# On a theorem of Littlewood

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**Abstract.** In 1927 Littlewood constructed a bounded holomorphic function on the unit disc, having no tangential boundary limits almost everywhere. This theorem was the complement of a positive theorem of Fatou (1906), establishing almost everywhere non-tangential convergence of bounded holomorphic functions. There are several generalizations of Littlewood's theorem whose proofs are based on the specific properties of holomorphic functions. Applying real variable methods, we extend these theorems to general convolution operators.

*Key words*: Fatou theorem, Littlewood theorem, Poisson kernel.

## 1. Introduction

The following remarkable theorems of Fatou [8] play a significant role in the study of boundary value problems of analytic and harmonic functions.

Let $\mathbb{T} = \mathbb{R}/2\pi$ and $D = \{z \in \mathbb{C} : |z| < 1\}$.

**Theorem A** (Fatou, 1906) *Any bounded analytic function on the unit disc $D$ has non-tangential limits at almost all boundary points.*

**Theorem B** (Fatou, 1906) *If a function of bounded variation $\mu(t)$ is differentiable at $x_0 \in \mathbb{T}$, then the Poisson integral*

$$\frac{1}{2\pi} \int_{\mathbb{T}} P_r(x - t) d\mu(t)$$

*converges non-tangentially to $\mu'(x_0)$ as $r \to 1$.*

Littlewood [15] made an important complement to these results, proving essentiality of non-tangential approach in Fatou's theorems. The following statement of Littlewood's theorem is equivalent to the original one. It is fitted to the further statement of the present paper.

**Theorem C** (Littlewood, 1927) *If a continuous function $\lambda(r) : [0, 1) \to (0, \infty)$ satisfies the conditions*

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$$\lim_{r \to 1} \lambda(r) = 0 \ \ and \ \ \lim_{r \to 1} \lambda(r)/(1 - r) = \infty, \qquad (1.1)$$

*then there exists a bounded analytic function $f(z)$, $z \in D$, such that the boundary limit*

$$\lim_{r \to 1} f\big(re^{i(x + \lambda(r))}\big),$$

*does not exist at almost every $x \in \mathbb{T}$.*

There are various generalizations of these theorems in different aspects. A simple proof of Theorem C was given by Zygmund [27]. In [16] Lohwater and Piranian proved, that in Littlewood's theorem almost everywhere divergence can be replaced to everywhere and the example function can be a Blaschke product. That is

**Theorem D** (Lohwater and Piranian)    *If a continuous function $\lambda(r)$ satisfies* (1.1)*, then there exists a Blaschke product $B(z)$ such that the limit*

$$\lim_{r \to 1} B\big(re^{i(x + \lambda(r))}\big),$$

*does not exist for any $x \in \mathbb{T}$.*

In [1] Aikawa obtained a similar everywhere divergence theorem for bounded harmonic functions on the unit disc, giving a positive answer to a problem raised by Barth [6, p. 551].

**Theorem E** (Aikawa)    *If $\lambda(r)$ is continuous and satisfies the condition* (1.1)*, then there exists a bounded harmonic function $u(z)$ on the unit disc, such that the limit*

$$\lim_{r \to 1} u\big(re^{i(x + \lambda(r))}\big),$$

*does not exist for any $x \in \mathbb{T}$.*

As it is noticed in [1] this theorem implies theorem A. Indeed, if $u(z)$ is a harmonic function obtained in Theorem E and $v(z)$ is its harmonic conjugate, then the holomorphic function $\exp(u + iv)$ has the same divergence property as $u(z)$ does.

Related questions were considered also in higher dimensions. Korani [14] extended Fatou's theorem for the Poisson-Szegö integral. Littlewood

type theorems for the higher dimensional Poisson integral was established by Aikawa [1], [2] and for the Poisson-Szegö integral by Hakim-Sibony [9], Hirata [10]. In [6] Nagel and Stein proved that the Poisson integral on the upper half space of $\mathbb{R}^{n+1}$ has the boundary limit at almost every point within a certain approach region, which is not contained in any non-tangential approach regions. Sueiro [8] extended Nagel-Stein's result for the Poisson-Szegö integral. Almost everywhere convergence over tangential tress (family of curves) were investigated by Di Biase [4], Di Biase-Stokolos-Svensson-Weiss [5].

Sjögren ([23], [24], [25]), Rönning ([19], [20], [21]), Katkovskaya-Krotov ([12]), Krotov ([13]), Brundin [7], Mizuta-Shimomura [17], Aikawa [3] studied fractional Poisson integrals with respect to the fractional power of the Poisson kernel. In [12] and [3] higher dimensional cases of such integrals are studied.

The present paper is the continuation of the authors investigations in [11]. In [11] we introduce $\lambda(r)$-convergence, where $\lambda(r)$ is a function

$$\lambda(r) : (0,1) \to (0,\infty) \text{ with } \lim_{r \to 1} \lambda(r) = 0. \tag{1.2}$$

For a given $x \in \mathbb{T}$ we denote by $\lambda(r,x)$ the interval $[x - \lambda(r), x + \lambda(r)]$. If $\lambda(r) \geq \pi$ we assume that $\lambda(r,x) = \mathbb{T}$. Let $F_r(x)$ be a family of functions from $L^1(\mathbb{T})$, where $r$ varies in $(0,1)$. We say $F_r(x)$ is $\lambda(r)$-convergent at a point $x \in \mathbb{T}$ to a value $a \in \mathbb{R}$, if

$$\lim_{r \to 1} \sup_{\theta \in \lambda(r,x)} |F_r(\theta) - a| = 0. \tag{1.3}$$

We shall denote this relation by

$$\lim_{r \to 1 : \, \theta \in \lambda(r,x)} F_r(\theta) = a.$$

We shall say $F_r(x)$ is $\lambda(r)$-divergent at $x \in \mathbb{T}$, if there is no $a \in \mathbb{R}$ satisfying (1.3).

This definition generalizes the non-tangential convergence. For example, in Fatou's Theorems A and B we have a. e. $\lambda(r)$-convergence, if $\lambda(r)$ satisfies the condition (1.7).

We say the family of kernels $\varphi = \{\varphi_r(x) \geq 0 : 0 < r < 1\} \subset L^\infty(\mathbb{T})$, forms an approximative identity (AI), if they satisfy the conditions

**(a):** $\varphi_r(x)$ is even and decreasing on $[0, \pi]$,
**(b):** $\|\varphi_r\|_1 \to 1$ as $r \to 1$,
**(c):** $\varphi_r(x) \to 0$ as $r \to 1, 0 < |x| < \pi$.

We denote by $\mathrm{BV}(\mathbb{T})$ the family of functions of bounded variation on $\mathbb{T}$. In [11] we have investigated a. e. $\lambda(r)$-convergence properties of the integrals

$$\Phi_r(x, d\mu) = \int_{\mathbb{T}} \varphi_r(x - t) d\mu(t), \quad \mu \in \mathrm{BV}(\mathbb{T}), \tag{1.4}$$

$$\Phi_r(x, f) = \int_{\mathbb{T}} \varphi_r(x - t) f(t) dt, \quad f \in L^p(\mathbb{T}), \quad 1 \le p \le \infty. \tag{1.5}$$

The quantity

$$\alpha = \alpha(\lambda, \varphi) = \limsup_{r \to 1} \lambda(r) \|\varphi_r\|_\infty \tag{1.6}$$

plays a significant role in the investigations of (1.4) and (1.5) with $p = 1$. It is proved

**Theorem F** ([11])    *Let $\{\varphi_r\}$ be an AI.*
   *1) If $\alpha(\lambda, \varphi) < \infty$ and $\mu(t) \in \mathrm{BV}(\mathbb{T})$ is differentiable at $x_0$, then*

$$\lim_{r \to 1: t \in \lambda(r, x_0)} \Phi_r(t, d\mu) = \mu'(x_0).$$

   *2) If $\alpha(\lambda, \varphi) = \infty$, then there exists a positive function $f \in L^1(\mathbb{T})$ such that*

$$\limsup_{r \to 1: t \in \lambda(r, x)} \Phi_r(t, f) = \infty$$

*for all $x \in \mathbb{T}$.*

Theorem F implies that an admissible approach region for $\Phi_r$ is determined in terms of the finiteness of the quantity (1.6). It is interesting, that this rate depends only on the values $\|\varphi_r\|_\infty$. If $\varphi_r$ is the Poisson kernel

$$P_r(t) = \frac{1 - r^2}{1 - 2r \cos t + r^2},$$

then one can easily check that $\|\varphi_r\|_\infty \sim 1/(1-r)$ and the condition $\alpha(\lambda, \varphi) <$

$\infty$ coincides with the well-known condition

$$\limsup_{r\to 1} \lambda(r)/(1-r) < \infty, \tag{1.7}$$

guaranteeing non-tangential convergence in the unit disc. So the first part of Theorem F generalizes Fatou's theorem.

In the same paper [11] the authors presented a necessary and sufficient condition for a. e. $\lambda(r)$-convergence of the integrals (1.5) with $p = \infty$. This condition is described by another quantity

$$\beta = \beta(\lambda, \varphi) = \limsup_{\delta\to 0}\limsup_{r\to 1} \int_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t)dt,$$

which contains more information about $\{\varphi_r\}$ than $\alpha(\lambda, \varphi)$ does. That is

**Theorem G** ([11])    *Let $\{\varphi_r(x)\}$ be an arbitrary AI.*
   *1) If $\beta(\lambda, \varphi) = 0$, then for any $f \in L^\infty(\mathbb{T})$ we have*

$$\lim_{r\to 1:\,\theta\in\lambda(r,x)} \Phi_r(\theta, f) = f(x)$$

*at any Lebesgue point $x \in \mathbb{T}$.*
   *2) If $\beta(\lambda, \varphi) > 0$, then there exists a set $E \subset \mathbb{T}$, such that $\Phi_r(x, \mathbb{I}_E)$ is $\lambda(r)$-divergent at any $x \in \mathbb{T}$.*

Observe that $\alpha(\lambda, \varphi) < \infty$ implies $\beta(\lambda, \varphi) = 0$, which may be deduced directly or by using the results of Theorems F and G. One can easily check, that in the case of Poisson kernels for a given function (1.2) the value of $\beta(\lambda, \varphi)$ is either 1 or 0. Besides the condition $\beta(\lambda, \varphi) = 0$ is equivalent to (1.7) and $\beta(\lambda, \varphi) = 1$ coincides with

$$\limsup_{r\to 1} \lambda(r)/(1-r) = \infty. \tag{1.8}$$

We note, that the second part of Theorem G does not imply Theorems C and E, because they provide the everywhere divergence of $\Phi_r(x + \lambda(r), \mathbb{I}_E)$, as $r \to 1$, along tangential curves, not within tangential regions.

The purpose of the present paper is to prove Littlewood type theorems for the operators (1.5). We shall obtain such theorems for more general kernels, than the approximative identity. Consider a family of kernels $\varphi =$

$\{\varphi_r(x) \geq 0 : 0 < r < 1\}$, satisfying the properties **(b)** and

**(d):** $m(v) = \sup_{0 < r < v} \|\varphi_r\|_\infty < \infty$ for any $0 < v < 1$.

We introduce another quantity

$$\beta^* = \beta^*(\lambda, \varphi) = \limsup_{\delta \to 0} \liminf_{r \to 1} \int_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t) dt \leq \beta(\lambda, \varphi).$$

We prove the following theorems.

**Theorem 1.1**    *Let $\{\varphi_r\}$ be a family of nonnegative functions with properties* **(b)** *and* **(d)**. *If a function* (1.2) *is continuous and satisfies the condition $\beta^*(\lambda, \varphi) > 1/2$, then there exists a measurable set $E \subset \mathbb{T}$ such that*

$$\limsup_{r \to 1} \Phi_r(x + \lambda(r), \mathbb{I}_E) - \liminf_{r \to 1} \Phi_r(x + \lambda(r), \mathbb{I}_E) \geq 2\beta^* - 1$$

*at every point $x \in \mathbb{T}$.*

In the case of the Poisson kernel under the condition (1.8) we have $\beta^*(\lambda, \varphi) = 1 > 1/2$. Therefore Theorem 1.1 implies the following generalization of Theorems C and E, giving additional information about the divergence character.

**Corollary 1.2**    *If a continuous function* (1.2) *satisfies* (1.1), *then there exists a harmonic function $u(z)$ on the unit disc with $0 \leq u(z) \leq 1$, such that*

$$\limsup_{r \to 1} u\big(re^{i(x+\lambda(r))}\big) = 1, \quad \liminf_{r \to 1} u\big(re^{i(x+\lambda(r))}\big) = 0$$

*at any point $x \in \mathbb{T}$.*

The higher dimensional case of this corollary and Theorems C and E was considered by Hirata [10]. We construct also a Blaschke product with Littlewood type divergence condition as in Theorem 1.1, which generalizes Theorem D. In this case a stronger condition $\beta^*(\lambda, \varphi) = 1$ is required. So we prove.

**Theorem 1.3**    *If a family of nonnegative functions $\{\varphi_r\}$ satisfies* **(b)**, **(d)**, *the function* (1.2) *is continuous and $\beta^*(\lambda, \varphi) = 1$, then there exists a function $B \in L^\infty(\mathbb{T})$, which is the boundary function of a Blaschke product,*

*such that the limit*

$$\lim_{r \to 1} \Phi_r(x + \lambda(r), B)$$

*does not exist for any $x \in \mathbb{T}$.*

In the definition of $\lambda(r)$-convergence the range of the parameter $r$ is $(0, 1)$ with the "limit point" 1, that is we consider the convergence or divergence properties when $r \to 1$. We take the limit point equal to 1 to compare our results with the boundary properties of analytic and harmonic functions in the unit disc. Certainly it is not essential in the theorems. In general there is no need to imagine the meaning of $\lambda(r)$-convergence geometrically. Instead of $(0, 1)$ we could take any interval (finite or infinite) $(a, b)$ with a limit point $r_0 \in [a, b]$. In this settings $\lambda(r)$ satisfies

$$\lambda(r) : (a, b) \to (0, \infty) \text{ with } \lim_{r \to r_0} \lambda(r) = 0.$$

instead of (1.2) and the properties **(b)** and **(d)** of the family of kernels $\varphi_r$, used in the formulations of new results, will take the forms

**(b'):** $\|\varphi_r\|_1 \to 1$ as $r \to r_0$,
**(d'):** $m(\delta) = \sup_{r \in (a,b) \setminus (r_0 - \delta, r_0 + \delta)} \|\varphi_r\|_\infty < \infty$ for any $\delta > 0$.

A theorem analogous to Theorem 1.1 may be considered also for the integrals

$$\Phi_r(x, f) = \int_{\mathbb{R}} \varphi_r(t - x) f(t) dt, \quad f \in L^1(\mathbb{R}), \quad r > 0, \qquad (1.9)$$

where the family of kernels $\varphi = \{\varphi_r(x) \geq 0 : 0 < r < \infty\} \subset L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ satisfy the conditions

**(b''):** $\|\varphi_r\|_1 \to 1$ as $r \to 0$,
**(d''):** $m(\delta) = \sup_{r > \delta} \|\varphi_r\|_\infty < \infty$ for any $\delta > 0$.

For a function

$$\lambda(r) : (0, \infty) \to (0, \infty) \text{ with } \lim_{r \to 0} \lambda(r) = 0 \qquad (1.10)$$

we define

$$\beta^* = \beta^*(\lambda, \varphi) = \limsup_{\delta \to 0} \liminf_{r \to 0} \int_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t)dt.$$

**Theorem 1.4**　　*Let* $\{\varphi_r(x) : 0 < r < \infty\}$ *be a family of nonnegative functions on* $\mathbb{R}$ *with properties* **(b")** *and* **(d")**. *If* (1.10) *is continuous and* $\beta^*(\lambda, \varphi) > 1/2$, *then there exists a measurable set* $E \subset \mathbb{R}$ *such that*

$$\limsup_{r \to 0} \Phi_r(x + \lambda(r), \mathbb{I}_E) - \liminf_{r \to 0} \Phi_r(x + \lambda(r), \mathbb{I}_E) \geq 2\beta^* - 1, \quad x \in \mathbb{R}.$$
$$(1.11)$$

We note that for any positive function $\Phi(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\|\Phi\|_1 = 1$ the kernels

$$\varphi_r(x) = \frac{1}{r}\Phi\left(\frac{x}{r}\right) \tag{1.12}$$

satisfy the conditions **(b")** and **(d")**. The condition

$$\lim_{r \to 0} \lambda(r)/r = \infty \tag{1.13}$$

geometrically means tangential approach. One can easily check, that for the Poisson kernel and for (1.12) the relation (1.13) is equivalent to the conditions

$$\lim_{r \to 0} \lambda(r)/r = \infty \Leftrightarrow \beta^*(\lambda, \varphi) = 1 \Leftrightarrow \beta^*(\lambda, \varphi) > 0.$$

Therefore, if the kernels in (1.9) coincide with (1.12) and $\lambda(r)$ satisfies (1.13), then Theorem 1.4 implies the everywhere "strong" $\lambda(r)$-divergence of integrals (1.9), which covers the one-dimensional case of a theorem obtained by Aikawa in [3].

P. Sjögren ([23], [24], [25]), J.-O. Rönning ([19], [20], [21]), I. N. Katkovskaya and V. G. Krotov ([12]) considered the square root Poisson integrals

$$\mathcal{P}_r^0(x, f) = \frac{1}{c(r)} \int_{\mathbb{T}} [P_r(x - t)]^{1/2} f(t)dt, \tag{1.14}$$

where

$$c(r) = \int_{\mathbb{T}} [P_r(t)]^{1/2} dt$$

is the normalizing coefficient. They proved that

$$\lim_{r \to 1: \, \theta \in \lambda(r,x)} \mathcal{P}_r^0(\theta, f) = f(x) \text{ a.e.} \tag{1.15}$$

whenever $f \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$, and

$$\lambda(r) \leq \begin{cases} c(1-r)\left( \log \dfrac{1}{1-r} \right)^p & \text{if } 1 \leq p < \infty, \\ c(\mu)(1-r)^\mu \text{ for any } 0 < \mu < 1 & \text{if } p = \infty, \end{cases} \tag{1.16}$$

where $c(\mu) > 0$ is a constant, depended only on $\mu$. The case of $p = 1$ was proved in [23], $1 < p \leq \infty$ were considered in [19], [20]. The cases $p = 1$ and $p = \infty$ are consequences of Theorem F with an additional information about the points where the convergence occurs. Indeed, it is easy to observe that the kernels

$$\varphi_r(x) = \frac{[P_r(x)]^{1/2}}{c(r)}$$

of the operators (1.14) satisfy all conditions **(a)**–**(d)**. Besides, as it is mentioned in [19, p. 223],

$$\varphi_r(x) \sim \psi_r(x) = \begin{cases} \dfrac{1}{2(1-r)\log(1/(1-r))} & \text{if } |x| < 1-r, \\ \dfrac{1}{2|x|\log(1/(1-r))} & \text{if } 1-r \leq |x| \leq \pi. \end{cases} \tag{1.17}$$

This implies

$$\|\varphi_r\|_\infty \sim \frac{1}{2(1-r)\log(1/(1-r))},$$

and therefore, according to Theorem F, we conclude that the condition in (1.16), corresponding to the case $p = 1$, is necessary and sufficient to have the relation (1.15) for any function $f \in L^1(\mathbb{T})$. This reproves the result of [23] and establishes the optimality of such estimate for $\lambda(r)$.

Now suppose $\lambda(r)$ satisfies the condition (1.16) with $p = \infty$. A simple calculation shows that for such $\lambda(r)$ and for the kernels (1.17) we have $\beta(\lambda, \varphi) = \beta^*(\lambda, \varphi) = 0$. Hence, according to the first part of Theorem G, we get the result of the paper [19] for $p = \infty$. Taking $\lambda(r) = (1 - r)^\mu$ with a fixed $0 < \mu < 1$ we shall get $\beta(\lambda, \varphi) = \beta^*(\lambda, \varphi) = 1 - \mu > 0$ and, applying the second part of Theorem G, we conclude the optimality of the bound (1.16) in the case $p = \infty$ too. If $0 < \mu < 1/2$, then we have $\beta^* > 1/2$. In this case, applying Theorem 1.4, we get Littlewood type strong divergence of the integrals (1.14) for some indicator function $f = \mathbb{I}_E$.

## 2. Proof of Theorem 1.1

We shall consider the sets

$$U(n, \delta) = \bigcup_{j=0}^{n-1} \left( \frac{\pi(2j - \delta)}{n}, \frac{\pi(2j + \delta)}{n} \right) \subset \mathbb{T} \tag{2.1}$$

in the proofs of this as well as the next theorems. Observe that, using the definition of $\beta^* > 1/2$ and the property **(d)**, we may choose positive numbers $\delta_k$, $u_k$, $v_k$ $(k \in \mathbb{N})$, satisfying

$$\delta_k < 2^{-k-5}, \quad 1 > v_k > u_k \to 1, \quad 0 < 3\lambda(v_k) \le \lambda(u_k) < \pi, \tag{2.2}$$

$$\int_{-\delta_k \lambda(u_k)}^{\delta_k \lambda(u_k)} \varphi_r(t)dt > \beta^*(1 - 2^{-k}), \quad u_k < r < 1, \quad k = 1, 2, \ldots, \tag{2.3}$$

$$\sum_{j \ge k+1} \delta_j < \frac{1}{10\pi \cdot 2^k m(v_k)}, \tag{2.4}$$

where $m(v)$ is defined in **(d)**. Denote

$$U_k = U(n_k, 5\delta_k), \quad n_k = \left[ \frac{5\pi}{\lambda(u_k)} \right], \quad k \in \mathbb{N}, \tag{2.5}$$

and define the sequence of measurable sets $E_k \subset \mathbb{T}$ by

$$E_1 = U_1, \tag{2.6}$$

$$E_k = \begin{cases} E_{k-1} \setminus U_k & \text{if } k \text{ is even,} \\ E_{k-1} \cup U_k & \text{if } k \text{ is odd.} \end{cases} \qquad (2.7)$$

It is easy to observe that if $k < m$, then

$$E_m \subset E_k \bigcup \left( \bigcup_{j=k+1}^{m} U_i \right),$$

$$E_m \setminus \bigcup_{j=k+1}^{m} U_j = E_k \setminus \bigcup_{j=k+1}^{m} U_j.$$

These relations imply

$$E_k \triangle E_m \subset \bigcup_{j=k+1}^{m} U_j$$

and therefore we get

$$\|\mathbb{I}_{E_k} - \mathbb{I}_{E_m}\|_1 = |E_k \triangle E_m| \leq \sum_{j \geq k+1} |U_j| \leq 10\pi \sum_{j \geq k+1} \delta_j. \qquad (2.8)$$

This and (2.4) impliy that $\mathbb{I}_{E_n}(t)$ converges to a function $f(t)$ in $L^1$-norm. Using Egorov's theorem, we conclude that $f(t) = \mathbb{I}_E(t)$ for some measurable set $E \subset \mathbb{T}$. Tending $m$ to infinity, from (2.8) we get

$$|E \triangle E_k| \leq 10\pi \sum_{j \geq k+1} \delta_j. \qquad (2.9)$$

Take an arbitrary $x \in \mathbb{T}$. There exists an integer $1 \leq j_0 \leq n_k$ such that

$$\frac{2\pi j_0}{n_k} - x \in \left[ \frac{2\pi}{n_k}, \frac{4\pi}{n_k} \right] \subset \left[ \frac{\lambda(u_k)}{3}, \lambda(u_k) \right] \subset [\lambda(v_k), \lambda(u_k)]$$

and therefore, since $\lambda(r)$ is continuous, we may find a number $r$, $u_k \leq r \leq v_k$, such that

$$\lambda(r) = \frac{2\pi j_0}{n_k} - x. \qquad (2.10)$$

If $k \in \mathbb{N}$ is odd, then according to the definition of $E_k$ we get

$$E_k \supset U_k \supset I = \left( \frac{\pi(2j_0 - 5\delta_k)}{n_k}, \frac{\pi(2j_0 + 5\delta_k)}{n_k} \right).$$

Thus, using (2.3), (2.10) as well as the definition of $n_k$ from (2.5), we conclude

$$\Phi_r(x + \lambda(r), \mathbb{I}_{E_k}) \geq \int_I \varphi_r(x + \lambda(r) - t)dt$$

$$= \int_I \varphi_r\left( \frac{2\pi j_0}{n_k} - t \right)dt$$

$$= \int_{-5\pi\delta_k/n_k}^{5\pi\delta_k/n_k} \varphi_r(t)dt$$

$$\geq \int_{-\delta_k\lambda(u_k)}^{\delta_k\lambda(u_k)} \varphi_r(t)dt > \beta^*(1 - 2^{-k}). \qquad (2.11)$$

From (2.4) and (2.9) it follows that

$$\left| \Phi_r(t, \mathbb{I}_E) - \Phi_r(t, \mathbb{I}_{E_k}) \right| \leq |E \triangle E_k| \cdot m(v_k)$$

$$< 10\pi m(v_k) \sum_{j \geq k+1} \delta_j < 2^{-k}, \quad t \in \mathbb{T}, \quad 0 < r < v_k,$$

and hence from (2.11) we obtain

$$\limsup_{r \to 1} \Phi_r(x + \lambda(r), \mathbb{I}_E) \geq \beta^*. \qquad (2.12)$$

If $k \in \mathbb{N}$ is even, then we have $E_k \cap U_k = \varnothing$ and therefore $E_k \cap I = \varnothing$. Thus we get

$$\Phi_r(x + \lambda(r), \mathbb{I}_{E_k}) \leq \int_{\mathbb{T}} \varphi_r(x + \lambda(r) - t)dt - \int_I \varphi_r(x + \lambda(r) - t)dt$$

$$\leq \|\varphi_r\|_1 - \int_{-\delta_k\lambda(u_k)}^{\delta_k\lambda(u_k)} \varphi_r(t)dt \leq \|\varphi_r\|_1 - \beta^*(1 - 2^{-k})$$

and similarly we get

$$\liminf_{r\to 1} \Phi_r(x + \lambda(r), \mathbb{I}_E) \leq 1 - \beta^*. \tag{2.13}$$

Relations (2.12) and (2.13) complete the proof of theorem.

## 3. Proof of Theorem 1.3

The following finite Blaschke products

$$b(n, \delta, z) = \frac{z^n - \rho^n}{\rho^n z^n - 1} = \prod_{k=0}^{n-1} \frac{z - \rho e^{2\pi i k/n}}{\rho e^{2\pi i k/n} z - 1}, \quad \rho = e^{-\sqrt{\delta}/n}. \tag{3.1}$$

plays a significant role in the proof of Theorem 1.3. Similar products were used in the proof of Theorem D too. If $z = e^{ix}$, then (3.1) defines a continuous function in $H^\infty(\mathbb{T})$. We shall use the set $U(n, \delta)$ defined in (2.1). The following lemma shows that on $U(n, \delta)$ the function (3.1) is approximative $-1$, and outside of $U(n, \sqrt[4]{\delta})$ is approximative 1.

**Lemma 3.1** *There exists an absolute constant $C > 0$ such that*

$$\left| b(n, \delta, e^{ix}) + 1 \right| \leq C\sqrt{\delta}, \quad x \in U(n, \delta), \tag{3.2}$$

$$\left| b(n, \delta, e^{ix}) - 1 \right| \leq C\sqrt[4]{\delta}, \quad x \in \mathbb{T} \setminus U(n, \sqrt[4]{\delta}). \tag{3.3}$$

*for any $0 < \delta < 1$.*

*Proof.* Deduction of these inequalities based on the inequalities

$$\frac{1}{3} \leq |e^{ix} - 1| \leq 2.$$

If $x \in U(n, \delta)$, then we have

$$\left| b(n, \delta, e^{ix}) + 1 \right| = \left| \frac{(e^{inx} - 1)(\rho^n + 1)}{\rho^n e^{inx} - 1} \right| \leq \frac{4\pi\delta}{1 - e^{-\sqrt{\delta}}},$$

$$\leq \frac{4e\pi\delta}{e^{\sqrt{\delta}} - 1} \leq \frac{8e\pi\delta}{\sqrt{\delta}} \leq C\sqrt{\delta}. \tag{3.4}$$

If $x \in \mathbb{T} \setminus U(n, \sqrt[4]{\delta})$, then $e^{inx} = e^{i\alpha}$ with $\pi\sqrt[4]{\delta} < |\alpha| < \pi$. Thus we obtain

$$\left| b(n, \delta, e^{ix}) - 1 \right| = \left| \frac{(e^{inx} + 1)(1 - \rho^n)}{\rho^n e^{inx} - 1} \right| \leq \frac{2(e^{\sqrt{\delta}} - 1)}{|e^{inx} - e^{\sqrt{\delta}}|}$$

$$\leq \frac{4\sqrt{\delta}}{|e^{inx} - 1| - |e^{\sqrt{\delta}} - 1|} \leq \frac{4\sqrt{\delta}}{\pi \sqrt[4]{\delta}/2 - 2\sqrt{\delta}} \leq C \sqrt[4]{\delta}. \quad (3.5)$$

$\square$

*Proof of Theorem 1.3.*　　As in the proof of Theorem 1.1 we may choose numbers $\delta_k$, $u_k$, $v_k$ ($k \in \mathbb{N}$), satisfying (2.2), (2.3) and the condition

$$\sum_{j \geq k+1} \sqrt[4]{\delta_j} < \frac{1}{10\pi \cdot 2^k m(v_k)} \tag{3.6}$$

instead of (2.4). Then we denote

$$b_k(x) = b(n_k, \delta_k, e^{ix}), \quad n_k = \left[ \frac{6\pi}{\lambda(u_k)} \right], \quad k \in \mathbb{N}, \tag{3.7}$$

and

$$B_k(x) = \prod_{j=1}^{k} b_j(x), \quad B(x) = \prod_{j=1}^{\infty} b_j(x).$$

The convergence of the infinite product follows from the bound (3.10), which will be obtained below. Observe that in the process of selection of the numbers (2.2) we were free to define $\delta_k > 0$ as small as needed. Besides, taking $u_k$ to be close to 1 we may get $n_k$ as big as needed. Using these notations and Lemma 3.1, aside of the conditions (2.2), (2.3) and (3.6) we can additionally claim the bounds

$$\omega(2\pi/n_k, B_{k-1}) = \sup_{|x-x'| < 2\pi/n_k} |B_{k-1}(x) - B_{k-1}(x')| < 2^{-k}, \tag{3.8}$$

$$|b_k(x) + 1| < 2^{-k}, \quad x \in U(n_k, 6\delta_k), \tag{3.9}$$

$$|b_k(x) - 1| < 2^{-k}, \quad x \in \mathbb{T} \setminus U(n_k, \sqrt[4]{\delta_k}). \tag{3.10}$$

From (3.10) we get

$$|B(x) - B_k(x)| = \left| \prod_{j \geq k+1} b_j(x) - 1 \right|$$

$$\leq \prod_{j \geq k+1} (1 + 2^{-j}) - 1 < 2^{-k+1},$$

$$x \in \mathbb{T} \setminus \bigcup_{j \geq k+1} U\left(n_j, \sqrt[4]{\delta_j}\right). \qquad (3.11)$$

Take an arbitrary $x \in \mathbb{T}$. There exists an integer $1 \leq j_0 \leq n_k$ such that

$$\frac{2\pi j_0}{n_k} - x \in \left[\frac{2\pi}{n_k}, \frac{4\pi}{n_k}\right] \subset \left[\frac{2\pi}{n_k}, \frac{5\pi}{n_k}\right] \subset \left[\frac{\lambda(u_k)}{3}, \lambda(u_k)\right] \subset [\lambda(v_k), \lambda(u_k)],$$

where the inclusions follow from the definition of $n_k$ (see (3.7)) and from the inequality $3\lambda(v_k) \leq \lambda(u_k) < \pi$ coming from (2.2). Thus since $\lambda(r)$ is continuous, we may find numbers $u_k \leq r'_k \leq r''_k \leq v_k$, such that

$$\lambda(r'_k) = \frac{2\pi j_0}{n_k} - x, \quad \lambda(r''_k) = \frac{2\pi j_0}{n_k} + \frac{\pi}{n_k} - x. \qquad (3.12)$$

For the set

$$e = \bigcup_{j \geq k+1} U\left(n_j, \sqrt[4]{\delta_j}\right),$$

we have

$$|e| = 10\pi \sum_{j \geq k+1} \sqrt[4]{\delta_j}.$$

To simplify the further estimations, without loss of generality we may replace the relation $\|\phi_r\|_1 \to 1$ by $\|\phi_r\|_1 = 1$. So taking $r \in [u_k, v_k]$, from (3.6) and (3.11) we conclude

$$\left| \Phi_r(x, B) - \Phi_r(x, B_k) \right|$$

$$\leq \int_e \varphi_r(x - t)|B(t) - B_k(t)|dt + 2^{-k+1} \int_{\mathbb{T} \setminus e} \varphi_r(x - t)dt$$

$$\leq 20\pi m(v_k) \sum_{j \geq k+1} \sqrt[4]{\delta_j} + 2^{-k+1}$$

$$\leq 2 \cdot 2^{-k} + 2^{-k+1} = 4 \cdot 2^{-k}, \quad x \in \mathbb{T}. \tag{3.13}$$

If

$$t \in I = (-\delta_k \lambda(u_k), \delta_k \lambda(u_k)) \subset \left( -\frac{6\pi\delta_k}{n_k}, \frac{6\pi\delta_k}{n_k} \right),$$

then we have

$$\frac{2\pi j_0}{n_k} - t \in U(n_k, 6\delta_k),$$

$$\frac{2\pi j_0}{n_k} + \frac{\pi}{n_k} - t \in \mathbb{T} \setminus U(n_k, \sqrt[4]{\delta_k}).$$

Then, using these relations together with (3.9) and (3.8), we get

$$\left| B_k\left( \frac{2\pi j_0}{n_k} - t \right) + B_{k-1}\left( \frac{2\pi j_0}{n_k} \right) \right|$$

$$\leq \left| B_{k-1}\left( \frac{2\pi j_0}{n_k} - t \right) \right| \left| b_k\left( \frac{2\pi j_0}{n_k} - t \right) + 1 \right|$$

$$+ \left| B_{k-1}\left( \frac{2\pi j_0}{n_k} - t \right) - B_{k-1}\left( \frac{2\pi j_0}{n_k} \right) \right|$$

$$< 2^{-k} + 2^{-k} = 2^{-k+1} \tag{3.14}$$

and

$$\left| B_k\left( \frac{2\pi j_0}{n_k} + \frac{\pi}{n_k} - t \right) - B_{k-1}\left( \frac{2\pi j_0}{n_k} \right) \right|$$

$$\leq \left| B_{k-1}\left( \frac{2\pi j_0}{n_k} + \frac{\pi}{n_k} - t \right) \right| \left| b_k\left( \frac{2\pi j_0}{n_k} + \frac{\pi}{n_k} - t \right) - 1 \right|$$

$$+ \left| B_{k-1}\left( \frac{2\pi j_0}{n_k} + \frac{\pi}{n_k} - t \right) - B_{k-1}\left( \frac{2\pi j_0}{n_k} \right) \right|$$

$$< 2^{-k} + 2^{-k} = 2^{-k+1}. \tag{3.15}$$

On the other hand we have

$$\left| \Phi_{r'_k}(x + \lambda(r'_k), B_k) + B_{k-1}\left(\frac{2\pi j_0}{n_k}\right) \right|$$

$$= \left| \int_{\mathbb{T}} \varphi_{r'_k}(t) B_k(x + \lambda(r'_k) - t)dt + B_{k-1}\left(\frac{2\pi j_0}{n_k}dt\right) \right|$$

$$= \left| \int_{\mathbb{T}} \varphi_{r'_k}(t)\left[ B_k\left(\frac{2\pi j_0}{n_k} - t\right) + B_{k-1}\left(\frac{2\pi j_0}{n_k}\right) \right]dt \right|$$

$$\leq \left| \int_{I} \varphi_{r'_k}(t)\left[ B_k\left(\frac{2\pi j_0}{n_k} - t\right) + B_{k-1}\left(\frac{2\pi j_0}{n_k}\right) \right]dt \right|$$

$$+ \left| \int_{I^c} \varphi_{r'_k}(t)\left[ B_k\left(\frac{2\pi j_0}{n_k} - t\right) + B_{k-1}\left(\frac{2\pi j_0}{n_k}\right) \right]dt \right|$$

$$= A + B. \tag{3.16}$$

Then from (3.14) we get

$$A \leq 2^{-k+1} \int_{I} \varphi_{r'_k}(t)dt \leq 2^{-k+1}. \tag{3.17}$$

By the hypotheses of Theorem 1.3 we have $\beta^* = 1$. So from (2.3) we get

$$\int_{I} \varphi_{r'_k}(t) = \int_{-\delta_k \lambda(u_k)}^{\delta_k \lambda(u_k)} \varphi_r(t)dt > 1 - 2^{-k}$$

and therefore

$$\int_{I^c} \varphi_{r'_k}(t) \leq \|\varphi_{r'_k}\|_1 - 1 + 2^{-k} = 2^{-k}. \tag{3.18}$$

From (3.12) and (3.18) we get

$$B \leq 2 \cdot 2^{-k}. \tag{3.19}$$

Thus, using (3.16), (3.17) and (3.19), we obtain

$$\left| \Phi_{r'_k}(x + \lambda(r'_k), B_k) + B_{k-1}\left(\frac{2\pi j_0}{n_k}\right) \right| \leq 4 \cdot 2^{-k}. \tag{3.20}$$

Similarly, using (3.15), we conclude

$$\left|\Phi_{r_k''}(x+\lambda(r_k''),B_k)-B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)\right|\le 4\cdot 2^{-k}. \qquad (3.21)$$

Since $|B_{k-1}(2\pi j_0/n_k)|=1$, from (3.13), (3.20) and (3.21) it follows that

$$\left|\Phi_{r_k'}(x+\lambda(r_k'),B)-\Phi_{r_k''}(x+\lambda(r_k''),B)\right|$$

$$=\Bigg| -2B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)$$

$$+\Phi_{r_k'}(x+\lambda(r_k'),B_k)+B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)$$

$$-\Phi_{r_k''}(x+\lambda(r_k''),B_k)+B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)$$

$$+\Phi_{r_k'}(x+\lambda(r_k'),B)-\Phi_{r_k'}(x+\lambda(r_k'),B_k)$$

$$+\Phi_{r_k''}(x+\lambda(r_k''),B_k)-\Phi_{r_k''}(x+\lambda(r_k''),B)\Bigg|$$

$$\ge 2-4\cdot 2^{-k}-4\cdot 2^{-k}-4\cdot 2^{-k}-4\cdot 2^{-k}$$

$$=2-16\cdot 2^{-k},$$

which implies the divergence of $\Phi_r(x+\lambda(r),B)$ at a point $x$. $\qquad\square$

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