 paper [6], where Hytönen extended the space of RBMO, defined by Tolsa [18], to metric measure setting. There still exist many open problems on function spaces and boundedness of operators; see, for example, [4], [19].

2. A counterexample

To prove Theorem 1.11, we need a setup.

2.1. The space we work on

First, we define a set X on which we work. To this end, we define X_0.

Definition 2.1 Denote by $\Delta(z,r)$ the open ball centered at $z \in \mathbb{C}$ of radius $r > 0$;

$$\Delta(z,r) := \{w \in \mathbb{C} : |w - z| < r\}. \tag{2.1}$$

Define A_k to be the boundary of $\Delta(0, 3^{-k})$ for $k \in \mathbb{N} \cup \{0\}$;

$$A_k := \{z \in \mathbb{C} : |z| = 3^{-k}\}. \tag{2.2}$$

Finally, we let

$$X_0 := \{0\} \cup \bigcup_{k=0}^{\infty} A_k \subset \mathbb{C}. \tag{2.3}$$

We equip X_0 with a measure. We denote by $\mathcal{H}^1$ denote the Hausdorff measure on $\mathbb{C}$ with the dimension 1 and by $\mathcal{B}_0$ the Borel sets in X_0.

Definition 2.2 ([13, Definition 2.6])

(1) Let $\gamma \in \mathbb{R}$ be such that $\gamma \sum_{k=0}^{\infty} \frac{\mathcal{H}^1(A_k)}{(k!)^k} = 1$.

(2) One defines a function $w_0 : X_0 \to [0, \infty)$ by $w_0 := \gamma \sum_{k=0}^{\infty} \frac{1}{(k!)^k} \chi_{A_k}$.

(3) Define a measure on X_0 by $\mu_0 := w_0 \, d\mathcal{H}^1$, so that $(X_0, \mathcal{B}_0, \mu_0)$ is a probability space.

Note that A_k is an annulus, that X_0 is the union and that X is a countable product of X_0.

We recall a "singular" metric on X given in [13, Definition 2.3]. We denote by $[a]$ the largest integer not greater than a.

Definition 2.3 ([13, Definition 2.3])

(1) Define $X := {X_0}^{\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}$ based on (2.1)–(2.3).

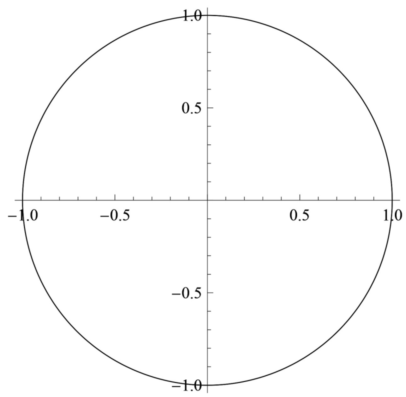

Figure 1. We draw graphs of A_0 and X_0.

(2) One assumes that $N_0 > 100$ is an integer such that $\log_9 N_0$ is an integer.
(3) We define, for $\delta > 0$, $N(\delta) := \max(0, [\log_{N_0}(\delta^{-1})])$.
(4) Define the distance $d(\{x_j\}_{j=1}^\infty, \{y_j\}_{j=1}^\infty)$ between $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ by

$$d\big(\{x_j\}_{j=1}^\infty, \{y_j\}_{j=1}^\infty\big) := \inf\{\delta > 0 : |x_j - y_j| \le \delta \text{ for all } j \le N(\delta)\}.$$

(5) For each $k \in \mathbb{N}$, define a sphere S_k by:

$$S_k := (A_k)^{N(3^{-k})} \times X_0 \times X_0 \times \cdots .$$

(6) One also defines $\mathbf{o} := (0, 0, \dots) \in X$.

As we have seen in [13, Lemma 2.4], (X, d) is a bounded metric space such that

$$d_X := \sup_{x,y \in X} d(x, y) = 2.$$

Note also that χ_{S_k} is μ-measurable. Let $r > 0$ and $\mathbf{x} = (x_1, x_2, \dots) \in X$. Then

$$B(\mathbf{x}, r) = \Delta(x_1, r) \times \Delta(x_2, r) \times \cdots \times \Delta(x_{N(r)}, r) \times X_0 \times X_0 \times \cdots \tag{2.4}$$

as we have seen in [13, Lemma 2.5]. Note that (2.4) is valid no matter where $\mathbf{x}$ lies.

Now we make (X, d) into a probability space.

Definition 2.4 ([13, Definition 2.1])

(1) For $j = 1, 2, \ldots$, denote by p_j the jth projection from X to X_0.
(2) By using Definition 2.2, make X into a probability space $(X, \mathcal{B}, \mu)$ such that $\{p_j\}_{j=1}^{\infty}$ is an independent and identically distributed sequence with the distribution μ_0, or equivalently, for all Borel sets $A \subset X_0$ and $j = 1, 2, \ldots$,

$$\mu(\underbrace{X_0 \times \cdots \times X_0}_{(j-1)-\text{fold}} \times A \times X_0 \times X_0 \times \cdots) = \mu_0(A).$$

In Section 2, we adopt the following convention: We shall denote by z without subindex the point in $\mathbb{C}$. The bold letters such as $\mathbf{x}, \mathbf{y}, \mathbf{z}$ stand for points in X. When we add j and we write x_j, y_j, z_j and so on, they denote complex numbers of the jth component of elements in X.

We recall two important quantitative estimates on μ.

Lemma 2.5 ([13, Lemma 2.11]) *We have*

$$\mu(B(\mathbf{o}, r)) \le \mu(B(\mathbf{x}, 10r))$$

for all $\mathbf{x} \in X$ *and* $r > 0$.

Lemma 2.6 ([13, Lemma 2.12]) *Suppose that* $2a = \log_3 N_0$. *There exists* $C > 0$ *such that*

$$\mu(B(\mathbf{o}, 2.2 \times 3^{-(2j-1)a})) \le C\mu(S_{(2j-1)a})$$

for all $j \in \mathbb{N}$.

To prove our main result, we will need two other auxiliary estimates, which can not be found in [13]:

Lemma 2.7 *Let* $1 \le q < p < \infty$. *Suppose that* $2a = \log_3 N_0$. *Then*

$$\liminf_{k \to \infty} \left(\frac{1}{\pi} \cos^{-1} \frac{2}{3} \right)^{N(3^{-k+1})(1/p-1/q)} \left(\frac{\mu(S_{k-1})}{\mu(S_k)} \right)^{1/p-1/q} = 0 \qquad (2.5)$$

Proof. By [13, (2.21)], we have

$$\mu(S_k) = \left(\frac{2\pi\gamma}{(3 \cdot k!)^k}\right)^{N(3^{-k})}.$$

Recall that we are assuming $N_0 > 100$. If $k \in \mathbb{N} \cap [3a, \infty)$ is such that k/a is an odd multiple of positive integers, then

$$\begin{aligned} N(3^{-k}) &= \max(1, [\log_{N_0} 3^k]) = \max\left(1, \left[\frac{k}{2a}\right]\right) = \frac{k-a}{2a} \\ N(3 \times 3^{-k}) &= \max(1, [\log_{N_0} 3^k - \log_{N_0} 3]) \\ &= \max\left(1, \left[\frac{k}{2a} - \log_{N_0} 3\right]\right) = \frac{k-a}{2a}. \end{aligned}$$

This implies

$$\begin{aligned} 0 &\le \liminf_{k \to \infty} \left(\frac{1}{\pi}\cos^{-1}\frac{2}{3}\right)^{N(3^{-k+1})(1/p-1/q)} \left(\frac{\mu(S_{k-1})}{\mu(S_k)}\right)^{1/p-1/q} \\ &\le \lim_{k \to \infty} \left(\frac{1}{\pi}\cos^{-1}\frac{2}{3}\right)^{N(3^{-(2k-1)a+1})(1/p-1/q)} \left(\frac{\mu(S_{(2k-1)a-1})}{\mu(S_{(2k-1)a})}\right)^{1/p-1/q} \\ &= \lim_{k \to \infty} \left(\frac{1}{\pi}\cos^{-1}\frac{2}{3}\right)^{N(3^{-(2k-1)a+1})(1/p-1/q)} \\ &\quad \times \left(\frac{(3 \cdot [(2k-1)a]!)^{(2k-1)a}}{(3 \cdot [(2k-1)a-1]!)^{(2k-1)a-1}}\right)^{N(3^{-(2k-1)a})(1/p-1/q)} \\ &= 0. \end{aligned}$$

So, (2.5) follows. □

Lemma 2.8 *Let $R > 3$ and $(x_0, y_0) \in \mathbb{R}^2$. In $\mathbb{R}^2$, suppose that two circles $x^2 + y^2 = 1$ and $(x - x_0)^2 + (y - y_0)^2 = R^2$ intersect. Then the arclength L of the arc given by*

$$\{(x, y) : x^2 + y^2 = 9, (x - x_0)^2 + (y - y_0)^2 \le R^2\}$$

is greater than or equal to

$$6\cos^{-1}\frac{2}{3}.$$

Proof. A rotation allows us to assume $y_0 = 0 < x_0$. Two circles $x^2+y^2=1$ and $(x-x_0)^2+y^2=R^2$ intersect if and only if

$$R-1 < x_0 < R+1.$$

Let us set

$$\Theta(x_0,R) := |\{\theta \in [-\pi,\pi] : (3\cos\theta - x_0)^2 + (3\sin\theta)^2 \le R^2\}|.$$

Then L is given by

$$L = 3\Theta(x_0,R). \tag{2.6}$$

An arithmetic shows that

$$\Theta(x_0,R) = 2\cos^{-1}\left(\frac{{x_0}^2+9-R^2}{6x_0}\right).$$

Since $x_0 < R+1$ and $R > 3$, we have

$$\begin{aligned}\Theta(x_0,R) &\ge 2\cos^{-1}\left(\frac{2R+10}{6(R+1)}\right)\\ &= 2\cos^{-1}\left(\frac{1}{3}+\frac{4}{3(R+1)}\right) \ge 2\cos^{-1}\left(\frac{2}{3}\right).\end{aligned} \tag{2.7}$$

(2.6) and (2.7) yield the desired lower bound of L. □

2.2. Conclusion of the proof of Theorem 1.11

Assume that $\mathcal{M}^p_q(2,\mu) = \mathcal{M}^p_q(6,\mu)$. Then by the closed graph theorem, there exists a constant $C > 0$ such that

$$\|f\|_{\mathcal{M}^p_q(2,\mu)} \le C\|f\|_{\mathcal{M}^p_q(6,\mu)} \tag{2.8}$$

for all $f \in \mathcal{M}^p_q(2,\mu) = \mathcal{M}^p_q(6,\mu)$. In fact, let T be a mapping given by $f \in \mathcal{M}^p_q(6,\mu) \mapsto f \in \mathcal{M}^p_q(2,\mu)$. Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence in $\mathcal{M}^p_q(6,\mu)$ which is convergent to f in $\mathcal{M}^p_q(6,\mu)$ and that $\{Tf_n\}_{n=1}^\infty$ is a sequence in $\mathcal{M}^p_q(2,\mu)$ which is convergent to g in $\mathcal{M}^p_q(2,\mu)$. Since

$\{f_n\}_{n=1}^{\infty} = \{Tf_n\}_{n=1}^{\infty}$ converges to g in $\mathcal{M}_q^p(2,\mu)$, we can say that $\{f_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{M}_q^p(6,\mu)$ which is convergent to g in $\mathcal{M}_q^p(6,\mu)$. Since $\mathcal{M}_q^p(6,\mu)$ is a normed space, we have $f = g$. Hence T is a closed operator. Since $\mathcal{M}_q^p(6,\mu)$ and $\mathcal{M}_q^p(2,\mu)$ are Banach spaces, T is a continuous operator. So (2.8) follows.

Let k be a fixed integer of the form $(2j-1)a$ with $j \in \mathbb{N}$ and $a := \log_9 N_0$. We need estimates of $\|\chi_{S_k}\|_{\mathcal{M}_q^p(2,\mu)}$ and $\|\chi_{S_k}\|_{\mathcal{M}_q^p(6,\mu)}$. The estimate we need for $\|\chi_{S_k}\|_{\mathcal{M}_q^p(2,\mu)}$ is as follows:

$$\begin{aligned}\|\chi_{S_k}\|_{\mathcal{M}_q^p(2,\mu)} &\geq \mu(B(\mathbf{o}, 2.2\cdot 3^{-k}))^{1/p-1/q}\mu(B(\mathbf{o}, 1.1\cdot 3^{-k})\cap S_k)^{1/q}\\ &= \mu(B(\mathbf{o}, 2.2\cdot 3^{-k}))^{1/p-1/q}\mu(S_k)^{1/q}\\ &\geq C\mu(S_k)^{1/p},\end{aligned}$$

where for the last inequality we have used Lemma 2.6. As a consequence,

$$\|\chi_{S_k}\|_{\mathcal{M}_q^p(2,\mu)} \geq C\mu(S_k)^{1/p}. \tag{2.9}$$

Meanwhile, we need an estimate for $\|\chi_{S_k}\|_{\mathcal{M}_q^p(6,\mu)}$. Let $B = B(\mathbf{x}, r)$ be a ball which intersects S_k.

We distinguish two cases.

(1) If $r \leq 2^{-1}\cdot 3^{-k}$, then $6B \supset B \supset B\cap S_k$ and hence

$$\mu(6B)^{1/p-1/q}\mu(B\cap S_k)^{1/q} \leq \mu(B\cap S_k)^{1/p}.$$

Write $\mathbf{x} = (x_1, x_2, \ldots, x_{N(3^{-k})}, \ldots)$. Recall that

$$B = \Delta(x_1, r)\times\Delta(x_2, r)\times\cdots\times\Delta(x_{N(r)}, r)\times X_0\times X_0\times\cdots$$

according to (2.4). Thus,

$$\begin{aligned}&\mu(B\cap S_k)\\ &\quad= \mu\big((\Delta(x_1, r)\times\Delta(x_2, r)\\ &\qquad\qquad\times\cdots\times\Delta(x_{N(3^{-k})}, r)\times X_0\times X_0\times\cdots)\cap S_k\big)\\ &\quad= \mu_0(\Delta(x_1, r)\cap A_k)\mu_0(\Delta(x_2, r)\cap A_k)\cdots\mu_0(\Delta(x_{N(3^{-k})}, r)\cap A_k).\end{aligned}$$

We shall estimate $\mu_0(\Delta(x_j, r) \cap A_k)$ for $j = 1, 2, \ldots, N(3^{-k})$. Suppose that $\Delta(x_j, r)$ and A_k meet at a point y_j. Then $\Delta(x_j, r) \subset \Delta(y_j, 2r)$ and hence $\Delta(x_j, r) \cap A_k \subset \Delta(y_j, 2r) \cap A_k$. Since $r \le 2^{-1} \cdot 3^{-k}$, it follows that

$$\mu(\Delta(x_j, r) \cap A_k) \le \mu(\Delta(y_j, 2r) \cap A_k) \le \mu(\Delta(y_j, 3^{-k}) \cap A_k) = \frac{1}{3}\mu(A_k).$$

Using this geometric observation, we have

$$\mu(B \cap S_k) \le \underbrace{\frac{1}{3}\mu_0(A_k) \cdot \frac{1}{3}\mu_0(A_k) \cdots \frac{1}{3}\mu_0(A_k)}_{N(3^{-k})\text{ times}} = 3^{-N(3^{-k})}\mu(S_k).$$

Hence we have

$$\mu(6B)^{1/p-1/q}\mu(B \cap S_k)^{1/q} \le 3^{-(N(3^{-k}))/p}\mu(S_k)^{1/p}.$$

(2) If $r > 2^{-1} \cdot 3^{-k}$, then Lemma 2.8 yields

$$\mu(6B \cap S_{k-1}) \ge \left(\frac{1}{\pi}\cos^{-1}\frac{2}{3}\right)^{N(3^{-k+1})}\mu(S_{k-1}).$$

Hence

$$\begin{aligned}
&\mu(6B)^{1/p-1/q}\mu(B \cap S_k)^{1/q} \\
&\quad \le \mu(6B \cap S_{k-1})^{1/p-1/q}\mu(B \cap S_k)^{1/q} \\
&\quad \le \left(\frac{1}{\pi}\cos^{-1}\frac{2}{3}\right)^{N(3^{-k+1})(1/p-1/q)}\mu(S_{k-1})^{1/p-1/q}\mu(S_k)^{1/q}.
\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}
&\mu(6B)^{1/p-1/q}\mu(B \cap S_k)^{1/q} \\
&\quad \le \max\Bigg(3^{-(N(3^{-k}))/p}, \\
&\qquad\qquad \left(\frac{1}{\pi}\cos^{-1}\frac{2}{3}\right)^{N(3^{-k+1})(1/p-1/q)}\left(\frac{\mu(S_{k-1})}{\mu(S_k)}\right)^{1/p-1/q}\Bigg)\mu(S_k)^{1/p}.
\end{aligned}$$

Since the ball B is arbitrary, we obtain

$$\begin{aligned}
&\|\chi_{S_k}\|_{\mathcal{M}^p_q(6,\mu)} \\
&\quad \le C \max\Biggl(3^{-(N(3^{-k}))/p}, \\
&\qquad\qquad \left(\frac{1}{\pi}\cos^{-1}\frac{2}{3}\right)^{N(3^{-k+1})(1/p-1/q)} \left(\frac{\mu(S_{k-1})}{\mu(S_k)}\right)^{1/p-1/q}\Biggr)\|\chi_{S_k}\|_{\mathcal{M}^p_q(2,\mu)}
\end{aligned}$$

thanks to (2.9). This is a contradiction to (2.8), since

$$\begin{aligned}
&\liminf_{k\to\infty} \max\Biggl(3^{-(N(3^{-k}))/p}, \\
&\qquad\qquad \left(\frac{1}{\pi}\cos^{-1}\frac{2}{3}\right)^{N(3^{-k+1})(1/p-1/q)} \left(\frac{\mu(S_{k-1})}{\mu(S_k)}\right)^{1/p-1/q}\Biggr) = 0
\end{aligned}$$

by Lemma 2.7.

Acknowledgement The authors are grateful to the anonymous referee for his/her careful reading, which increased readability of the paper.

References

[1] Assouad P., *Plongements lipschitziens dans* $\mathbb{R}^n$. Bulletin de la S. M. F. **111** (1983), 429–448.

[2] Cao Y. H. and Zhou J., Morrey spaces for nonhomogeneous metric measure spaces, Abstr. Appl. Anal. 2013, Art. ID 196459, 8 pp.

[3] Coifman R. R. and Weiss G., gAnalyse harmonique non-commutative sur certains espaces homogènes", Étude de certaines intègrales singulières, Lecture Notes in Mathematics **242**, Springer-Verlag, Berlin-New York, 1971.

[4] Fu X., Lin H., Yang Da. and Yang Do., *Hardy spaces* H^p *over nonhomogeneous metric measure spaces and their applications.* Sci. China Math. **58** (2015), 309–388.

[5] Hedberg L. I., *On certain convolution inequalities.* Proc. Amer. Math. Soc. **36** (1972), 505–510.

[6] Hytönen T., *A framework for non-homogeneous analysis on metric spaces, and the RBMO space of Tolsa.* Publ. Mat. **54** (2010), 485–504.

[7] Journé J. L., Calderón-Zygmund operators, pseudodifferential operators and the Cauchy integral of Calderón, Lecture Notes in Mathematics, 994,

Springer-Verlag, Berlin, 1983.

[8] Lu Y., Yang D. and Yuan W., *Morrey-Sobolev spaces on metric measure spaces.* Potential Anal. **41** (2014), 215–243.

[9] Mizuta Y., Shimomura T. and Sobukawa T., *Sobolev's inequality for Riesz potentials of functions in non-doubling Morrey spaces.* Osaka J. Math. **46** (2009), 255–271.

[10] Morrey C. B., *On the solutions of quasi-linear elliptic partial differential equations.* Trans. Amer. Math. Soc. **43** (1938), 126–166.

[11] Nazarov F., Treil S. and Volberg A., Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces, Internat. Math. Res. Notices 1998, no. 9, 463–487.

[12] Sawano Y., *Sharp estimates of the modified Hardy-Littlewood maximal operator on the nonhomogeneous space via covering lemmas.* Hokkaido Math. J. **34** (2005), 435–458.

[13] Sawano Y. and Shimomura T., *Sobolev embeddings for Riesz potentials of functions in non-doubling Morrey spaces of variable exponents.* Collect. Math. **64** (2013), 313–350.

[14] Sawano Y. and Tanaka H., *Morrey spaces for non-doubling measures.* Acta Math. Sin. (Engl. Ser.) **21** (2005), 1535–1544.

[15] Sawano Y. and Tanaka H., *Predual spaces of Morrey spaces with non-doubling measures.* Tokyo J. Math. **32** (2009), 471–486.

[16] Sihwaningrum I. and Sawano Y., *Weak and strong type estimates for fractional integral operators on Morrey spaces over metric measure spaces.* Eurasian Math. J. **4** (2013), 76–81.

[17] Terasawa Y., Outer measures and weak type $(1,1)$ estimates of Hardy-Littlewood maximal operators, J. Inequal. Appl. 2006, Art. ID 15063, 13 pp.

[18] Tolsa X., *BMO, H^1, and Calderón-Zygmund operators for non doubling measures.* Math. Ann. **319** (2001), 89–149.

[19] Yang Da., Yang Do. and Hu G., The Hardy space H^1 with non-doubling measures and their applications, Lecture Notes in Mathematics, **2084**, Springer, Cham, 2013.

[20] Yuan W., Lu Y. and Yang D., *Fractional Hajłasz-Morrey-Sobolev spaces on quasi-metric measure spaces.* Studia Math. **226** (2015), 95–122.

Yoshihiro Sawano
Department of Mathematics and Information Sciences
Tokyo Metropolitan University
1-1 Minami-Ohsawa, Hachioji, Tokyo, 192-0397, Japan
E-mail: ysawano@tmu.ac.jp
yoshihiro-sawano@celery.ocn.ne.jp

Tetsu Shimomura
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524, Japan
E-mail: tshimo@hiroshima-u.ac.jp

Hitoshi Tanaka
Research and Support Center on Higher Education
for the hearing and Visually Impaired
National University Corporation Tsukuba University of Technology
Kasuga 4-12-7, Tsukuba City 305-8521, Japan
E-mail: htanaka@k.tsukuba-tech.ac.jp