A tower condition characterizing normality

Lars Kadison

Abstract. We define left relative H-separable tower of rings and continue a study of these begun by Sugano. It is proven that a progenerator extension has right depth 2 if and only if the ring extension together with its right endomorphism ring is a left relative H-separable tower. In particular, this applies to twisted or ordinary Frobenius extensions with surjective Frobenius homomorphism. For example, normality for Hopf subalgebras of finite-dimensional Hopf algebras is also characterized in terms of this tower condition.

Key words: Frobenius extension, H-separable extension, normal subring, induced characters, subring depth, Hopf subalgebra.

1. Introduction and Preliminaries

Depth two is a bimodule condition on subrings that is equivalent to usual notions of normality for subgroups [4], Hopf subalgebras [2] and semisimple complex subalgebra pairs [6]; in addition, depth two is a condition of normality for a ring extension that makes it a Galois extension with respect to a right bialgebroid coaction [16], [18]. The right depth two condition on an algebra extension $A \supseteq B$ is that the natural A-B-bimodule $A \otimes_B A$ is isomorphic to a direct summand of a natural A-B-bimodule $A \oplus \cdots \oplus A$: in symbols this is ${}_A A \otimes_B A_B \oplus * \cong {}_A A_B^n$. If A is a finite-dimensional Hopf algebra and B is a Hopf subalgebra of A, it is shown in [2] that $A \supseteq B$ has right (or left) depth 2 if and only if B is a normal Hopf subalgebra of A (i.e., B is invariant under either the left or right adjoint actions). If A is a finite-dimensional group algebra $\mathbb{C}G$, its module theory is determined by the character theory of G, and the right depth two condition on a group subalgebra $B = \mathbb{C}H$ in A is determined by a matrix inequality condition on the induction-restriction table for the irreducible characters of G and subgroup H; for more on this, depth greater than 2 as well as modular representations, see [5], [6], [4], [7], [8], [9].

For the reasons just given we call a subring $B \subseteq A$ satisfying the right depth 2 condition above a right normal subring (and the ring extension

$A \supseteq B$ a right normal extension), the details appearing in Definition 1.7. We find a characterization of normal Frobenius extensions with surjective Frobenius homomorphism (such as finite Hopf-Galois extensions with surjective trace map [15]), in terms of an old one-sided H-separability condition on a tower of rings $A \supseteq B \supseteq C$ appearing in Sugano's [25, Lemma 1.2]. This condition is interesting and we gather into "Sugano's Theorem" (Theorem 1.8) the results for a tower satisfying this condition in [25], providing a different proof. We show in Section 3 that a ring extension $A \supseteq B$ with the natural module A_B a progenerator, together with its right endomorphism ring $\mathrm{End}\, A_B$, forms a tower satisfying Sugano's condition, called "left relative H-separable," if and only if $A \supseteq B$ is a right normal extension. In Corollary 3.5 it is noted that Theorem 3.2 establishes the same tower characterization of normality for β-Frobenius extensions with surjective Frobenius homomorphism. For example, an arbitrary Hopf subalgebra of a finite-dimensional Hopf algebra is such a twisted Frobenius extension: then Corollary 3.6 characterizes a normal Hopf subalgebra in terms of its right endomorphism algebra. A new proof that right normality is equivalent to left normality for Frobenius extensions with the surjectivity condition is noted in Corollary 3.4.

1.1.   H-separable extensions

A ring extension $A \supseteq B$ is H-separable if $A \otimes_B A \oplus * \cong A^n$ as natural A-bimodules [11]. The notion of H-separability extends certain nice results for Azumaya algebras to ring extensions. For example, the Azumaya isomorphism of the enveloping algebra and the endomorphism algebra is extended for an H-separable ring extension $A \supseteq B$ to a bimodule isomorphism, $A \otimes_B A \cong \mathrm{Hom}(R_Z, A_Z)$ where R is the centralizer of B in A and Z is the center of A [10]. One also shows that $A \supseteq B$ is a separable extension, and if this is additionally split, that R is a separable algebra over Z [10], [25], [14]. Any bimodule M over A has a generalized Azumaya isomorphism $M^A \otimes_Z R \cong M^B$ between the A- and B-centralized elements of M.

H-separable extension theory was one of the motivational models for [16] which extends to ring theory the notion of depth 2 for free Frobenius extensions in [15] (see [16, Examples 3.6, 4.8, 5.8], another toy model being Lu's Hopf algebroids on an algebra). Examples of H-separable extensions come from tensoring Azumaya algebras with other algebras, or looking at certain subalgebras within Azumaya algebras; certainly group algebra and

Hopf algebra extensions are trivial if H-separable, which is true generally [22] but easier to prove with characters (see Proposition 2.6 in this paper). The more general notion of depth two ring extension welcomes examples from Hopf-Galois extensions including normal Hopf subalgebras; indeed, depth two ring extension is equivalent, with one other condition (balanced module), to a Galois extension with bialgebroid coactions [16], [18], where bialgebroid is the good generalization of bialgebra from algebras to ring extensions.

An H-separable extension $A \supseteq B$ is the H-depth $n = 1$ case of odd minimal H-depth $d_H(B, A) = 2n - 1$ where $A^{\otimes_B n} \sim A^{\otimes_B (n+1)}$ as A-bimodules ([20], see below in this section for H-equivalent modules). We show in Propositon 2.2 that a relative separable and H-separable tower $A \supseteq B \supseteq C$ has equality of minimal H-depth, $d_H(B, A) = d_H(C, A)$. We note a different proof of Sugano's theorem 1.8 (cf. [25]) that shows that in such a relative H-separable tower there is a close relation between depth 1 (centrally projective) and H-depth 1 (H-separable) extensions $B \supseteq C$ and $A^C \supseteq A^B$, as well as split and separable extensions.

The following unpublished characterization of H-separable extensions is useful below.

Proposition 1.1 *Let $A \mid B$ be a ring extension. Then $A \mid B$ is H-separable if and only if for each A-module N, its restriction and induction satisfies $\operatorname{Ind}_B^A \operatorname{Res}_B^A N \oplus * \cong N^m$ for some $m \in \mathbb{N}$ via two natural transformations. Consequently, if $A \mid B$ is H-separable and A-modules V, W satisfy $V_B \oplus * \cong W_B$, then $V_A \oplus * \cong W_A^m$ for some $m \in \mathbb{N}$.*

Proof. The second statement follows from the fact that $A \mid B$ is also separable, so that $V \otimes_B A \to V$, $v \otimes a \mapsto va$ is a natural split epi. Note that $V \otimes_B A \oplus * \cong W \otimes_B A$, so that the second statement follows from the characterization in the first statement.

($\Rightarrow$) Since $A \otimes_B A \oplus * \cong A^n$ as A-bimodules, the implication follows from tensoring this by $N \otimes_A -$. Naturality follows from looking more carefully at the mappings, starting with a module homomorphism $g : N_A \to N'_A$. Another characterization of H-separability is that there are elements $e_i \in (A \otimes_B A)^A$ and $r_i \in A^B$ $(i = 1, \ldots, n)$ such that $1 \otimes_B 1 = \sum_i r_i e_i$. For each module M_A, define natural transformations $\tau_M : M \otimes_B A \to M_A^n$ by $\tau_M(m \otimes_B a) = (mr_1 a, \ldots, mr_n a)$, and $\sigma_M : M_A^n \to M \otimes_B A$ by $\sigma_M(m_1, \ldots, m_n) = \sum_i m_i e_i$; note that $\sigma_M \tau_M = \operatorname{id}_{M \otimes_B A}$ and the naturality

commutative square follows readily.

($\Leftarrow$) Let $N = A$ in the hypothesis using natural transformations (as above) σ_N and τ_N. Then there are A-bimodule homomorphisms (from naturality) $\tau_A : A \otimes_B A \to A^m$ and $\sigma_A : A^m \to A \otimes_B A$ such that $\sigma_A(\tau_A(1 \otimes_B 1))) = 1 \otimes_B 1$. Then $\tau_A(1 \otimes_B 1) = (r_1, \ldots, r_m) \in (A^B)^m$ and, denoting the canonical basis of A^m by $\{c_1, \ldots, c_m\}$, $\sigma_A(c_i) = e_i \in (A \otimes_B A)^A$. Since τ_A is a section of σ_A, the equation $1 \otimes_B 1 = \sum_{i=1}^m r_i e_i$ follows; thus $A \mid B$ is H-separable. $\qquad\square$

1.2. Left relative separable ring towers

In this paper a *tower of rings* $A \supseteq B \supseteq C$ is a unital associative ring A with subring B, and C a subring of B, so that $1_C = 1_B = 1_A$, which is denoted by 1. Sugano in [25] defines B to be a *left relative separable extension of C in A* (or briefly, a left relative separable tower) if the B-A-epimorphism $\mu : B \otimes_C A \to A$, defined by $\mu(b \otimes a) = ba$, is split; equivalently, there is a B-central element $e \in (B \otimes_C A)^B$ such that $e^1 e^2 = 1$ (where $e = e^1 \otimes e^2$ is modified Sweedler notation suppressing a finite sum of simple tensors). Similarly one defines B to be a right relative separable extension of C in A by requiring $\mu : A \otimes_C B \to A$ to be a split A-B-epimorphism. The next lemma notes that a separable extension $B \supseteq C$ always give rise to a left and right relative separable extension of C in any over-ring A.

Lemma 1.2 *Let $A \supseteq B \supseteq C$ be a tower of rings. If $B \supseteq C$ is a separable extension, then B is a left and right relative separable extension of C in A. Conversely, if $A \supseteq B$ is a split extension and B is a left or right relative separable extension of C in A, then $B \supseteq C$ is a separable extension.*

Proof. Let $e \in (B \otimes_C B)^B$ satisfy $\mu(e) = e^1 e^2 = 1$, the separability condition on $B \supseteq C$. Then $e \in (B \otimes_C A)^B \cap (A \otimes_C B)^B$ defines mappings $a \mapsto ae$ and $a \mapsto ea$ splitting $\mu_r : A \otimes_C B \to A$ and $\mu_\ell : B \otimes_C A \to A$, respectively.

Suppose $E : {}_B A_B \to {}_B B_B$ is a bimodule projection (equivalently, $E(1) = 1$), and $\mu : B \otimes_C A \to A$ is a split B-A-epimorphism (by $\sigma : A \to B \otimes_C A$). Then $e = \sigma(1)$ is in $(B \otimes_C A)^B$ satisfying $e^1 e^2 = 1$. Note then that $e^1 \otimes_C E(e^2)$ is a separability element for $B \supseteq C$. A similar argument for a right relative separable tower shows that $B \supseteq C$ is separable. $\qquad\square$

For example, suppose $G > H > J$ is a tower of finite groups (i.e., H and J are subgroups of G where $J \subseteq H$). Since group algebra extensions

are always split, the lemma implies that a tower of group algebras $KG \supseteq KH \supseteq KJ$ over a commutative ring K is left or right relative separable if and only if $KH \supseteq KJ$ is separable if and only if $|H : J|1$ is an invertible element in K.

The lemma below provides nontrivial examples of relative separable towers; its proof is easy and therefore omitted.

Lemma 1.3 *Suppose $A \supseteq C$ is a separable extension with separability element $e \in B \otimes_C A$ for some intermediate subring B of A containing C. Then B is a left relative separable extension of C in A.*

Example 1.4 Let K be a commutative ring and $A = M_n(K)$ the full K-algebra of $n \times n$ matrices. Let e_{ij} denote the matrix units $(i, j = 1, \ldots, n)$. Any of the n elements $e_j = \sum_{i=1}^n e_{ij} \otimes_K e_{ji}$ are separability idempotents for A. This is also an example of a (symmetric) Frobenius algebra with trace map $T : A \to K$ having dual bases e_{ij}, e_{ji}.

Let B_1, B_2 denote the upper and lower triangular matrix algebras respectively (both of rank $n(n+1)/2$). Then B_1 and B_2 are right, respectively left, relative separable algebras in $M_n(K)$, since $e_1 \in A \otimes_K B_1 \cap B_2 \otimes_K A$.

1.3. Left relative H-separable ring towers

In the same paper [25], Sugano considers a related condition on a tower of rings $A \supseteq B \supseteq C$, given by the condition on B-A-bimodules,

$$_B B \otimes_C A_A \oplus * \cong {}_B A_A^n, \tag{1}$$

for some $n \in \mathbb{N}$, i.e., $B \otimes_C A$ is isomorphic to a direct summand of $A \oplus \cdots \oplus A$ as natural B-A-bimodules. We define B to be a *left relative H-separable extension of C in A* (or briefly refer to a *left relative H-separable tower*) if it satisfies the condition in (1). Note that if $A = B$ the condition in (1) is that of H-separability of $B \supseteq C$; if $B = C$, the condition becomes trivially satisfied by any ring extension $A \supseteq B$. We note a lemma similar to the one above.

Lemma 1.5 *Let $A \supseteq B \supseteq C$ be a tower of rings. If $B \supseteq C$ is an H-separable extension, then B is a left (and right) relative H-separable extension of C in A.*

Proof. Given the H-separability condition on the natural B-B-bimodules, $B \otimes_C B \oplus * \cong B^m$, we tensor this from the right by the additive functor

$- \otimes_B A_A$. After a cancellation of the type $B \otimes_B A \cong A$, the condition in (1) results. Tensoring similarly by additive functor $A \otimes_B -$ from the category of B-B-bimodules into the category of A-B-bimodules results in the obvious right relative H-separable extension condition ($_A A \otimes_C B_B \oplus * \cong {}_A A_B^n$) on B over C in A. $\qquad \square$

Note that if B is an H-separable extension of C, then B is left and right relative separable and relative H-separable extension of C in any over-ring A, since H-separable extensions are separable extensions [14] (and applying both lemmas).

1.4. Preliminaries on subring normality and depth

Let A be a unital associative ring. The category of right modules over A will be denoted by $\mathcal{M}_A$. Two modules M_A and N_A are *H-equivalent* (or similar) if $M \oplus * \cong N^q$ and $N \oplus * \cong M^r$ for some $r, q \in \mathbb{N}$ (sometimes briefly denoted by $M \sim N$). It is well-known that H-equivalent modules have Morita equivalent endomorphism rings.

Let B be a subring of A (always supposing $1_B = 1_A$). Consider the natural bimodules $_A A_A$, $_B A_A$, $_A A_B$ and $_B A_B$ where the last is a restriction of the preceding, and so forth. Denote the tensor powers of $_B A_B$ by $A^{\otimes_B n} = A \otimes_B \cdots \otimes_B A$ for $n = 1, 2, \ldots$, which is also a natural bimodule over B and A in any one of four ways; set $A^{\otimes_B 0} = B$ which is only a natural B-B-bimodule.

Definition 1.6 If $A^{\otimes_B (n+1)}$ is H-equivalent to $A^{\otimes_B n}$ as X-Y-bimodules, one says $B \subseteq A$ (or $A \supseteq B$) has

- depth $2n + 1$ if $X = B = Y$;
- left depth $2n$ if $X = B$ and $Y = A$;
- right depth $2n$ if $X = A$ and $Y = B$;
- H-depth $2n - 1$ if $X = A = Y$.

(Valid for even depth and H-depth if $n \geq 1$ and for odd depth if $n \geq 0$.) Note that $B \subseteq A$ having depth n implies it has depth $n + 1$. Similarly if $B \subseteq A$ has H-depth $2n - 1$, then it has H-depth $2n + 1$ (and depth $2n$). Define minimum depth $d(B, A)$, and minimum H-depth $d_H(B, A)$ to be the least depth, or H-depth, satisfied by $B \subseteq A$; if $B \subseteq A$ does not have finite depth, equivalently finite H-depth, set $d(B, A) = d_H(B, A) = \infty$.

For example, $B \subseteq A$ has depth 1 iff $_B A_B$ and $_B B_B$ are H-equivalent

[2]. Equivalently, $_B A_B \oplus * \cong {}_B B_B^n$ for some $n \in \mathbb{N}$ [21]. This in turn is equivalent to there being $f_i \in \text{Hom}(_B A_B, {}_B B_B)$ and $r_i \in A^B$ such that $\text{id}_A = \sum_i f_i(-)r_i$, the classical central projectivity condition [23]. In this case, it is easy to show that A is ring isomorphic to $B \otimes_{Z(B)} A^B$ where $Z(B), A^B$ denote the center of B and centralizer of B in A. From this we deduce immediately that a centrally projective ring extension $A \supseteq B$ (equivalently, depth 1 extension) has centers satisfying $Z(B) \subseteq Z(A)$, a condition of Burciu that characterizes depth 1 for a semisimple complex subalgebra pair $B \subseteq A$. Depth 1 subgroups are normal with one other condition on centralizers that depends on the commutative ground ring [3].

For another and important example of depth, the subring $B \subset A$ has right depth 2 iff $_A A_B$ and $_A A \otimes_B A_B$ are similar; equivalently,

$$_A A \otimes_B A_B \oplus * \cong {}_A A_B^n \tag{2}$$

for some $n \in \mathbb{N}$. If $A = KG$ is a group algebra of a finite group G, over a commutative ring K, and $B = KH$ is the group algebra of a subgroup $H < G$, then $B \subseteq A$ has right depth 2 iff H is a normal subgroup of G iff $B \subseteq A$ has left depth 2 [4]; a similar statement is true for a Hopf subalgebra $R \subseteq H$ of finite index and over any field [4]. For this and further reasons mentioned in the first paragraphs of this section we propose the following terminology that is consistent with the literature on normality of subobjects [4], [2], [6] and of Galois extensions [16], [18].

Definition 1.7 Suppose that $B \subseteq A$ is a subring pair. We say that B is a right (or left) *normal subring* of A if $B \subseteq A$ satisfies the right (or left) depth 2 condition above. Similarly, if $B \to A$ is a ring homomorphism, we say that the ring extension $A \mid B$ is a right (or left) normal extension if the bimodules induced by $B \to A$ satisfy the right (or left) depth 2 condition. A normal extension or normal subring is both left and right normal.

For example, centrally projective, or depth 1, ring extensions are normal extensions. As a corollary of [21, Theorem 3.2] we know that a QF extension is left normal if and only if it is right normal (extending the equivalence of left and right normality for Frobenius extensions in [16]). The Galois theory of a normal extension $A \supseteq B$ with the additional condition that A_B is a balanced module (with respect to its endomorphism ring $\text{End } A_B$) is briefly summarized as follows: the ring $T := (A \otimes_B A)^B \cong \text{End} \, _A A \otimes_B A_A$ has right

bialgebroid structure (and is left projective) over the centralizer subring $A^B := R$ [16] with coaction on A, denoted by $a \mapsto a_{(0)} \otimes_R a_{(1)} \in A \otimes_R T$, having coinvariant subring B, such that

$$A \otimes_B A \xrightarrow{\cong} A \otimes_R T \tag{3}$$

given by the canonical Galois mapping $a \otimes_B c \mapsto ac_{(0)} \otimes_R c_{(1)}$ with inverse given by $a \otimes_R t \mapsto at^1 \otimes_B t^2$ [18].

1.5. Sugano's theorem

Compiling results in [25] into a theorem and using the terminology of depth, we provide a different proof (except in (7) below). Let $A \supseteq B \supseteq C$ be a tower of rings, and consider the centralizers $D := A^C \supseteq A^B := R$.

Theorem 1.8 *Suppose B is a left relative H-separable extension of C in A; i.e., ${}_BB \otimes_C A_A \oplus * \cong {}_BA_A^n$. Then the following hold*:

(1) *D is a left finitely generated projective module over its subring R;*
(2) *as natural B-A-bimodules, $B \otimes_C A \cong \text{Hom}({}_RD, {}_RA)$ via $b \otimes a \longmapsto (d \mapsto bda)$;*
(3) *if B is a split extension of C, then D is a separable extension of R;*
(4) *if $d(C,B) = 1$, then $d_H(R,D) = 1$;*
(5) *if $B \supseteq C$ is a separable extension, then $D \supseteq R$ is a split extension;*
(6) *if $d_H(C,B) = 1$, then $d(R,D) = 1$;*
(7) *if $A^R = B$ (i.e. B has the double centralizer property in A) and B_C is a finitely generated projective module, then the isomorphism given in (2) restricts to $B \otimes_C B \cong \text{Hom}({}_RD_R, {}_RA_R)$ and (a) $d_H(C,B) = 1$ iff $d(R,D) = 1$; (b) $B \supseteq C$ is separable iff $D \supseteq R$ is split.*

Proof. The relative H-separability condition (1) on the tower $A \supseteq B \supseteq C$ is clearly equivalent to the two conditions

- there are $g_1, \ldots, g_n \in \text{Hom}({}_BB \otimes_C A_A, {}_BA_A) \cong D$ via $g_i \mapsto g_i(1 \otimes_C 1)$, and $f_1, \ldots, f_n \in \text{Hom}({}_BA_A, {}_BB \otimes_C A_A) \cong (B \otimes_C A)^B$ via $f_i \mapsto f_i(1)$ such that $\sum_{i=1}^n f_i \circ g_i = \text{id}_{B \otimes_C A}$;
- there are $e_i \in (B \otimes_C A)^B$ and $d_i \in D$ for $i = 1, \ldots, n$ such that $1 \otimes_C 1 = \sum_{i=1}^n e_i d_i$,

since we define $e_i = f_i(1)$ and $d_i = g_i(1 \otimes_C 1) \in A^C$. We will make use of the equation

$$1 \otimes_C 1 = \sum_i e_i^1 \otimes e_i^2 d_i \tag{4}$$

in almost every step of the proof below.

(1) Given $d \in D$, $d = \sum_i e_i^1 d e_i^2 d_i$. Define $h_i \in \mathrm{Hom}(_R D, {}_R R)$ by $h_i(d) = e_i^1 d e_i^2$, thus $d = \sum_i h_i(d) d_i$ is a finite projective bases equation.

(2) An inverse to $b \otimes_C a \mapsto \lambda_b \circ \rho_a$ is given by sending $f \in \mathrm{Hom}(_R D, {}_R A)$ into $\sum_i e_i f(d_i)$.

(3) Given a bimodule projection $E : B \to C$, note that applying E to Eq. (4) yields $1 = \sum_i E(e_i^1) e_i^2 d_i$. At the same time, a computation shows that $\sum_i E(e_i^1) e_i^2 \otimes_R d_i \in (D \otimes_R D)^D$.

(4) Trivially $D \cong \mathrm{Hom}(_C C_C, {}_C A_C)$, while $D \otimes_R D \cong \mathrm{Hom}(_C B_C, {}_C A_C)$ via $d \otimes_R d' \mapsto \lambda_d \circ \rho_{d'}$ (with inverse given by $g \mapsto \sum_i g(e_i^1) e_i^2 \otimes_R d_i$ for each $g \in \mathrm{Hom}(_C B_C, {}_C A_C)$). The mapping $\mu : D \otimes_R D \to D$ corresponds under these isomorphisms to restriction $r : \mathrm{Hom}(_C B_C, {}_C A_C) \to \mathrm{Hom}(_C C_C, {}_C A_C)$. If we have the depth one condition $_C B_C \oplus * \cong {}_C C_C^m$, then after applying the additive functor $\mathrm{Hom}(-, {}_C A_C)$ and the (D-bimodule) isomorphisms just considered, we obtain $D \otimes_R D \oplus * \cong D^m$ as D-D-bimodules, the H-depth one condition.

(5) If $e \in (B \otimes_C B)^B$ satisfies $e^1 e^2 = 1$, then the mapping in (2) applied to e is a bimodule projection in $\mathrm{Hom}(_R D_R, {}_R R_R)$.

(6) If B is an H-separable extension of C, there are t elements $z_i \in (B \otimes_C B)^B$ and t elements $r_i \in B^C$ such that $1 \otimes_C 1 = \sum_{i=1}^t z_i r_i$ [14]. But $B^C \subseteq A^C = D$ and $d \mapsto z_i^1 d z_i^2$ defines t mappings $h_i \in \mathrm{Hom}(_R D_R, {}_R R_R)$ such that $d = \sum_i h_i(d) r_i$, a centrally projective bases equation for $D \supseteq R$, thus $d(R, D) = 1$.

(7) First note that $_A A \otimes_R D_D \cong {}_A \mathrm{Hom}(B_C, A_C)_D$ via $a \otimes d \mapsto \lambda_a \circ \rho_d$ (with inverse $f \mapsto \sum_i f(e_i^1) e_i^2 \otimes_R d_i$). The isomorphism in (2) restricts to the composite isomorphism (using Proposition 20.11 in [1]) of

$$_B B \otimes_C B_B \cong {}_B B \otimes_C \mathrm{Hom}(_A A_R, {}_A A_R)_B$$
$$\cong {}_B \mathrm{Hom}(_A \mathrm{Hom}(B_C, A_C)_R, {}_A A_R)$$

(since $A^R = B(\cong \mathrm{Hom}(_A A, {}_A A)^R)$ and B_C is finite projective)

$$\cong {}_B \mathrm{Hom}(_A A \otimes_R D_R, {}_A A_R)_B \cong \mathrm{Hom}(_R D_R, {}_R \mathrm{Hom}(_A A, {}_A A)_R)$$
$$\cong {}_B \mathrm{Hom}(_R D_R, {}_R A_R)_B.$$

(8) (7a ⇐) Suppose $h_j \in \mathrm{Hom}(_RD_R, {}_RR_R)$ and $w_j \in D^R$ satisfy $\mathrm{id}_D = \sum_j h_j(-)w_j$. Then using the isomorphism in (7) there are $e_j \in (B \otimes_C B)^B \cong \mathrm{Hom}(_RD_R, {}_RR_R)$ such that $h_j(d) = e_j^1 d e_j^2$ for all $d \in D$. Note that $w_j \in D^R \subseteq A^R = B$ and $w_j \in D = A^C$, whence $w_j \in B^C$. It follows from the isomorphism (7) that $1 \otimes_C 1 = \sum_j e_j w_j$ an equivalent condition for H-separability, $d_H(C, B) = 1$.

(9) (7b ⇐) Given a projection $E : {}_RD_R \to {}_RR_R$ one notes that $E \in \mathrm{Hom}(_RD_R, {}_RA_R)^B \cong (B \otimes_C B)^B$, so that there is $e \in (B \otimes_C B)^B$ such that $E(d) = e^1 d e^2$ for all $d \in D$. In particular, $e^1 e^2 = E(1) = 1$. $\qquad \square$

Since H-separability implies separability for ring extensions, we might expect some mild condition should imply the same for towers of rings. The next corollary addresses this question.

Corollary 1.9 *Suppose $A \supseteq B \supseteq C$ is a left relative H-separable tower satisfying $D = A^C$ is a left split extension of $R = A^B$. Then $A \supseteq B \supseteq C$ is left relative separable.*

Proof. Applying (2) of Sugano's theorem, note that $\mu : B \otimes_C A \to A$ corresponds to the A-dual of the inclusion $\iota : {}_RR \to {}_RA$, which is $\iota^* : \mathrm{Hom}(_RD, {}_RA) \to \mathrm{Hom}(_RR, {}_RA) \cong {}_BA_A$. If ι is a split monic, then ι^* and μ are split B-A-epimorphisms. $\qquad \square$

Corollary 1.10 *Suppose K is a commutative ring, A is a K-algebra with B a K-subalgebra satisfying the B-A-bimodule generator condition $B \otimes_K A \oplus * \cong A^n$ (for some $n \in \mathbb{N}$). Let R be the centralizer A^B. The following holds:*

(1) *$_RA$ is a finite projective module;*

(2) *$B \otimes_K A \cong \mathrm{End}\,{}_RA$ via $b \otimes a \mapsto \lambda_b \circ \rho_a$;*

(3) *if B has a K-linear projection onto $K1$, then A is a separable extension of R;*

(4) *if B is finite projective as a K-module, then A is an H-separable extension of R;*

(5) *if B is a separable K-algebra, then A is a progenerator B-A-bimodule and $A \supseteq R$ is a split extension;*

(6) *if B is an Azumaya algebra with center Z such that $Z \otimes_K Z \cong Z$ (via μ), then A is centrally projective over its subalgebra R;*

(7) *if $A^R = B$ and B is a finite projective K-module, then the isomorphism*

in (2) *restricts to* $B \otimes_K B \cong \operatorname{End} {}_R A_R$ *and* (5), (6) *become iff statements.*

Proof. The proof follows from Sugano's theorem by letting $C = K1$, the unit subalgebra in A and B. In (6) and (7) we make use of Sugano's characterization of an H-separable K-algebra B as being Azumaya over its center Z subject to the condition $\mu : Z \otimes_K Z \xrightarrow{\cong} Z$. In (5), the bimodule ${}_B A_A$ is already noted to be a generator, and it is finite projective, since given any B-A-bimodule ${}_B M_A$ and B-A-epimorphism $\phi : M \to A$, ϕ is split by $a \mapsto e^1 m e^2 a$ where $e \in B^e$ is a separability idempotent and $\phi(m) = 1$. $\square$

A converse to Lemma 1.5 is given in the following. The hypothesis of cleft extension in the corollary is fullfilled for example by any finite-dimensional A with nilradical J and separable subalgebra $B \cong A/J$ (using Wedderburn's Principal Theorem).

Corollary 1.11 *Suppose* $\pi : {}_B A_B \to {}_B B_B$ *is a ring epimorphism splitting* $A \supseteq B$ *(a so-called cleft extension), and* C *is a subring of* B *such that the left relative H-separable tower condition holds. Then* $B \supseteq C$ *is H-separable* *(i.e.,* $d_H(C, B) = 1$*).*

Proof. Apply $\operatorname{id}_B \otimes_C \pi$ to the decomposition of $1 \otimes_C 1$ given in Eq. (4). We obtain $1 \otimes_C 1 = \sum_i e_i^1 \otimes \pi(e_i^2) \pi(d_i)$ where each $e_i^1 \otimes_C \pi(e_i^2) \in (B \otimes_C B)^B$ and each $\pi(d_i) \in B^C$: possessing Casimir elements and centralizer elements like these characterizes H-separability of B over C. $\square$

2. Subring depth in a relative separable tower

The progenerator condition in Corollary 1.10 is used again in the hypothesis of the proposition below.

Proposition 2.1 *Suppose a finitely generated projective* K*-algebra* A *has subalgebra* B *such that* A *is a progenerator* B*-A-bimodule. Then* $A \supseteq B$ *is left normal.*

Proof. Since $\mu : B \otimes_K A \to A$ splits, ${}_B A_A \oplus * \cong B \otimes_K A$; thus tensoring by $A \otimes_B -$ we obtain $A \otimes_B A \oplus * \cong A \otimes_K A$ as natural A-bimodules. Since B is a separable algebra, any B-module is K-relative projective, whence by the hypothesis on A, ${}_B A \oplus * \cong {}_B B^m$ and so $A \otimes_B A \oplus * \cong B \otimes_K A^m$ as B-A-bimodules. Since ${}_B A_A$ is a generator, it follows that $B \otimes_K A \oplus * \cong {}_B A_A^q$, whence ${}_B A \otimes_B A_A \oplus * \cong {}_B A_A^{mq}$, the left depth 2 condition on $A \supseteq B$. $\square$

Proposition 2.2 *Suppose a tower of rings $A \supseteq B \supseteq C$ satisfies the left relative H-separability condition ${}_B B \otimes_C A_A \oplus * \cong {}_B A_A^n$ and the left relative separability condition ${}_B A_A \oplus * \cong {}_B B \otimes_C A_A$. Then $d_H(B, A) = d_H(C, A)$.*

Proof. Tensoring by ${}_A A \otimes_B -$ the left relative H-separability condition yields ${}_A A \otimes_C A_A \oplus * \cong {}_A A \otimes_B A_A^n$. Tensoring by ${}_A A \otimes_B -$ the left relative separability condition above yields $A \otimes_B A \oplus * \cong A \otimes_C A$ as natural A-bimodules, whence $A \otimes_B A$ and $A \otimes_C A$ are H-equivalent as A-A-bimodules.

Suppose that $A^{\otimes_B n}$ is H-equivalent to $A^{\otimes_C n}$ for any $m > n \geq 2$. Then $A^{\otimes_B (m-1)}$ and $A^{\otimes_C (m-1)}$ are H-equivalent, so $A \otimes_C A^{\otimes_B (m-1)}$ and $A^{\otimes_C m}$ are H-equivalent, as are $(A \otimes_C A) \otimes_B \cdots \otimes_B A$ and $(A \otimes_B A) \otimes_B \cdots \otimes_B A$. It follows from this inductive argument that $A^{\otimes_B m}$ and $A^{\otimes_C m}$ are H-equivalent as A-bimodules for any $m > 1$.

Suppose $A \supseteq B$ has H-depth 1, equivalently, A and $A \otimes_B A$ are H-equivalent, which is equivalent to A and $A \otimes_C A$ being H-equivalent iff $A \supseteq C$ has H-depth 1. From the definition of H-depth in Section 1 and the H-equivalences noted above, $A \supseteq B$ has H-depth n iff $A \supseteq C$ has H-depth n for any $n \geq 1$. $\square$

We improve on [17, Theorem 2.3] next.

Proposition 2.3 *Suppose B is an Azumaya K-algebra and subalgebra of a finitely generated projective K-algebra A. Then $A \supseteq B$ has depth 1.*

Proof. Since B is Azumaya, it is well-known that B is a progenerator B^e-module (e.g. [14]). Since B is a separable K-algebra, B^e is a semisimple extension of $K1$. Then ${}_B A_B$ is K-relative projective, therefore ${}_B A_B$ is finite projective since A is projective over K. Thus $A \oplus * \cong B \otimes_K B^m$ for some $m \in \mathbb{N}$. But $B^e \oplus * \cong {}_B B_B^n$ for some $n \in \mathbb{N}$ since ${}_B B_B$ is a generator. Putting these together, ${}_B A_B \oplus * \cong {}_B B_B^{mn}$. $\square$

2.1. Higman-Jans-like theorem

Higman's theorem in [10] states that a finite-dimensional group algebra kG, where k is a field of positive characteristic p, has finite representation type if and only if the Sylow p-subgroup of G is cyclic. The proof was teased apart by Jans in [13] into two statements about the property of finite representation type of a subalgebra pair of Artin algebras going up or down according to whether A is a split or separable extension of B; e.g., a separable and finitely generated extension $A \supseteq B$ of Artin algebras where B

has finitely many isoclasses of indecomposable modules implies that also A has finite representation type; see also [24, pp. 173-174]. In generalizing this theorem, we first need a lemma characterizing left relative separable towers of rings $A \supseteq B \supseteq C$ in terms of modules.

Lemma 2.4 *B is a left relative separable extension of C in A if and only if for each module $_AM$, the mapping $\mu_M : B \otimes_C M \to M$ given by $b \otimes_C m \mapsto bm$ splits naturally as a left B-module epimorphism.*

Proof. ($\Rightarrow$) This is clear from tensoring the split epi $\mu : B \otimes_C A \to A$ by $- \otimes_A M$ to obtain the split epi μ_M. ($\Leftarrow$) Apply the hypothesis to $M = A$ and use naturality to obtain a split B-A-bimodule epi $\mu : B \otimes_C A \to A$. $\square$

Let A be an Artin algebra, $A - \mathrm{mod}$ denote the category of finitely generated left A-modules, and add M denote the category of summands of finite sums of copies of a module M.

Theorem 2.5 *Suppose $A \supseteq B \supseteq C$ is a left relative separable tower of Artin algebras, where $_BA$ and $_CB$ are finitely generated. Suppose $C - \mathrm{mod}$ has finitely many isoclasses of indecomposable representatives $V_1, \ldots, V_n$. Then the restriction functor $\mathrm{Res}_B^A : A - \mathrm{mod} \to B - \mathrm{mod}$ factors through the subcategory* add $\bigoplus_{i=1}^n B \otimes_C V_i$.

Proof. Given $M \in A - \mathrm{mod}$, its restrictions $_BM$ and $_CM$ are finitely generated. By the lemma, $_BM$ is isomorphic to a direct summand of $B \otimes_C M$. Since the restriction $_CM \cong \bigoplus_{i=1}^n n_i V_i$ for some nonnegative integers n_i, one obtains $\mathrm{Res}_B^A M \oplus * \cong \bigoplus_{i=1}^n n_i B \otimes_C V_i$, which is expressible as a Krull-Schmidt decomposition into finitely many indecomposable B-module summands of $B \otimes_C V_1, \ldots, B \otimes_C V_n$. $\square$

Of course if $A = B$ and $B \supseteq C$ is a separable finitely generated extension, then the theorem recovers Jans's, "B has finite representation type if C has."

2.2. Triviality of Relatively H-separable Group Algebra Towers

We next note that towers of finite complex group algebras that are left or right relative H-separable extensions are just arbitrary group algebra extensions.

Proposition 2.6 *Let $A = \mathbb{C}G \supseteq B = \mathbb{C}H \supseteq C = \mathbb{C}J$ where $G > H > J$ is a tower of subgroups of a finite group G. Then $A \supseteq B \supseteq C$ is a left or*

right relative H-separable tower of algebras if and only if $H = J$.

Proof. Given ${}_B B \otimes_C A_A \oplus * \cong {}_B A_A^n$, we tensor this with simple left A-modules, i.e. G-modules, and make use of their (irreducible) characters. Let $\psi \in \mathrm{Irr}(G)$ and $\phi \in \mathrm{Irr}(H)$. From the relative H-separable condition above it follows that

$$\langle \mathrm{Ind}_J^H \mathrm{Res}_J^G \psi, \phi \rangle_H \leq n \langle \mathrm{Res}_H^G \psi, \phi \rangle_H.$$

Letting $\psi = 1_G$, note that $\mathrm{Res}_J^G 1_G = 1_J$ for instance, so that

$$\langle \mathrm{Ind}_J^H 1_J, \phi \rangle_H \leq n \langle 1_H, \phi \rangle_H.$$

This last inner product is zero if $\phi \neq 1_H$, so that also $\langle \mathrm{Ind}_J^H 1_J, \phi \rangle_H = 0$. If $\phi = 1_H$, then $\langle \mathrm{Ind}_J^H 1_J, 1_H \rangle_H = \langle 1_J, \mathrm{Res}_J^H 1_H \rangle_J = 1$ by Frobenius reciprocity. From the orthonormal expansion of $\mathrm{Ind}_J^H 1_J$ in terms of $\mathrm{Irr}(H)$, it follows that $\mathrm{Ind}_J^H 1_J = 1_H$. Comparing degrees, it follows that $|H : J| = 1$, whence $H = J$. The proof using the right relative H-separability condition is a similar use of characters of right modules. The converse is of course trivial. $\square$

We have seen in Lemma 1.2 that a tower $A \supseteq B \supseteq C$ of arbitrary finite group algebras is always left or right relative separable if $B \supseteq C$ is a separable extension (iff $|H : J|$ is invertible in the ground ring). This follows from the fact that group algebra extensions are split extensions (since given a subgroup $H < G$, the difference set $G - H$ is closed under multiplication by H).

3. A Characterization of normality for progenerator ring extensions

The next proposition provides an alternative characterization of the left relative H-separable condition for a tower $A \supseteq B \supseteq C$ where B_C is finitely generated and projective.

Proposition 3.1 *Suppose $A \supseteq B \supseteq C$ is a tower of rings such that the natural module B_C is finite projective. Then the left relative H-separable condition (1) is equivalent to the condition, ($\exists n \in \mathbb{N}$:)*

$$
{}_A \mathrm{Hom}(B_C, A_C)_B \oplus * \cong {}_A A_B^n. \tag{5}
$$

Proof. Apply $\mathrm{Hom}(-_A, A_A)$, an additive functor from the category of B-A-bimodules into the category of A-B-bimodules, to (1), $_B B \otimes_C A_A \oplus * \cong {}_B A_A^n$. Note that $\mathrm{Hom}(B \otimes_C A_A, A_A) \cong \mathrm{Hom}(B_C, A_C)$ as natural A-B-bimodules via $F \mapsto F(- \otimes_C 1)$ with inverse

$$f \longmapsto (b \otimes_C a \mapsto f(b)a)$$

for every $f \in \mathrm{Hom}(B_C, A_C)$. Since $_A \mathrm{Hom}(A_A, A_A)_B \cong {}_A A_B$, the condition (5) follows without the assumption that B_C is finite projective.

Assuming that B_C is finite projective, it follows that $B \otimes_C A_A$ is finite projective and therefore reflexive. Then $\mathrm{Hom}(_A \mathrm{Hom}(B_C, A_C), {}_A A) \cong B \otimes_C A$ as natural B-A-bimodules. It follows reflexively that condition (5) implies condition (1). $\qquad\square$

The next theorem provides many interesting classes of examples of relative H-separable towers of rings.

Theorem 3.2 *Suppose $B \supseteq C$ is a ring extension and has the natural module B_C a progenerator. Let $A := \mathrm{End}\, B_C$ and $B \hookrightarrow A$ given by the left regular representation $b \mapsto \lambda_b$. Then $B \supseteq C$ is right normal if and only if the tower $A \supseteq B \supseteq C$ is left relative H-separable.*

Proof. ($\Rightarrow$) This direction of the proof only requires that B_C is finite projective. Given the right normality condition,

$$_B B \otimes_C B_C \oplus * \cong {}_B B_C^m \tag{6}$$

for some $m \in \mathbb{N}$, apply the bimodule $_A B_C$ and the additive functor $_A \mathrm{Hom}(-_C, B_C)$ to this. Note that the hom-tensor adjoint relation implies that $\mathrm{Hom}(B \otimes_C B_C, B_C) \cong \mathrm{Hom}(B_C, \mathrm{Hom}(B_C, B_C)_C)$ as natural A-B-bimodules. This implies condition (5), equivalent by the proposition to (1).

($\Leftarrow$) Since we assume B_C is a progenerator, the rings C and A are Morita equivalent, with bimodules $_A B_C$ and $_C \mathrm{Hom}(B_C, C_C)_A$ forming a Morita context. In particular, $\mathrm{Hom}(B_C, C_C) \otimes_A B_C \cong C$ as C-bimodules and $B \otimes_C \mathrm{Hom}(B_C, C_C) \cong A$ as A-bimodules.

Supposing that condition (5) holds on the tower $C \subseteq B \hookrightarrow A$, we substitute $_A A_B = {}_A \mathrm{Hom}(B_C, B_C)_B$ in this condition and apply the hom-tensor adjoint relation with the last Morita isomorphism to obtain:

$$\mathrm{Hom}(B \otimes_C B_C, {}_A B_C)_B \oplus * \cong {}_A B \otimes_C \mathrm{Hom}(B_C, C_C)_B^n.$$

Tensor this from the left by the additive functor ${}_C \mathrm{Hom}(B_C, C_C) \otimes_A -$ and using the (cancellation) isomorphism of Morita pointed out above. We obtain ${}_C \mathrm{Hom}(B \otimes_C B_C, \mathrm{Hom}(B_C, C_C) \otimes_A B_C)_B \oplus * \cong$

$$ {}_C \mathrm{Hom}(B \otimes_C B_C, C_C)_B \oplus * \cong {}_C \mathrm{Hom}(B_C^n, C_C)_B$$

since $B \otimes_C B_C$ is finite projective and one may apply the well-known natural isomorphism [1, Proposition 20.10]. Now by reflexivity of the projective modules B_C^n and $B \otimes_C B_C$, we apply to this last isomorphism the additive functor $\mathrm{Hom}(-, {}_C C)$ from the category of C-B-bimodules into the category of B-C-bimodules and obtain the condition (6) with $n = m$. □

The left and right relative H-separable conditions on a tower $A \supseteq B \supseteq C$ are equivalent if $B \supseteq C$ is a Frobenius extension, i.e., ${}_C B_B \cong {}_C \mathrm{Hom}(B_C, C_C)_B$ as bimodules and B_C has finite projective bases $\{b_i\} \subset B$, $\{\phi_i\} \subset \mathrm{Hom}(B_C, C_C)$, $(i = 1, \ldots, m)$.

Proposition 3.3 *If $B \supseteq C$ is a Frobenius extension, then a tower $A \supseteq B \supseteq C$ is left relative H-separable if and only if it is right relative H-separable.*

Proof. We make use of the equivalent condition for left relative H-separable tower in Proposition 3.1. We note that $\mathrm{Hom}(B_C, A_C) \cong A \otimes_C \mathrm{Hom}(B_C, C_C)$ via $f \mapsto \sum_i f(b_i) \otimes_C \phi_i$, with inverse given by the "one-point projections" mapping $a \otimes_C \psi \mapsto a\psi(-)$. Observe that this mapping is an A-B-bimodule isomorphism. It follows from the Frobenius condition ${}_C B_B \cong {}_B \mathrm{Hom}(B_C, C_C)_B$ that the right relative H-separable condition is satisfied by $A \supseteq B \supseteq C$. □

As a corollary of this proposition and Theorem 3.2, we note that the right normality condition for Frobenius extensions with surjective Frobenius homomorphism is equivalent to left normality condition, another proof in this case of [16].

Corollary 3.4 *If $B \supseteq C$ is a Frobenius extension, where B_C is a generator, then $B \supseteq C$ is left normal if and only if it is right normal.*

3.1. A characterization of normality for Frobenius extensions

In this subsection we characterize normal (twisted) Frobenius extensions together with their endomorphism rings as being relative H-separable towers. We find it convenient to change notation to $A \supseteq B$ being the Frobenius ring extension and $E := \operatorname{End} A_B$ being the top ring in the tower $B \subseteq A \hookrightarrow E$ where $A \hookrightarrow E$ is given by $a \mapsto \lambda_a$ and $\lambda_a(x) = ax$.

Suppose $\beta : B \to B$ is a ring automorphism of B. Denote a B-module M_B as M_β if twisted by β as follows: $m \cdot b = m\beta(b)$. Recall that a β-Frobenius (ring) extension $A \supseteq B$ is characterized by having a (Frobenius) homomorphism $F : {}_B A_B \to {}_\beta B_B$ satisfying $F(b_1 a b_2) = \beta(b_1)F(a)b_2$ for each $b_1, b_2 \in B$, $a \in A$. Dual bases $\{x_i\}, \{y_i\}$ in A satisfy $\sum_{i=1}^n x_i F(y_i a) = a$ and $\sum_{i=1}^n \beta^{-1}(F(ax_i))y_i = a$ for each $a \in A$. Equivalently, A_B is finite projective and $A \cong {}_\beta \operatorname{Hom}(A_B, B_B)$ as B-A-bimodules: see [14] for more details and references.

For example, if β is an inner automorphism, then $A \supseteq B$ is an (ordinary) Frobenius extension, such as a group algebra extension of a group G and subgroup H of finite index n. (Suppose $g_1, \dots, g_n$ are the right coset representatives of H in G, K an arbitrary commutative ring, then the group algebra $A = KG$ is a Frobenius extension of the group subalgebra $B = KH$ with $F : A \to B$ the obvious projection defined by $F(\sum_{g \in G} a_g g) = \sum_{h \in H} a_h h$ and dual bases $x_i = g_i^{-1}, y_i = g_i$.)

Corollary 3.5 *Suppose $A \supseteq B$ is a β-Frobenius extension with surjective Frobenius homomorphism $F : A \to B$. Let $E := \operatorname{End} A_B$ and embed $A \hookrightarrow E$ via the left regular representation $\lambda_a(x) = ax$. Then the tower of rings $B \subseteq A \hookrightarrow E$ is left relative H-separable if and only if $B \subseteq A$ is right normal.*

Proof. Since $F : A \to B$ is assumed surjective, it follows that A_B (and ${}_B A$ by using equivalently $\beta^{-1} \circ F$) is a generator. It also follows from the hypothesis of Frobenius extension that A_B (and ${}_B A$) are finite projective. Apply Theorem 3.2 to conclude that the left relative H-separable tower condition on $B \subseteq A \hookrightarrow E$ is equivalent to the right normality condition on $B \subseteq A$. $\qquad \square$

Recall that a Hopf subalgebra R is normal in a Hopf algebra H if R is stable under the left and right adjoint actions of H on R. For group algebra extensions this specializes to the usual notion of normal subgroup.

Corollary 3.6 *A Hopf subalgebra R of a finite-dimensional Hopf algebra H is normal if and only if the tower of algebras $R \subseteq H \hookrightarrow \operatorname{End} H_R$ is left relative H-separable.*

Proof. This follows from Corollary 3.5 and theorems that H is a β-Frobenius extension of R (Oberst-Schneider), and another that H_R is free (Nichols-Zoeller). Also, as remarked in the introduction, the equivalence of the normality condition for a Hopf subalgebra $R \subseteq H$ with the depth 2 condition on the ring extension $R \subseteq H$ follows from [2]. $\square$

3.2. Galois correspondence proposal

Again let E denote $\operatorname{End} A_B$. The condition on the tower $B \subseteq A \hookrightarrow E$ in the next corollary is called the rD3 condition in [19]. The depth three condition on $A \supseteq B$ is that ${}_B A \otimes_B A_B \oplus * \cong {}_B A_B^m$ for some $m \in \mathbb{N}$. Below we apply the same Frobenius coordinate system as above, but we may assume that the twist automorphism $\beta = \operatorname{id}_B$.

Corollary 3.7 *Suppose $A \supseteq B$ is a Frobenius extension with surjective Frobenius homomorphism. Then $A \supseteq B$ has depth 3 if and only if ${}_E E \otimes_A E_B \oplus * \cong {}_E E_B^m$ for some $m \in \mathbb{N}$.*

Proof. The proof is similar to the proof of Theorem 3.2, but using the E-A-bimodule isomorphism $E \xrightarrow{\cong} A \otimes_B A$ given by $f \mapsto \sum_i f(x_i) \otimes y_i$, with inverse mapping given by $a \otimes_B a' \mapsto \lambda_a \circ F \circ \lambda_{a'}$. The rest of the proof is left to the reader. $\square$

The two conditions of "depth three" and "depth two" on a tower go up and down as follows. The short proof is left to the reader as an exercise using Proposition 1.1.

Proposition 3.8 *Suppose $A \supseteq B \supseteq C \supseteq D$ is a tower of unital subrings. If $A \supseteq B$ has depth 1 and $A \supseteq B \supseteq C$ is right relative H-separable, then $B \supseteq C \supseteq D$ satisfies the rD3 condition, ${}_B B \otimes_C B_D \oplus * \cong {}_B B_D^n$ for some $n \in \mathbb{N}$. If $B \supseteq D$ has H-depth 1 and $B \supseteq C \supseteq D$ satisfies the rD3 condition, then $A \supseteq B \supseteq C$ is right relative H-separable.*

In [25] the left relative separable and H-separable conditions on towers of rings are used by Sugano for Galois correspondence in an H-separable extension in terms of centralizers. A final thought is to ask if results for Galois correspondence of a normal extension in [19] (in terms of endomorphism

rings) may be improved with the use of the tower condition studied in this paper.

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Departamento de Matematica

Faculdade de Ciências da Universidade do Porto

Rua Campo Alegre 687

4169-007 Porto

E-mail: lkadison@fc.up.pt