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On the global existence and asymptotic behavior of solutions of reaction-diffusion equations

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1. Introduction

Let Ω be a bounded domain with smooth boundary Γ in R^{n}. Let $\beta\geq 1$, and $\mu_{j}>0$ $(j=1,2)$. R. Martin posed a problem on the existence and uniform bounds of solutions $u=\{u_{1}, u_{2}\}$ of the reaction-diffusion equation of the form:

$$(1)\qquad \begin{cases} \dfrac{\partial u_{1}}{\partial t}=\mu_{1}\Delta u_{1}-u_{1}u_{2}^{\beta} \\ \dfrac{\partial u_{2}}{\partial t}=\mu_{2}\Delta u_{2}+u_{1}u_{2}^{\beta} \end{cases} \qquad x\in\Omega,\ t>0$$

under various boundary conditions and non-negative initial data; this equation is related to the Rosenzweig-MacArthur equation in ecology (see J. Maynard-Smith [5]; D. Conway and A. Smoller [2]). N. Alikakos [1] obtained L^{∞}-bounds of solutions of (1) subject to the homogeneous Neumann boundary condition under the assumption $1\leq\beta<(n+2)/n$, and gave a positive partial answer to it. The purpose of the present paper is to give a complete answer to the problem of Martin.

We consider a solution $u=\{u_{1}, u_{2}\}$ of the more general type of reaction-diffusion equations

$$(2)\qquad \frac{\partial u_{j}}{\partial t}=\mu_{j}\Delta u_{j}+f_{j}(u),\ x\in\Omega,\ t>0\ (j=1,2);$$

subject to the boundary condition:

$$(3)\qquad \alpha_{j}(x)\frac{\partial u_{j}}{\partial n}+\left(1-\alpha_{j}(x)\right)u_{j}=0,\ x\in\Gamma,\ (j=1,2);$$

and with the initial condition:

$$(4)\qquad u_{j}|_{t=0}=a_{j}(x),\qquad x\in\Omega\ (j=1,2),$$

($\partial/\partial n$ denotes differentiation in the direction of the exterior normal to Γ). Here we make the following assumptions:

ASSUMPTION 1. *$\alpha_{j}(x)$ $(j=1,2)$ is a non-negative C^{2}-function on Γ such* *that* $0 \leq \alpha_j(x) \leq 1$ $(x \in \Gamma)$.

ASSUMPTION 2. $a_j(x)$ $(j=1, 2)$ *is a non-negative C^2-function on* $\bar{\Omega}$, *satisfying* (3);

ASSUMPTION 3. $f_j(y)$ $(j=1, 2)$ *is a C^1-function on* $\bar{R}^2_+ = \{y=(y_1, y_2)\,;\, y_1 \geq 0,\ y_2 \geq 0\}$ *such that*

i) $-f_1(y)$, $f_2(y)$ *are non-negative and* $f_1((0, s)) = f_2((s, 0)) = 0$ *for* $s \geq 0$;

ii) *there is a monotonically increasing function* $\omega(s)$ $(s \geq 0)$ *and a positive constant* r *with*

$$
\begin{aligned}
&f_2(y) \leq \omega(y_1)\,(y_2 + y_2{}^r)\,, \\
&f_2(y) \leq \omega(y_1)\big|f_1(y)\big|\,.
\end{aligned}
\qquad y = (y_1, y_2) \in \bar{R}^2_+\,, \tag{5}
$$

We note that $f_1(y) = -y_1 y_2^\beta$, $f_2(y) = y_1 y_2^\beta$ (see (1)) satisfies the assumption 3. Our result is now given by the following

THEOREM. *Let the assumptions* 1, 2 *and* 3 *hold. Then the initial-boundary value problem* (2), (3), (4) *has a unique global solution* $u = \{u_1, u_2\}$. *Moreover, u tends to some constant vector of the form* $c = \{c_1, c_2\}$, *as* $t \to \infty$, *uniformly on* $\bar{\Omega}$, *where* $c_j \geq 0$ *and* $f_j(c_j) = 0$ $(j=1, 2)$.

The following corollary follows immediately from the theorem above, and gives a complete answer to the problem of Martin.

COROLLARY. *Let the assumptions* 1 *and* 2 *hold. Then the initial-boundary value problem* (1), (3), (4) *has a unique global solution* $u = \{u_1, u_2\}$. *Moreover, u tends to a constant vector* $c = \{c_1, c_2\}$, *as* $t \to \infty$, *uniformly on* $\bar{\Omega}$, *where* $c_j \geq 0$, $c_1 c_2 = 0$, *and*

$$
c_1 + c_2 = \int_0^\infty \int_\Gamma \left(\frac{\partial u_1}{\partial n} + \frac{\partial u_2}{\partial n} \right) dS_x\, dt + \int_\Omega \big(a_1(x) + a_2(x)\big)\, dx\,. \tag{6}
$$

In the next two sections we shall give some a priori estimates, which are the core of the proof of our theorem. In the final section we shall give the proof of the theorem.

2. L^p bounds of solutions

Here we shall give some a priori estimates for a solution u of (2), (3), (4) which is supposed to exist in the interval $[0, T)$. We first see that u_j $(j=1, 2)$ is non-negative;

$$
u_j(x, t) \geq 0,\ \ x \in \Omega,\ \ 0 \leq t < T,\ \ (j=1, 2) \tag{7}
$$

since the initial data $a_j(x)$ is non-negative, and since $f_j(u)$ has, by the assumption 3 (i), the form : $f_j(u)=c_j(u)\,u_j$ where $c_j(u)=c_j(u(x,t))$ is some cotinuous function of x, t. By the maximum principle,

$$(8) \qquad 0\le u_1(x,t)\le \|a_1\|_{L^\infty}$$

since $f_1(u)\le 0$ by the assumption 3. To get the L^∞-bounds for u is the essential part of the proof of our theorem. This follows from the following key a priori estimate : for $p>1$,

$$(9) \qquad \|u_2(t)\|_{L^p}\le M_p\,, \qquad 0\le t<T\,,$$

M_p being a constant independent of T (we shall simply write $u_j(t)$ for $u_j(x,t)$), which we shall prove in a series of lemmas ; $\|\cdot\|_{L_p}$ denotes the usual L^p-norm over Ω. In doing so, we set

$$g_p(u_2)=(1+u_2)^p\,, \qquad u_2\ge 0$$

for $p>0$. We note

$$(10) \qquad g_p'(u_2)\le (p/|p-1|)^{1/2}g_p(u_2)^{1/2}\left|g_p''(u_2)\right|^{1/2}$$

for $p\ne 1$. We denote the inner product of $L^2(\Omega)$ by $(\cdot,\cdot)$, and define $(\nabla b, c\nabla d)$ by :

$$(\nabla b, c\nabla d)=\sum_{i=1}^{n}\int_\Omega \frac{\partial b(x)}{\partial x_i}c(x)\frac{\partial d(x)}{\partial x_i}dx$$

for scalar functions b, c and d on Ω.

LEMMA 1. *Let $u=\{u_1, u_2\}$ be a solution of* (2), (3), (4) *in* $[0, T)$. *Then for w in $W^{1,2}(\Omega)$ and $0\le t<T$,*

$$(11) \qquad \left(\Delta u_j(t), w\right)=\int_\Gamma \frac{\partial u_j(x,t)}{\partial n}w(x)\,dS_x-\left(\nabla u_j(t), \nabla w\right).$$

Moreover, if w is non-negative, then

$$(12) \qquad \left(\Delta u_j(t), w\right)\le -\left(\nabla u_j(t), \nabla w\right), \qquad 0\le t<T\,,$$

($W^{1,2}(\Omega)$ is the usual Sobolev space).

PROOF. By integration by parts, we have (11). By (3) and (7), it is easy to see that

$$(13) \qquad \partial u_j(x,t)/\partial n\le 0\,, \qquad x\in\Omega,\ 0\le t<T,\ j=1,2\,,$$

from which (12) follows by (11).

LEMMA 2. *Let u be as in Lemma* 1. *Then*

$$\int_\Omega u_j(x,t)\,dx-\int_0^t\int_\Gamma \frac{\partial u_j(x,s)}{\partial n}dS_x ds-\int_0^t\int_\Omega f_j\big(u(x,s)\big)\,dxds \tag{14}$$
$$=\int_\Omega a_j(x)\,dx\,,\qquad 0\le t<T,\ j=1,2\,.$$

PROOF. Integrating (2) with respect to x, t over $\Omega\times[0,t]$, we have, by (11) with $w=1$, (14).

LEMMA 3. *Let u be as in Lemma* 1. *Then*

$$\begin{aligned}&\big(g_p(u_2(t)),1\big)-\mu_2\int_0^t\int_\Gamma \frac{\partial u_2(x,s)}{\partial n}g'_p\big(u_2(x,s)\big)\,dS_x ds\\ &\quad=\big(g_p(a_2),1\big)-\mu_2\int_0^t\big(\nabla u_2, g''_p(u_2)\nabla u_2\big)\,ds+\int_0^t\big(f_2(u),g'_p(u_2)\big)\,ds\,,\\ &\qquad\qquad 0\le t<T\,.\end{aligned} \tag{15}$$

PROOF. By (2) with $j=2$, and (12),

$$\begin{aligned}\frac{d}{dt}\big(g_p(u_2(t)),1\big)&=\Big(g'_p\big(u_2(t)\big),\frac{\partial u_2(t)}{\partial t}\Big)\\ &=\mu_2\big(g'_p(u_2(t)),\Delta u_2(t)\big)+\big(g'_p(u_2(t)),f_2(u(t))\big)\end{aligned}$$

Integrating the above inequality in t, we have, by (11), (15).

LEMMA 4. *Let u be as in Lemma* 1. *Let $p\neq 1$. Then*

$$\begin{aligned}\int_0^t\big(f_2(u),g_p(u_2)\big)\,ds&\le M+M\int_0^t\big(\nabla u_2,|g''_p(u_2)|\nabla u_2\big)\,ds\\ &\quad+M\int_0^t\big(g'_p(u_2),f_2(u)\big)\,ds\end{aligned} \tag{16}$$
$$0\le t<T$$

M being a constant independent of T (which may depend on $\|a_j\|_{L^\infty}$, $j=1,2$)

PROOF. We shall denote by the same M various constant independent of T (which may depend on p, $\|a_1\|_{L^\infty}$, $\|a_2\|_{L^\infty}$). By (2) and (12), $(u_j=u_j(t))$

$$\begin{aligned}&\frac{d}{dt}\big(u_1+u_1^2,g_p(u_2)\big)\\ &\quad=\big((1+2u_1)\big(\mu_1\Delta u_1+f_1(u)\big),g_p(u_2)\big)\\ &\qquad+\big(u_1+u_1^2,g'_p(u_2)\big(\mu_2\Delta u_2+f_2(u)\big)\big)\\ &\quad\le-(\mu_1+\mu_2)\big(\nabla u_1,(1+2u_1)g'_p(u_2)\nabla u_2\big)-2\mu_1\big(\nabla u_1,g_p(u_2)\nabla u_1\big)\\ &\qquad+\big(f_1(u)+2u_1f_1(u),g_p(u_2)\big)-\mu_2\big(\nabla u_2,(u_1+u_1^2)g''_p(u_2)\nabla u_2\big)\end{aligned} \tag{17}$$

$$+\left(u_1+u_1^2,\, g_p'(u_2)f_2(u)\right)$$
$$(\equiv I_1+I_2+I_3+I_4+I_5)\,.$$

We shall estimate each term on the right-side of (17). By the Schwarz inequality, (10), and (8),

$$I_1+I_2\leq M\left(\nabla u_2,\,(1+2u_1)^2\left|g_p''(u_2)\right|\nabla u_2\right)$$
$$\leq M\left(\nabla u_2,\,\left|g_p''(u_2)\right|\nabla u_2\right).$$

By the assumption 3 (i) and (ii), $\left(M_0=\omega(\|a_1\|_{L^\infty})/(1+2\|a_1\|_{L^\infty})\right)$

$$I_3\leq\left(1+2\|a_1\|_{L^\infty}\right)\left(f_1(u),\,g_p(u_2)\right)\leq -M_0^{-1}\left(f_2(u),\,g_p(u_2)\right).$$

By (8),

$$I_4\leq M\left(\nabla u_2,\,\left|g_p''(u_2)\right|\nabla u_2\right);\ I_5\leq M\left(g_p'(u_2), f_2(u)\right).$$

Collecting all the estimates above, we see:

$$\text{the left hand side of (17)}\leq -M_0^{-1}\left(f_2(u),\,g_p(u_2)\right)+M\left(g_p'(u_2), f_2(u)\right)$$
$$+M\left(\nabla u_2,\,\left|g_p''(u_2)\right|\nabla u_2\right).$$

Integrating the above inequality in t over $[0, t]$, we get (16).

LEMMA 5. *Let u be as in Lemma 1. Let $0<\theta<1$. Then*

$$\int_0^t\left(f_2\left(u(s)\right),\, g_{k-\theta}(s)\right)ds\leq M_k\,.\qquad 0\leq t<T,\ (k=1, 2, \cdots),\tag{18}$$

M_k being a constant independent of T.

PROOF. We first show (18) holds for $k=1$. By (14) with $j=1$, the assumption 3 (ii), and (13), $(M_0'=\omega(\|a_1\|_{L^\infty}))$

$$0\leq\int_0^t\left(f_2(u),\,1\right)ds\leq -M_0'\int_0^t\left(f_1(u),\,1\right)ds\leq M_0'(a_1,\,1)\,.\tag{19}$$

Hence the first and second terms on the left hand side of (14) with $j=2$ are bounded by some constant (independent of T). Hence the left hand side of (15) with $p=1-\theta$ is bounded by some constant (independent of T). Since the left hand side of (18) with $k=1$ is, by (16) with $p=1-\theta$, bounded by the left hand side of (15) with $p=1-\theta$, we see that (18) holds for $k=1$; note $g_p''(u_2)$ is negative for $p=1-\theta$. We suppose that (18) holds for $k=m$. Then since $g_p''(u_2)>0$ for $p>1$, it follows from (15) and (16) that

$$\int_0^t \left(f_2(u), g_p(u_2)\right) ds \leq M_2 + M_2 \int_0^t \left(f_2(u), g_p'(u_2)\right) ds,$$

M_2 being a constant independent of T, which shows (18) holds for $k=m+1-\theta$. By induction on k, the proof of the lemma is completed.

LEMMA 6. *For any $p>1$, we have*

$$\|u_2(t)\|_{L^p} \leq M, \qquad 0 \leq t < T, \tag{9}$$

M_p being a constant independent of T.

PROOF. From (13), (15) and (18), the estimate (9) follows.

3. L^∞-bounds of solutions

A) Let $p>\max\{n, 2\}$. We then define the operator $A_{j,p}$ in $L^p(\Omega)$ by:

$$D(A_{j,p}) = \left\{u \in W^{2,p}(\Omega)\,;\ \alpha_j(x)\frac{\partial u}{\partial n} + \left(1-\alpha_j(x)\right) u = 0 \text{ on } \Gamma\right\};$$

$$A_{j,p}u = -\mu_j \Delta u$$

where $W^{2,p}(\Omega)$ is the usual Sobolev space; We shall simply write A_j for $A_{j,p}$ unless otherwise stated. Here we recall the basic properties of the A_j (For the detail see, e.g., H. Tanabe [7]). We know that the estimate

$$||u||_{2,p} \leq M\left\{||u||_{L^p} + ||A_j u||_{L^p}\right\}, \qquad u \in D(A_j),$$

holds (M: constant) where $||\cdot||_{k,p}$ is the norm of the Sobolev space $W^{k,p}(\Omega)$. Furthermore, $-A_j$ generates the holomorphic semi-groups $\{e^{-tA_j}\}_{t>0}$ in $L^p(\Omega)$. We also know that the spectrum of A_j consists of non-negative eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ with finite multiplicity: $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$. Since any eigenvalue λ_i of $A_{j,p}$ is also eigenvalue of the self-adjoint operator $A_{j,2}$ in $L^2(\Omega)$, it is easy to see that the first eigenvalue λ_1 of A_j is positive if $\alpha_j(x) \not\equiv 1$; and zero if $\alpha_j(z) \equiv 1$. For $f \in L^p(\Omega)$, we set

$$P_j f = \begin{cases} \int_\Omega f(x)\, dx/|\Omega| & (\text{if } \alpha_j(x) \equiv 1) \\ 0 & (\text{if } \alpha_j(x) \not\equiv 1). \end{cases}$$

Then P_j is the projection operator onto the eigenspace corresponding to the first eigenvalue $\lambda_1=0$ if $\alpha_j(x) \equiv 1$. Define the operator Q_j in $L^p(\Omega)$ by

$$Q_j = I - P_j$$

(I denotes the identity operator). Then

LEMMA 7. *Let* $0<\theta\leq 1$. *For* $0\leq s\leq t$, *we have*

(20) $$||e^{-tA_j}||\leq M;$$

(21) $$||Q_j e^{-tA_j}||\leq Me^{-\beta t};$$

(22) $$||A_j e^{-tA_j}||\leq Mt^{-1}e^{-\beta t};$$

(23) $$||e^{-tA_j}-e^{-sA_j}||\leq M(t-s)^{\theta}s^{-\theta}e^{-\beta s};$$

(24) $$||\nabla e^{-tA_j}-\nabla e^{-sA_j}||\leq M(t-s)^{\theta/2}s^{-(\theta+1)/2}e^{-\beta s};$$

(25) $$||e^{-tA_j}w-e^{-sA_j}w||_{L^p}\leq M(t-s)\,e^{-\beta s}||A_j w||_{L^p},\quad w\in D(A_j);$$

(27) $$||\nabla e^{-tA_j}w-\nabla e^{-sA_j}w||_{L^p}\leq M(t-s)^{1/2}e^{-\beta s}||A_j w||_{L^p},\quad w\in D(A_j),$$

($j=1,2$) *where* M, β *are positive constants independent of* t; $||\cdot||$ *denotes the operator norm in* $L^p(\Omega)$.

PROOF. The proofs of the inequalities above are standard in the theory of semi-groups. We omitt the subscript j. Since the spectrum of AQ in $QL^p(\Omega)$ consists of positive eigenvalues, and since $AQ=O$, (21) and (22) follow from the representation of the holomorphic semi-groups by the Duford-Taylor integral. Since $Pe^{-tA}=P$, using (21) just proved, we have (20). (23) and (25) follow from (21), (22) and the following inequalities:

$$||e^{-tA}w-e^{-sA}w||_{L^p}^{\theta}\leq(t-s)^{\theta}\left\|\frac{I-e^{-(t-s)A}}{(t-s)A}Q\right\|^{\theta}||Ae^{-sA}w||_{L^p}^{\theta};$$

$$||e^{-tA}w-e^{-sA}w||_{L^p}=||e^{-tA}w-e^{-sA}w||_{L^p}^{\theta}\,||e^{-tA}w-e^{-sA}w||_{L^p}^{1-\theta}.$$

Since A has a bounded inverse in $QL^p(\Omega)$, we have

$$||Qw||_{L^p}\leq M||Aw||_{L^p},\qquad w\in D(A).$$

Hence, using the interpolation theorem:

$$||\nabla w||_{L^p}^2\leq M||Qw||_{L^p}||w||_{2,p}$$

(see, e.g., S. Mizohata [6: Theorem 3.26]) we find

(28) $$||\nabla w||_{L^p}^2\leq M||Qw||_{L^p}\,||Aw||_{L^p};$$

(29) $$||\nabla w||_{L^p}\leq M||Aw||_{L^p}.$$

Hence, by (28),

$$||\nabla e^{-tA}w-\nabla e^{-sA}w||_{L^p}\leq M||e^{-tA}w-e^{-sA}w||_{L^p}^{1/2}\left\|(I-e^{-(t-s)A})\,Ae^{-sA}w\right\|_{L^p}^{1/2}.$$

from which (24) follows by (22) and (23). Similarly, (27) can be proved by (29), (22), and (25).

Let $q>2$. Let $f(t)$, $0\leq t<T$, be an L^q-function of t with values in $L^p(\Omega)$. We then define the function $v_j(t)$ by:

$$v(t)=e^{-tA_j}a_j+\int_0^t e^{-(t-s)A_j}f(s)\,ds\,, \qquad j=1,2\,. \tag{30}$$

Then:

LEMMA 8. *Let v_j, f be as above. Let $0<\theta<1-2/q$. We have*; $(0\leq s\leq t<T)$

$$\left\|v_j(t)\right\|_{L^p}\leq M+M\int_0^t\left\|f(\sigma)\right\|_{L^p}d\sigma\,; \tag{31}$$

$$\left\|Q_jv_j(t)\right\|_{L^p}\leq Me^{-\beta t}+M\int_0^t e^{-\beta(t-\sigma)}\left\|f(\sigma)\right\|_{L^p}d\sigma\,; \tag{32}$$

$$\left\|\nabla v_j(t)\right\|_{L^p}\leq Me^{-\beta t}+M\int_0^t(t-\sigma)^{-1/2}e^{-\beta(t-\sigma)}\left\|f(\sigma)\right\|_{L^p}d\sigma\,; \tag{33}$$

$$\left\|v_j(t)-v_j(s)\right\|_{L^p}\leq M(t-s)+M\{(t-s)^{\theta_1}+(t-s)^{\theta}\}\left(\int_0^t\left\|f(\sigma)\right\|_{L^p}^q d\sigma\right)^{1/q}; \quad (\theta_1=1-1/q) \tag{34}$$

$$\left\|\nabla v_j(t)-\nabla v_j(s)\right\|_{L^p}\leq M(t-s)+M\{(t-s)^{\theta_2}+(t-s)^{\theta_3}\}\left(\int_0^t\left\|f(\sigma)\right\|_{L^p}^q d\sigma\right)^{1/q} \quad (\theta_2=\theta/2\,;\ \theta_3=1/2-1/q)\,. \tag{35}$$

PROOF. (31) and (32) follow easily from (20), (21). We shall show (35). We omitt the subscript j, and write v, A, etc. for v_j, A_j, etc. We have

$$\begin{aligned}v(t)-v(s)=&\int_s^t Pf(s)\,ds+Q(e^{-tA}-e^{-sA})\,a+\int_s^t Qe^{-(t-\sigma)A}f(\sigma)\,d\sigma\\&+\int_0^s Q(e^{-(t-\sigma)A}-e^{-(s-\sigma)A})f(\sigma)\,d\sigma\,, \qquad (\equiv I_1+I_2+I_3+I_4)\end{aligned} \tag{36}$$

By (28) and (22), we see

$$\|\nabla e^{-tA}\|\leq Mt^{-1/2}e^{-\beta t} \tag{37}$$

We have: $\nabla I_1=0$. By (25), $\|\nabla I_2\|_{L^p}\leq M(t-s)$. By (37) and the Hölder inequalty,

$$\begin{aligned}\|\nabla I_3\|_{L^p}&\leq M\int_s^t(t-\sigma)^{-1/2}e^{-\beta(t-\sigma)}\left\|f(\sigma)\right\|_{L^p}d\sigma\\&\leq M(t-s)^{\theta_3}\left(\int_0^t\left\|f(\sigma)\right\|_{L^p}^q d\sigma\right)^{1/q}.\end{aligned}$$

By (24) and the Hölder inequality,

$$||\nabla I_4||_{L^p} \leq M(t-s)^{\theta_2}\left(\int_0^t \left\|f(\sigma)\right\|_{L^p}^q d\sigma\right)^{1/q}.$$

Collecting all the estimates above, we get (35). Similarly, using (23), (25), and (37), we can prove (33) and (34).

LEMMA 9. *Let v_j, f be as in Lemma 8. Suppose that $f(t)$ is Hölder continuous for t $(0 \leq t < T)$:*

$$\left\|f(t)-f(s)\right\|_{L^p} \leq L\left(|t-s|^{\theta_4}+|t-s|^{\theta_5}\right) \tag{38}$$

where L, θ_4, θ_5 are constants, $0<\theta_4$, $\theta_5 \leq 1$. Then $v_j(t) \in D(A_j)$, and

$$\left\|A_j v_j(t)\right\|_{L^p} \leq M+ML+M\left\|f(t)\right\|_{L^p} \tag{39}$$

$(0 \leq t < T)$, M being a constant independent of T, f. Moreover, for any $\varepsilon>0$ we have $(\varepsilon \leq s < t < T)$

$$\left\|A_j v_j(t)-A_j v(s)\right\|_{L^p} \leq M(t-s)\,\varepsilon^{-1}+ML\left((t-s)^{\theta_4}+(t-s)^{\theta_5}\right) \tag{40}$$
$$+M(t-s)\,\varepsilon^{-1}\left\|f(t)\right\|_{L^p}.$$

(For the proof, see T. Kato [4: Lemma IX 1.28]).

B) We now proceed to the derivation of L^∞-bounds of solutions. Let $p>\max\{n, 2\}$. M denotes, as before, various constants independent of T (which may depend on p, $||a_j||_{2,p}$). We have:

LEMMA 10. *Let $u=\{u_1, u_2\}$ be as in Lemma 1. Then*

$$\left\|u_j(t)\right\|_{2,p} \leq M; \tag{41}$$

$$\left\|u_j(t)\right\|_{L^\infty} \leq M; \tag{42}$$

$$\int_0^t \left\|f_j\left(u(s)\right)\right\|_{L^p}^{3p} ds \leq M \tag{43}$$

for $0 \leq t < T$; $j=1, 2$.

PROOF. We claim: $(0 \leq t < T;\ j=1, 2)$

$$\left\|f_j\left(u(t)\right)\right\|_{L^q} \leq M \quad (q \geq 1); \tag{44}$$

$$\int_0^t \left\|f_j\left(u(s)\right)\right\|_{L^p}^{3p} ds \leq M\int_0^t \left|\left(f_j\left(u(s)\right), 1\right)\right| \left\|u(s)\right\|_{L^{(3p-1)/2}}^{3p-1} ds; \tag{45}$$

$$\left\|\nabla u_j(t)\right\|_{L^p} \leq M. \tag{46}$$

Indeed, by (5), (8), and (9), (44) holds for $j=2$. By the Hölder inequality,

(45) with $j=2$ follows from (44), and hence we have, by (19), (43) with $j=2$. Since $u_j(t)$ satisfies (30) with $f(t)$ replaced by $f_j(u(t))$:

$$u_j(t)=e^{-tA_j}a_j+\int_0^t e^{-(t-s)A_j}f_j\big(u(s)\big)\,ds \tag{47}$$

it follows from (33), (43) with $j=2$, and the Hölder inequality that (46) holds for $j=2$. (42) with $j=2$ now follows from (46), (9) and the Sobolev lemma. On the other hand, (42) with $j=2$, and (8) give (44) with $j=1$. Hence by the Hölder inequality, we have (45) with $j=1$, which shows (43) with $j=1$. From (45), (33), we can see, as before, that (46), (42) hold for $j=1$. By (34) and (43), $(q=3p\,;\ \theta=1/4)$

$$\big\|u_j(t)-u_j(s)\big\|_{L^p}\le M(t-s)+M(t-s)^{2/3}+M(t-s)^{1/4}$$

and so

$$\big\|f_j\big(u(t)\big)-f_j\big(u(s)\big)\big\|_{L^p}\le M(t-s)+M(t-s)^{2/3}+M(t-s)^{1/4}.$$

By (39) and (44), we have (41).

4. The proof of the theorem

Let $p>n+1$. Let $T>0$. Set $X_1=C([0,T]\,;\ W^{1,p}(\Omega))$. Then X_1 is the Banach space with an appropriate norm. We define a sequence $\{u^{(k)}\}_{k=1}^{\infty}$ in $X_1\times X_1$ inductively by: $(u^{(k)}=\{u_1^{(k)},u_2^{(k)}\})$

$$u_j^{(1)}(t)=e^{-tA_j}a_j\,;$$

$$u_j^{(k+1)}(t)=u_j^{(1)}(t)+\int_0^t e^{-(t-s)A_j}f_j\big(u^{(k)}(s)\big)\,ds$$

By the standard argument, we can see from (31)-(35) that $\{u^{(k)}(t)\}$ is a Cauchy sequence in $X_1\times X_1$ if T is sufficiently small. Let $u(t)$ be the limit of the Cauchy sequence $\{u^{(k)}\}$ in $X_1\times X_1$. Then the $u(t)$ is cleary a solution of (47). Hence, in the same way as in the proof of Lemma 9, we can show that $u_j(t)\in D(A_j)$, $A_ju_j(t)$ is Hölder continuous for $0<t<T$, $u_j(t)$ is continuously differentiable for $0<t<T$, its derivative $du(t)/dt$ is Hölder continuous for $0<t<T$, and $u_j(t)$ satisfies the (abstract) equation in $L^p(\Omega)$: $du_j(t)/dt+A_ju_j(t)=f_j(u(t))$ with $u_j(0)=a_j$. On the other hand this implies that $u_j\in W^{1,p}(\bar{\Omega}\times[0,T))$. Hence, by the Sobolev imbedding theorem, u is Hölder continuous on $\bar{\Omega}\times(0,T)$, and so is $f_j(u(t))$. Hence, by the existence and regularity theorem for solutions of paraboic equation (see S. Itô [3]) a classical solution v_j of $\partial v_j/\partial t=\mu_j\Delta v_j+f_j(u(x,t))$ with conditions (3) and (4) exists and coincides with u_j; compute $d\|v_j(t)-u_j(t)\|_{L^2}^2/dt$. This implies u_j

is a classical solution of (2), (3) and (4). The uniqueness of solution can be proved by a routin argument. Since we have a priori bounds (41), we have actually a unique global solution ; note it suffices to assume that $a_j \in W^{2,p}(\Omega)$.

We finally show the asymptotic behavior in t of the solution of (2), (3), (4). By (43), the right-hand sides of (32), (33) with $f(t)$ replaced by $f_j(u(t))$ tend to zero as $t \to \infty$. Hence $\|Q_j u_j(t)\|_{L^p} \to 0$, $\|\nabla u_j(t)\|_{L^p} \to 0$ as $t \to \infty$. By the Sobolev imbedding theorem, $Q_j u_j(t)$ converges to zero as $t \to \infty$, uniformly on $\bar{\Omega}$. By (13) and the assumption 3 (i), the second and third terms on the right-hand side of (14) with $j=1$ are monotone and bounded, and hence are convergent for $t \to \infty$. Hence $(u_1(t), 1)$ converges to some non-negative constant α_1 as $t \to \infty$. By (19), $(f_2(u(t)), 1)$ is integrable on $[0, \infty)$, since $-(f_1(u(t)), 1)$ is integrable on $[0, \infty)$. Hence by (14) with $j=2$, $(u_2(t), 1)$ converges to some non-negative constant α_2 as $t \to \infty$. Hence, $P_j u_j(t)$ converges to c_j as $t \to \infty$, where $c_j = \alpha_j / |\Omega|$. Hence, $u_j(t)$ converges to c_j as $t \to \infty$, uniformly on $\bar{\Omega}$; $c_j \geq 0$. Note $c_j = 0$ if $\alpha_j(x_0) < 1$ for some $x_0 \in \Gamma$. Since $\|f_j(u(t))\|_{L^p}^{3p}$ is, by (43), integrable on $[0, \infty)$, there is a sequence $\{t_k\}$, tending to the infinity, such that $\|f_j(u(t_k))\|_{L^p} \to 0$. Since $u_j(t) \to c_j$, uniformly on $\bar{\Omega}$ (as $t \to \infty$), we have: $\|f_j(c)\|_{L^p} = 0$ $(c = \{c_1, c_2\})$, which implies $f_j(c) = 0$. Thus the proof of the theorem is completed.

PROOF OF THE COROLLARY. It suffices to show (6). Adding (14) with $j=1$, and (14) with $j=2$, and letting t tend to the infinity, we have (6); note $f_1(u) + f_2(u) = 0$.

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