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Carleson inequalities on parabolic Hardy spaces

Hayato Nakagawa and Noriaki Suzuki

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Abstract. We study Carleson inequalities in a framework of parabolic Hardy spaces. Similar results for parabolic Bergman spaces are discussed in [NSY1] (see also [NSY2]), where τ-Carleson measures play an important roll. In the present case, T_τ-Carleson measures are useful. We give an relation between these measures.

Key words: Carleson inequality, parabolic operator, Hardy space, Carleson measure.

1. Introduction

For an integer $n \geq 1$, let $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^{n+1} \mid x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, t > 0\}$ denote the upper half space. For $0 < \alpha \leq 1$, let $L^{(\alpha)}$ be a parabolic operator

$$L^{(\alpha)} := \partial_t + (-\Delta_x)^\alpha, \quad \Delta_x := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We say that a continuous function u on $\mathbb{R}^{n+1}_+$ is an $L^{(\alpha)}$-harmonic function if $L^{(\alpha)}u = 0$ in the sense of distributions, which is defined later.

For $1 < p < \infty$, we denote by $h^p_\alpha := h^p_\alpha(\mathbb{R}^{n+1}_+)$ the set of all $L^{(\alpha)}$-harmonic functions with $\|u\|_{h^p_\alpha} < \infty$, where

$$\|u\|_{h^p_\alpha} := \sup_{t>0} \left(\int_{\mathbb{R}^n} |u(x,t)|^p dx \right)^{1/p}.$$

It is shown that h^p_α is a Banach space under the norm $\|\cdot\|_{h^p_\alpha}$ (see Section 2). We call h^p_α the α-parabolic Hardy space of order p.

Let $1 < p < \infty$ and $1 < q < \infty$. We say that a positive Borel measure μ on $\mathbb{R}^{n+1}_+$ satisfies a (p,q)-Carleson inequality on parabolic Hardy spaces if the inclusion mapping from h^p_α to $L^q(\mathbb{R}^{n+1}_+, d\mu)$ is bounded, that is, there exists a positive constant C such that

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$$\|u\|_{L^q(\mathbb{R}^{n+1}_+,\, d\mu)} \le C\|u\|_{h^p_\alpha} \tag{1}$$

holds for all $u \in h^p_\alpha$. To study (1), the following definition is useful.

Definition 1 Let μ be a positive Borel measure on $\mathbb{R}^{n+1}_+$ and τ be a positive number. We say that μ is a T_τ-Carleson measure if there exists a positive constant C such that

$$\mu(T^{(\alpha)}(x,t)) \le Ct^{(n/2\alpha)\tau} \tag{2}$$

holds for all $(x,t) \in \mathbb{R}^{n+1}_+$, where

$$T^{(\alpha)}(x,t) := \left\{(y,s) \in \mathbb{R}^{n+1}_+ \;\middle|\; |y-x|^{2\alpha} + s \le t\right\}. \tag{3}$$

We are now ready to state our main theorem.

Theorem 1 *Let $1 < p \le q < \infty$ and μ be a positive Borel measure on $\mathbb{R}^{n+1}_+$. Then μ satisfies a (p,q)-Carleson inequality if and only if μ is a $T_{q/p}$-Carleson measure.*

A Carleson inequality on parabolic Bergman spaces is already proved in [NSY1] (see also [NSY2]). We discuss a relation between two inequalities in Section 4. As a result, we will see that a positive Borel measure μ satisfies a (p,q)-Carleson inequality on parabolic Hardy spaces if and only if μ satisfies a (p',q')-Carleson inequality on parabolic Bergman spaces, where $(q/p)(n/2\alpha) = (q'/p')(n/2\alpha+1)$ (see Corollary 1 below).

Throughout the paper, we will use the same letter C to denote various positive constants; it may vary even within a line.

2. Preliminaries

In order to define an $L^{(\alpha)}$-harmonic function, we recall how the adjoint operator $\widetilde{L}^{(\alpha)} = -\partial_t + (-\Delta_x)^\alpha$ acts on $C^\infty_c(\mathbb{R}^{n+1}_+)$, where $C^\infty_c(\mathbb{R}^{n+1}_+)$ is the set of all C^∞-functions with compact support on $\mathbb{R}^{n+1}_+$. Since it is trivial when $\alpha = 1$, we only consider for $0 < \alpha < 1$ here. Then $(-\Delta_x)^\alpha$ is the convolution operator defined by $-c_{n,\alpha}\,\mathrm{p.f.}|x|^{-n-2\alpha}$, where

$$c_{n,\alpha} = 4^\alpha \pi^{-n/2}\Gamma((2n+\alpha)/2)/|\Gamma(-\alpha)|, \quad |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2},$$

and $\Gamma(\cdot)$ is the gamma function. Hence for $\varphi \in C_c^\infty(\mathbb{R}^{n+1}_+)$,

$$\widetilde{L}^{(\alpha)}\varphi(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) - c_{n,\alpha}\lim_{\delta\downarrow 0}\int_{|y|>\delta}(\varphi(x+y,t)-\varphi(x,t))|y|^{-n-2\alpha}dy.$$

A function h on $\mathbb{R}^{n+1}_+$ is said to be $L^{(\alpha)}$-harmonic if h is continuous,

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n}|h(x,t)|(1+|x|)^{-n-2\alpha}dxdt < \infty \tag{4}$$

for every $0 < t_1 < t_2 < \infty$ and $\iint_{\mathbb{R}^{n+1}_+} h \cdot \widetilde{L}^{(\alpha)}\varphi\, dxdt = 0$ holds for all $\varphi \in C_c^\infty(\mathbb{R}^{n+1}_+)$. Note that the condition (4) is equivalent to $\iint_{\mathbb{R}^{n+1}_+} |h \cdot \widetilde{L}^{(\alpha)}\varphi|\, dxdt < \infty$ for all $\varphi \in C_c^\infty(\mathbb{R}^{n+1}_+)$ (see [NSS1]).

We use the fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$, which is defined by

$$W^{(\alpha)}(x,t) = \begin{cases} (2\pi)^{-n}\displaystyle\int_{\mathbb{R}^n} e^{-t|\xi|^{2\alpha}}e^{ix\cdot\xi}\,d\xi & t>0 \\ 0 & t \le 0 \end{cases}$$

where $x\cdot\xi$ is the inner product of x and ξ. It is known that when $\alpha = 1/2$, $W^{(1/2)}$ coincides with the Poisson kernel on $\mathbb{R}^{n+1}_+$, that is, for $t > 0$,

$$W^{(1/2)}(x,t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}\frac{t}{(|x|^2+t^2)^{(n+1)/2}}. \tag{5}$$

Note also that $W^{(1)}(x,t) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$ is the Gauss kernel. We recall some properties of the fundamental solution (see [NSS1] and [NSS2] for details).

For any compact set K in $\mathbb{R}^{n+1}_+$, there exists a positive constant C such that

$$\inf_{(x,t)\in K} W^{(\alpha)}(x,t) > C. \tag{6}$$

For every positive t,

$$\int_{\mathbb{R}^n} W^{(\alpha)}(x,t)\,dx = 1 \tag{7}$$

and for any positive s, t with $0 < s < t$,

$$W^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} W^{(\alpha)}(x-y, t-s) W^{(\alpha)}(y,s)\, dy. \tag{8}$$

By a change of variables, we see

$$W^{(\alpha)}(x,t) = t^{-n/2\alpha} W^{(\alpha)}(t^{-1/2\alpha}x, 1). \tag{9}$$

The following estimate is useful: there exists a constant $C > 0$ such that

$$W^{(\alpha)}(x,t) \le C \frac{t}{(t+|x|^{2\alpha})^{n/2\alpha+1}}. \tag{10}$$

It is known that the usual harmonic Hardy space H^p on the upper half space is naturally equivalent with the space $L^p(\mathbb{R}^n)$. The same identity also holds in our case. For $f \in L^p(\mathbb{R}^n)$, we set

$$P^{(\alpha)}[f](x,t) := \int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t) f(y) dy. \tag{11}$$

Then we have the following proposition.

Proposition 1 *Let $1 < p < \infty$. For each $u \in h^p_\alpha$, there exists a unique function $f \in L^p(\mathbb{R}^n)$ such that $u = P^{(\alpha)}[f]$. Conversely, for any $f \in L^p(\mathbb{R}^n)$, $P^{(\alpha)}[f] \in h^p_\alpha$. Moreover, $\|P^{(\alpha)}[f]\|_{h^p_\alpha} = \|f\|_{L^p(\mathbb{R}^n)}$ holds.*

Proof. By (7), (8) and (10), the assertion follows from a quite similar proof to the usual harmonic Hardy space (cf. [S, p. 62 and p. 200]). Here we only check that $P^{(\alpha)}[f]$ is $L^{(\alpha)}$-harmonic when $f \in L^p(\mathbb{R}^n)$. Let q be such as $1/p + 1/q = 1$. For $0 < t_1 < t_2 < \infty$, by (10) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |P^{(\alpha)}[f](x,t)| (1+|x|)^{-n-2\alpha} dxdt \\
&\le \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (W^{(\alpha)}(x-y,t))^q dy \right)^{1/q} \\
&\quad \times \|f\|_{L^p(\mathbb{R}^n)} (1+|x|)^{-n-2\alpha} dxdt
\end{aligned}$$

$$\le C\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{t^q}{(|x-y|^{2\alpha}+t)^{(n/2\alpha+1)q}}dy\right)^{1/q}$$
$$\times\|f\|_{L^p(\mathbb{R}^n)}(1+|x|)^{-n-2\alpha}dxdt$$
$$\le C\left(\frac{\omega_{n-1}}{2\alpha}\int_0^\infty \zeta^{n/2\alpha-1}(1+\zeta)^{-(n/2\alpha+1)q}d\zeta\right)^{1/q}$$
$$\times\|f\|_{L^p(\mathbb{R}^n)}\int_{t_1}^{t_2}t^{-(n/2\alpha)(1/p)}dt\int_{\mathbb{R}^n}(1+|x|)^{-n-2\alpha}dx$$
$$<\infty$$

where ω_{n-1} is the surface area of unit sphere in $\mathbb{R}^n$. Hence by the Fubini theorem,

$$\iint_{\mathbb{R}_+^{n+1}} P^{(\alpha)}[f](x,t)\cdot\widetilde{L}^{(\alpha)}\varphi(x,t)\,dxdt$$
$$=\iint_{\mathbb{R}_+^{n+1}}\left(\int_{\mathbb{R}^n}W^{(\alpha)}(x-y,t)f(y)\,dy\right)\widetilde{L}^{(\alpha)}\varphi(x,t)\,dxdt$$
$$=\int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}_+^{n+1}}W^{(\alpha)}(x-y,t)\widetilde{L}^{(\alpha)}\varphi(x,t)\,dxdt\right)f(y)\,dy$$
$$=0,$$

because the fundamental solution is $L^{(\alpha)}$-harmonic. □

When $u=P_\alpha[f]$, then $|u(x,t)|\le\|W(x-\cdot,t)\|_{L^q(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}$ holds for $1/p+1/q=1$. This shows that a Cauchy sequence $\{u_n\}$ in h^p_α implies compact uniform convergence of $\{u_n\}$, so that h^p_α forms a Banach space. This proposition also shows that $h^p_{1/2}$ is just the usual harmonic Hardy space H^p.

3. Proof of Theorem 1

In this section, we will give a proof of Theorem 1. The "only if" part is not difficult. It follows from direct computations of integrals of $W^{(\alpha)}$. Let $1<p<\infty$, $1<q<\infty$ and suppose that μ satisfies a Carleson inequality (1). We fix $(x,t)\in\mathbb{R}_+^{n+1}$ and put $u(y,s)=W^{(\alpha)}(x-y,t+s)$. Then $u\in h^p_\alpha$.

In fact, by (10),

$$\begin{aligned}\|u\|_{h^p_\alpha}^p &= \sup_{s>0}\int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t+s)^p dy \\ &\le C\sup_{s>0}\int_{\mathbb{R}^n}\left(\frac{t+s}{(t+s+|x-y|^{2\alpha})^{n/2\alpha+1}}\right)^p dy \\ &= C\frac{\omega_{n-1}}{2\alpha}\sup_{s>0}(t+s)^{(n/2\alpha)(1-p)}\int_0^\infty \frac{\eta^{n/2\alpha-1}}{(1+\eta)^{(n/2\alpha+1)p}}d\eta \\ &\le Ct^{(n/2\alpha)(1-p)}\end{aligned}$$

On the other hand, since $|x-y|^{2\alpha}\le t-s<t+s$ for every $(y,s)\in T^{(\alpha)}(x,t)$, (6) and (9) show

$$u(y,s)=(t+s)^{-n/2\alpha}W^{(\alpha)}\left(\frac{x-y}{(t+s)^{1/2\alpha}},1\right)\ge C(t+s)^{-n/2\alpha}\ge C(2t)^{-n/2\alpha}$$

with some constant $C>0$. This implies that

$$\|u\|^q_{L^q(\mathbb{R}^{n+1}_+,d\mu)}\ge \iint_{T^{(\alpha)}(x,t)} u(y,s)^q d\mu(y,s)\ge Ct^{-(n/2\alpha)q}\mu(T^{(\alpha)}(x,t)).$$

Hence the inequality (1) gives us

$$t^{-n/2\alpha}\mu(T^{(\alpha)}(x,t))^{1/q}\le C\|u\|_{L^q(\mathbb{R}^{n+1}_+,d\mu)}\le C\|u\|_{h^p_\alpha}\le Ct^{(n/2\alpha)(1/p-1)},$$

which shows μ is a $T_{q/p}$-Carleson measure. Here we remark that we do not assume $p\le q$ in the above argument.

To show the "if" part, we use a Luecking's idea (see [L]). In the sequel, we denote by $B(x,r)$ the ball with center x and radius r in the boundary of upper half space, that is $B(x,r)=\{y\in\mathbb{R}^n \mid |x-y|<r\}$. For an open set E in $\mathbb{R}^n$, we set

$$\widehat{E}:=\{(x,t)\in\mathbb{R}^{n+1}_+ \mid B(x,t^{1/2\alpha})\subseteq E\}. \tag{12}$$

Let $(x,t)\in\mathbb{R}^{n+1}_+$. When $\alpha\le 1/2$, then $(t-s)^{1/2\alpha}\le t^{1/2\alpha}-s^{1/2\alpha}\le(2t-s)^{1/2\alpha}$ holds for $0<s\le t$. Hence $T^{(\alpha)}(x,t)\subset B\widehat{(x,t^{1/2\alpha})}\subset T^{(\alpha)}(x,2t)$.

When $\alpha > 1/2$, then $t^{1/2\alpha} - s^{1/2\alpha} \le (t-s)^{1/2\alpha} \le (2t)^{1/2\alpha} - s^{1/2\alpha}$ holds for $0 < s \le t$, and hence $\widehat{B(x, t^{1/2\alpha})} \subset T^{(\alpha)}(x,t) \subset \widehat{B(x, (2t)^{1/2\alpha})}$. Therefore

$$\widehat{B(x, (t/2)^{1/2\alpha})} \subset T^{(\alpha)}(x,t) \subset \widehat{B(x, (2t)^{1/2\alpha})} \tag{13}$$

holds for all $0 < \alpha \le 1$.

Let $1 < p < \infty$. We also use the maximal function Mf, which is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|\, dy.$$

for $f \in L^p(\mathbb{R}^n)$. It is known that

$$\|Mf\|_{L^p(\mathbb{R}^n)} \le C_{p,n} \|f\|_{L^p(\mathbb{R}^n)} \tag{14}$$

where $C_{p,n} = 2(5^n p/(p-1))^{1/p}$ (see [S, p. 5]).

Now we return to the proof of the "if" part. Assume that $1 < p \le q < \infty$ and μ is a $T_{q/p}$-Carleson measure. Then by (13),

$$\mu(\widehat{B(x, t^{1/2\alpha})}) \le C t^{(n/2\alpha)(q/p)} \tag{15}$$

with some constant $C > 0$. We use the following notations. For $u \in h^p_\alpha$ and $x \in \mathbb{R}^n$, we set

$$u^*(x) := \sup_{(y,s)\in\Omega(x)} |u(y,s)| \tag{16}$$

where $\Omega(x) := \{(y,s) \in \mathbb{R}^{n+1}_+ \mid |y-x| < s^{1/2\alpha}\}$ and for $\lambda > 0$, we set

$$E_\lambda := \{x \in \mathbb{R}^n \mid u^*(x) > \lambda\},$$
$$G_\lambda := \{(x,t) \in \mathbb{R}^{n+1}_+ \mid |u(x,t)| > \lambda\}.$$

Let $(x_0, t_0) \in G_\lambda$ and take any $z \in B(x_0, t_0^{1/2\alpha})$. Since $(x_0,t_0) \in \Omega(z)$, we have $u^*(z) > \lambda$, and hence $B(x_0, t_0^{1/2\alpha}) \subset E_\lambda$. This shows

$$G_\lambda \subset \widehat{E_\lambda}. \tag{17}$$

For subsets X and Y in $\mathbb{R}^n$, we denote by ∂X the boundary of X, by $\operatorname{diam}(X)$ the diameter of X and by $\operatorname{dist}(X,Y)$ the distance between X and Y. Since u^* is lower semicontinuous, E_λ is an open set. Hence we have the following Whitney decomposition;

$$E_\lambda = \bigcup_{k=1}^{\infty} Q_k, \tag{18}$$

where $\{Q_k\}$ are closed cubes whose sides are parallel to the axes and whose interiors are mutually disjoint, and satisfy

$$\operatorname{diam}(Q_k) \le \operatorname{dist}(Q_k, \partial E_\lambda) \le 4\operatorname{diam}(Q_k)$$

(see [S, p. 16]). Then there exists a constant $C > 0$ such that

$$\widehat{E_\lambda} \subset \bigcup_{k=1}^{\infty} \widehat{CQ_k} \tag{19}$$

where CQ_k is a cube with C times diameter and the common center as Q_k. In fact, take $(x,t) \in \widehat{E_\lambda}$. Since $B(x, t^{1/2\alpha}) \subset E_\lambda$, we choose $t_0 \ge t$ such that $\operatorname{dist}(B(x, t_0^{1/2\alpha}), \partial E_\lambda) = 0$. By (18), $x \in Q_{k_0}$ for some integer $k_0 \ge 1$. Let x_0 be the center of Q_{k_0} and $\widetilde{x}_0$ be a point in Q_{k_0} such that $\operatorname{dist}(\widetilde{x}_0, \partial E_\lambda) = \operatorname{dist}(Q_{k_0}, \partial E_\lambda)$. Then for any $y \in B(x, t_0^{1/2\alpha})$, we have

$$\begin{aligned}
|y - x_0| &\le |y - x| + |x - x_0| \\
&\le t_0^{1/2\alpha} + \operatorname{diam}(Q_{k_0}) \\
&\le |x - \widetilde{x}_0| + \operatorname{dist}(Q_{k_0}, \partial E_\lambda) + \operatorname{diam}(Q_{k_0}) \\
&\le \operatorname{diam}(Q_{k_0}) + 4\operatorname{diam}(Q_{k_0}) + \operatorname{diam}(Q_{k_0}) \\
&= 6\operatorname{diam}(Q_{k_0}).
\end{aligned}$$

This shows that $y \in CQ_{k_0}$, where $C = 12\sqrt{n}$. Since y is an arbitary point in $B(x, t_0^{1/2\alpha})$, we have $B(x, t^{1/2\alpha}) \subset B(x, t_0^{1/2\alpha}) \subset CQ_{k_0}$, and hence $(x,t) \in \widehat{CQ_{k_0}}$. This shows (19).

Next we estimate the L^p norm of u^*. There exists a positive constant C such that for every $u \in h^p_\alpha$,

$$\|u^*\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{h^p_\alpha}. \tag{20}$$

In fact, as in Proposition 1, we take $f \in L^p(\mathbb{R}^n)$ such that $u = P^{(\alpha)}[f]$ and let $x \in \mathbb{R}^n$. Take $(y,s) \in \Omega(x)$ and $z \in \mathbb{R}^n$ arbitrarily. Then

$$\begin{aligned} s + |x-z|^{2\alpha} &\leq s + (|x-y| + |y-z|)^{2\alpha} \leq s + (s^{1/2\alpha} + |y-z|)^{2\alpha} \\ &\leq (2^{2\alpha}+1)(s + |y-z|^{2\alpha}). \end{aligned}$$

Hence by (5), we have

$$\begin{aligned} |u(y,s)| &\leq \int_{\mathbb{R}^n} |f(z)| W^{(\alpha)}(y-z,s)\, dz \\ &\leq C \int_{\mathbb{R}^n} \frac{s|f(z)|}{(s+|y-z|^{2\alpha})^{n/2\alpha+1}}\, dz \\ &\leq C \int_{\mathbb{R}^n} \frac{s|f(z)|}{(s+|x-z|^{2\alpha})^{n/2\alpha+1}}\, dz \\ &= C \sum_{m=0}^{\infty} I_m, \end{aligned}$$

where

$$\begin{aligned} I_0 &:= \int_{|x-z|<s^{1/2\alpha}} \frac{s|f(z)|}{(s+|x-z|^{2\alpha})^{n/2\alpha+1}}\, dz \\ I_m &:= \int_{(2^{m-1}s)^{1/2\alpha} \leq |x-z| < (2^m s)^{1/2\alpha}} \frac{s|f(z)|}{(s+|x-z|^{2\alpha})^{n/2\alpha+1}}\, dz \\ &\qquad (m = 1, 2, \ldots). \end{aligned}$$

Then

$$I_0 \leq \int_{|x-z|<s^{1/2\alpha}} \frac{s|f(z)|}{s^{n/2\alpha+1}}\, dz = s^{-n/2\alpha} \int_{|x-z|<s^{1/2\alpha}} |f(z)|\, dz \leq Mf(x)$$

and

$$\begin{aligned} I_m &= \int_{(2^{m-1}s)^{1/2\alpha}\le|x-z|<(2^m s)^{1/2\alpha}} \frac{s|f(z)|}{(s+2^{m-1}s)^{n/2\alpha+1}}\,dz \\ &\le \frac{1}{(1+2^{m-1})^{n/2\alpha+1}}\frac{1}{s^{n/2\alpha}}\int_{|x-z|<(2^m s)^{1/2\alpha}}|f(z)|\,dz \\ &\le 2^{-m}2^{(n/2\alpha+1)}(2^m s)^{-n/2\alpha}\int_{|x-z|<(2^m s)^{1/2\alpha}}|f(z)|\,dz \\ &\le 2^{-m}2^{(n/2\alpha+1)}Mf(x) \end{aligned}$$

implies

$$\sum_{m=1}^{\infty} I_m \le 2^{(n/2\alpha+1)}Mf(x),$$

and hence $|u(y,s)| \le C\,Mf(x)$ holds. Since $(y,s)\in\Omega(x)$ is arbitrary, we have $u^*(x)\le CMf(x)$ for all $x\in\mathbb{R}^n$. This and (14) show (20).

Now we will finish the proof of the "if" part. By (15), we see $\mu(\widehat{Q}) \le C|Q|^{q/p}$ for every cube Q, and hence (17), (18) and (19) show

$$\mu(G_\lambda)\le\mu(\widehat{E_\lambda})\le\sum_{k=1}^{\infty}\mu(\widehat{CQ_k})\le C\sum_{k=1}^{\infty}|Q_k|^{q/p}\le C|E_\lambda|^{q/p}, \tag{21}$$

because $q/p\ge 1$, where $|G|$ denotes the volume of a Borel set G in $\mathbb{R}^n$. Then by (21)

$$\begin{aligned} \|u\|^q_{L^q(\mathbb{R}^{n+1}_+,d\mu)} &= \iint_{\mathbb{R}^{n+1}_+}|u(x,t)|^q\,d\mu(x,t) = q\int_0^\infty \mu(G_\lambda)\lambda^{q-1}d\lambda \\ &\le C\int_0^\infty |E_\lambda|^{q/p}\lambda^{q-1}d\lambda = C\sum_{k=-\infty}^{\infty}\int_{2^k}^{2^{k+1}}|E_\lambda|^{q/p}\lambda^{q-1}d\lambda \\ &\le C\sum_{k=-\infty}^{\infty}\int_{2^k}^{2^{k+1}}|E_{2^k}|^{q/p}2^{(k+1)(q-1)}d\lambda \\ &\le C\sum_{k=-\infty}^{\infty}2^{(k+1)q}|E_{2^k}|^{q/p}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\|u^*\|^p_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |u^*(x)|^p dx = p\int_0^\infty |E_\lambda|\lambda^{p-1}d\lambda \\ &\geq p\sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^k} |E_{2^k}|2^{(k-1)(p-1)}d\lambda \\ &= p\sum_{k=-\infty}^{\infty} 2^{(k-1)p}|E_{2^k}| = \frac{p}{2^{2p}}\sum_{k=-\infty}^{\infty} 2^{(k+1)p}|E_{2^k}|.\end{aligned}$$

Since $p \leq q$, we have

$$\begin{aligned}\|u\|^p_{L^q(\mathbb{R}^{n+1}_+,d\mu)} &\leq C\Big(\sum_{k=-\infty}^{\infty} 2^{(k+1)q}|E_{2^k}|^{q/p}\Big)^{p/q} \\ &\leq C\sum_{k=-\infty}^{\infty} 2^{(k+1)p}|E_{2^k}| \leq C\|u^*\|^p_{L^p(\mathbb{R}^n)}.\end{aligned}$$

This together with (20) gives us the Carleson inequality (1). This completes the proof.

4. A relation between T_τ-Carleson measures and τ-Carleson measures

We recall a result for parabolic Bergman spaces. For $1 \leq p < \infty$, we denote by $b^p_\alpha := b^p_\alpha(\mathbb{R}^{n+1}_+)$ the set of all $L^{(\alpha)}$-harmonic functions u with $\|u\|_{L^p(\mathbb{R}^{n+1}_+)} < \infty$, where

$$\|u\|_{L^p(\mathbb{R}^{n+1}_+)} := \Big(\iint_{\mathbb{R}^{n+1}_+} |u(x,t)|^p dxdt\Big)^{1/p}.$$

We call b^p_α the α-parabolic Bergman space of order p. As in the Hardy case, $b^p_{1/2}$ coincides with the usual harmonic Bergman space on the upper half space. Let μ be a positive Borel measure on $\mathbb{R}^{n+1}_+$ and τ be a positive number. We say that μ is a τ-Carleson measure if there exists a positive constant C such that

$$\mu(Q^{(\alpha)}(x,t)) \le Ct^{(n/2\alpha+1)\tau} \tag{22}$$

holds for all $(x,t) \in \mathbb{R}^{n+1}_+$, where

$$Q^{(\alpha)}(x,t) := \big\{(y_1, y_2, \dots, y_n, s) \in \mathbb{R}^{n+1}_+ \;\big|\; t \le s \le 2t,\\ |y_i - x_i| \le t^{1/2\alpha}/2, i = 1, 2, \dots, n\big\}.$$

Carleson inequalities on parabolic Bergman spaces are studied in [NSY1]: Let $1 \le p \le q < \infty$ and μ be a positive Borel measure on $\mathbb{R}^{n+1}_+$. Then μ is a q/p-Carleson measure if and only if there exists a positive constant C such that the inequality

$$\|u\|_{L^q(\mathbb{R}^{n+1}_+,\, d\mu)} \le C\|u\|_{L^p(\mathbb{R}^{n+1}_+)} \tag{23}$$

holds for all $u \in b^p_\alpha$.

We have the following proposition.

Proposition 2 *Let μ be a positive Borel measure on $\mathbb{R}^{n+1}_+$. For $\tau > 0$, we set $\tau_b := \tau(n/2\alpha)/(n/2\alpha + 1)$. Then*

(a) *if μ is a T_τ-Carleson measure, then μ is a τ_b-Carleson measure,*
(b) *if $\tau > 1$ and μ is a τ_b-Carleson measure, then μ is a T_τ-Carleson measure.*

Proof. Since $Q^{(\alpha)}(x,t) \subset T^{(\alpha)}(x, ((n/4)^\alpha + 2)t)$ for $(x,t) \in \mathbb{R}^{n+1}_+$, we have

$$\mu(Q^{(\alpha)}(x,t)) \le \mu\bigg(T^{(\alpha)}\bigg(x, \bigg(\bigg(\frac{n}{4}\bigg)^\alpha + 2\bigg)t\bigg)\bigg)\\ \le C\bigg(\bigg(\frac{n}{4}\bigg)^\alpha + 2\bigg)^{(n/2\alpha)\tau} t^{(n/2\alpha)\tau} \le Ct^{(n/2\alpha+1)\tau_b},$$

which shows (a). To show (b), we set

$$T_k := \big\{(y,s) \in T^{(\alpha)}(x,t) \;\big|\; t/2^{k+1} \le s \le t/2^k\big\}$$

for $k = 0, 1, 2, \dots$ and take a natural number $c(k)$ such that $(2^{(k+1/2\alpha+1)} + 1)^n \le c(k) \le 2^{((k+1)/2\alpha+2)n}$. Since

$$\left[\frac{(t-t/2^{k+1})^{1/2\alpha}}{(t/2^{k+1})^{1/2\alpha}}\right]+1\le 2^{((k+1)/2\alpha+1)}+1$$

where $[t]$ is the largest integer smaller than or equal to t, we can choose $c(k)$ points $\{(x_{k,i},t/2^{k+1})\}$ in T_k such that $T_k\subset\bigcup_{i=1}^{c(k)}Q^{(\alpha)}(x_{k,i},t/2^{k+1})$. Hence

$$\begin{aligned}\mu(T^{(\alpha)}(x,t))&\le\sum_{k=0}^{\infty}\mu(T_k)\le\sum_{k=0}^{\infty}\sum_{i=1}^{c(k)}\mu(Q^{(\alpha)}(x_{k,i},t/2^{k+1}))\\&\le\sum_{k=0}^{\infty}\sum_{i=1}^{c(k)}C(t/2^{k+1})^{(n/2\alpha+1)\tau_b}\\&\le C\sum_{k=0}^{\infty}2^{((k+1)/2\alpha+2)n}(t/2^{k+1})^{(n/2\alpha+1)\tau_b}\\&\le Ct^{(n/2\alpha)\tau}\sum_{k=0}^{\infty}2^{k(n/2\alpha-(n/2\alpha+1)\tau_b)}.\end{aligned}$$

Since $n/2\alpha-(n/2\alpha+1)\tau_b=(n/2\alpha)(1-\tau)<0$, μ is a T_τ-Carleson measure. □

Theorem 1 and Proposition 2 give us the following corollary.

Corollary 1 $1<p\le q<\infty$ *and let* $1\le p'\le q'<\infty$. *Suppose that* $(n/2\alpha)(q/p)=(n/2\alpha+1)(q'/p')$ *holds. Then for every positive Borel measure* μ *on* $\mathbb{R}^{n+1}_+$, *there exist positive constants* C *and* C' *such that* $\|u\|_{L^q(\mathbb{R}^{n+1}_+,\,d\mu)}\le C\|u\|_{h^p_\alpha}$ *holds for all* $u\in h^p_\alpha$ *if and only if* $\|u\|_{L^{q'}(\mathbb{R}^{n+1}_+,\,d\mu)}\le C'\|u\|_{L^{p'}(\mathbb{R}^{n+1}_+)}$ *holds for all* $u\in b^{p'}_\alpha$.

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Hayato Nakagawa
Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya, 466-8602, Japan
E-mail: m04026b@math.nagoya-u.ac.jp

Noriaki Suzuki
Department of Mathematics
Meijyo University
Tenpaku-ku, Nagoya, 468-8502, Japan
E-mail: suzukin@meijo-u.ac.jp