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The Euler limit and initial layer of the nonlinear Boltzmann equation

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1. Problem and main results

In his paper [11], Nishida proved the followings. The Cauchy problem for the nonlinear Boltzmann equation in the kinetic theory of rarefied gases has a unique classical solution f^{ε} locally in time t on an interval $[0, \tau]$ independent of the mean free path $\varepsilon>0$, if the initial distribution f_0 is sufficiently close to an absolute Maxwellian and is analytic in the space variable x. When $\varepsilon\rightarrow 0$, f^{ε} converges on $[0, \tau]$ to some f^{0}. For $t\in(0, \tau]$, the limit f^{0} is a local Maxwellian whose mass density $\rho(t, x)$, flow velocity $v(t, x)$, temperature $T(t, x)$ are unique solutions to the Cauchy problem with initial data specified by f_0 for the compressible Euler equation obtained from the Boltzmann equation as the first approximation to the Hilbert expansion.

The uniform existence in ε of the solution f^{ε} was established using an abstract nonlinear Cauchy-Kowalewski theorem developed by Nishida [10], Nirenberg [9] and Ovsjannikov [13]. The convergence of f^{ε} was shown under the additional assumption $\rho(0, x)>0$, based on the compactness argument supplemented with the uniqueness theorem for the relevant Euler equation.

In general the convergence is not uniform in t near $t=0$ and the limit f^{0} has a discontinuity at $t=0$. This singular behavior at $\varepsilon=t=0$ describes the initial layer of the solution of the Boltzmann equation, and the limit f^{0} plays a role of the outer solution in the theory of singular perturbations.

The aim of the present note is twofold. The first is to give a simplified proof which makes only use of the classical contraction mapping principle. The use of a rather complicated abstract theorem of [9], [10], [13] can be avoided by introducing a simple time-dependent norm which is analogous to that used by Asano [1] to obtain local solutions of the Boltzmann equation. As by-products, we can remove the assumption $\rho(0, x)>0$, and find a convergence rate. The second aim is concerned with the initial layer. It will be shown that a necessary and sufficient condition for the uniform convergence and for the continuity of f^{0} on $[0, \tau]$ is that the initial f_0 is itself a Maxwellian. This answers the Hilbert paradox that the Hilbert expansion solution to the Boltzmann equation is uniquely determined by the initial fluid state [6].

All our proofs are applicable to the initial boundary value problem for the Boltzmann equation in the half-space or in a rectangular domain with the specular reflection boundary condition, with an elementary modification stated in [6], [14], [15]. Then our results remains valid and it is worth noting that in these special cases no boundary layers appear.

The Cauchy problem for the Boltzmann equation is written as

$$(1.1)\qquad \begin{aligned} &\frac{\partial f}{\partial t} = -\xi\cdot\nabla_x f + \frac{1}{\varepsilon} Q[f,f], && (t,x,\xi)\in(0,\infty)\times \boldsymbol{R}^n\times\boldsymbol{R}^n, \\ &f|_{t=0} = f_0, && (x,\xi)\in \boldsymbol{R}^n\times\boldsymbol{R}^n. \end{aligned}$$

Here $f=f^\varepsilon(t,x,\xi)$ is the density of gas particles having the position x and velocity ξ at time t, $\cdot$ means the inner product in $\boldsymbol{R}^n$, $\varepsilon>0$ is the mean free path and Q expresses a quadratic integral operator representing the collision of gas particles and acting only on the variable ξ. The reader will be referred to [3] for the derivation of the Boltzmann equation as well as fundamental properties of Q. In this note only the gas of Grad's cutoff hard potential [5] will be dealt with.

When $\varepsilon\to 0$, (1.1) raises a singular perturbation problem. To find the corresponding reduced problem, suppose f^ε have a limit f^0 and $\varepsilon(f^\varepsilon_t+\xi\cdot\nabla_x f^\varepsilon)\to 0$ as $\varepsilon\to 0$. Then letting ε tend to 0 in (1.1), we find

$$(1.2)\qquad Q[f^0,f^0]=0\,.$$

According to [3], unique solutions to (1.2) are Maxwellians

$$(1.3)\qquad f^0=\frac{\rho}{(2\pi T)^{n/2}}\,e^{-|\xi-v|^2/2T}\,.$$

Here ρ, v, T are the mass density, flow velocity and temperature respectively, the hydrodynamical quantities independent of the individual particle velocity ξ. If they are constants in t and x, then f^0 of (1.3) is called an absolute Maxwellian, and if they depend on t and x, it is called a local Maxwellian. In order to distinguish them we denote the absolute Maxwellian by $g=g(\xi)$.

(1.2) does not determine ρ, v and T. To accomplish the reduced equation we need the functions

$$h_0(\xi)=1,\ h_j(\xi)=\xi_j\quad(1\le j\le n),\ h_{n+1}(\xi)=\frac{1}{2}|\xi|^2,$$

and note that ρ, v, T are determined by the moments of ξ with respect to

f^0, i. e.,

$$\rho v=\langle h_0, f^0\rangle\,,$$

(1. 4) $$\rho v_j=\langle h_j, f^0\rangle\,,\qquad 1\leq j\leq n\,,$$

$$\frac{n}{2}\rho T+\frac{1}{2}\rho|v|^2=\langle h_{n+1}, f^0\rangle\,,$$

where $\langle h,f\rangle=\int_{R^n} h(\xi)f(\xi)\,d\xi$. According to [3], h_j's are collision invaliants, i. e.,

$$\langle h_j, Q[f,f]\rangle=0\,,\qquad 0\leq j\leq n+1\,,$$

for any f, which, substituted into (1. 1), yields

(1. 5) $$\frac{\partial}{\partial t}\langle h_j, f^\varepsilon\rangle+\langle h_j, \xi\cdot\nabla_x f^\varepsilon\rangle=0\,,\qquad 0\leq j\leq n+1\,.$$

Passing to the limit as $\varepsilon\to 0$, using (1. 4) and taking into account that $\langle(\xi-v)h_{n+1}, f^0\rangle=\rho vT$, then lead to the system of nonlinear hyperbolic conservation laws

$$\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho v)=0\,,$$

(1. 6) $$\frac{\partial}{\partial t}(\rho v)+\nabla\cdot(\rho\, {}^t vv)+\nabla p=0\,,\qquad {}^t vv=(v_i v_j)\,,$$

$$\frac{\partial}{\partial t}\left(\rho\left(e+\frac{1}{2}|v|^2\right)\right)+\nabla\cdot\left(\rho\left(e+\frac{1}{2}|v|^2\right)v+pv\right)=0\,,$$

supplemented by the equation of state of the ideal gas

$$T=\frac{1}{\rho}p=\frac{2}{n}e\,,$$

where p, e are the pressure and internal energy per unit mass, respectively. (1. 6) may be considered as the compressible Euler equation derived from the Boltzmann equation. This is also obtained as the first approximation to the Hilbert expansion, see [3].

The justification of this asymptotic procedure requires, first of all, solutions to (1. 1) which exist on the interval $[0, \tau]$ independent of ε. Solutions in the large in time have been found in [12], [15] for initial data f_0 near absolute Maxwellians, but as $\varepsilon\to 0$, f_0 should be chosen indefinitely close to the Maxwellians. For arbitrary initials f_0 (rapidly decreasing in ξ), local solutions exists [1], [7], on the interval $[0, \tau^\varepsilon]$ where $\tau^\varepsilon>0$ depends on ε as well as f_0 and may tend to 0 with ε.

In order to obtain the desired solutions, therefore, we shall assume that f_0 is not only close to an absolute Maxwellian but also analytic in x. To make this precise and for later purposes, we prepare some notations and function spaces.

We denote by $\hat{u}=\mathscr{F}_x u$ the Fourier transform of a function $u\in\mathscr{S}'(\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n)$ with respect to x,

$$\hat{u}(k,\xi)=\mathscr{F}_x u(k,\xi)=(2\pi)^{-n/2}\int_{\boldsymbol{R}^n} e^{-ik\cdot x}u(x,\xi)\,dx,\quad k\in\boldsymbol{R}^n,\ i=\sqrt{-1},$$

and denote by $X_\beta^{\alpha,l}$ the Banach space of functions $u\in\mathscr{S}'(\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n)$ satisfying

$$\|u\|_{\alpha,l,\beta}\equiv\sup_{k,\xi\in\boldsymbol{R}^n}\left(1+|k|\right)^l\left(1+|\xi|\right)^\beta e^{\alpha|k|}|\hat{u}(k,\xi)|<\infty, \tag{1.7}$$

where α, l, $\beta\in\boldsymbol{R}$. We claim that if $\alpha>0$, $u\in X_\beta^{\alpha,l}$ is analytic in $x\in\boldsymbol{R}^n+iB_\alpha$ where $B_\alpha=\{x\in\boldsymbol{R}^n|\ |x|<\alpha\}$.

The space $\dot{X}_\beta^{\alpha,l}$ is the closed subspace of $X_\beta^{\alpha,l}$, hence by itself a Banach space with the norm (1.7), consisting of such u's that

$$\|\chi\left(|k|+|\xi|>R\right)u\|_{\alpha,l,\beta}\to 0\qquad(R\to 0)\,. \tag{1.8}$$

Here $\chi\ (|k|+|\xi|>R)$ is the characteristic function of the domain $|k|+|\xi|>R$ in $\boldsymbol{R}_k^n\times\boldsymbol{R}_\xi^n$. And we used and will use the convention that $\phi(k)u$ expresses $\mathscr{F}_x^{-1}\phi(k)\,\hat{u}(k,\xi)$ where $\phi(k)$ may depend also on t, ξ, etc, but not on x. Then we define the Banach space

$$\begin{aligned} &Y_\beta^{\alpha,\gamma,l}(I)=\left\{u=u(t)|e^{-\gamma t|k|}\,u(t)\in B^0(I;\dot{X}_\beta^{\alpha,l})\right\},\\ &|||u|||_{\alpha,\gamma,l,\beta,I}=\sup_{t\in I}\|u(t)\|_{\alpha-\gamma t,l,\beta}, \end{aligned} \tag{1.9}$$

where $\gamma\in\boldsymbol{R}$, $I\subset\boldsymbol{R}$ is an interval and $B^0(I;\ X)$ denotes the space of bounded continuous functions defined on I with values in a Banach space X. Similarly, $C^0(I;\ X)$ will denote the space of X-valued continuous functions on I.

Finally we also need the Banach space

$$\begin{aligned} &Z_{\beta,\tau}^{\alpha,\gamma,l}=B^0\left((0,\infty);\ Y_\beta^{\alpha,\gamma,l}([0,\tau])\right),\\ &|||u|||_{\alpha,\gamma,\beta,l,\tau}=\sup_{\epsilon>0,\ t\in[0,\tau]}\|u^\epsilon(t)\|_{\alpha-\gamma t,l,\beta}, \end{aligned} \tag{1.10}$$

where $\tau>0$. A time-dependent norm appears in (1.9) and (1.10).

Now we can state the main results of this note. Since (1.1) is to be solved near an absolute Maxwellian $g=g(\xi)$, we put

$$f=g+g^{1/2}u\,, \qquad f_0=g+g^{1/2}u_0\,,$$

and state our results in terms of u and u_0. The equation for u to solve will be presented in the next section. The first result is slightly stronger than that of [11] and will be proven in the next two sections.

THEOREM 1.1. *Let g be an absolute Maxwellian and suppose*

$$\alpha>0,\ l>n,\ \beta>\frac{n}{2}+1\,.$$

Then there exist positive numbers a_0, a_1 and for each initial data f_0 satisfying

$$u_0\in\dot{X}_\beta^{\alpha,l}\,, \qquad ||u_0||_{\alpha,l,\beta}<a_0\,,$$

the followings hold with some positive numbers γ and τ.

(i) *For each $\varepsilon>0$, (1.1) has a unique classical solution $f=f^\varepsilon$ on the time interval $[0,\tau]$ such that*

$$u^\varepsilon\in Z_{\beta,\tau}^{\alpha,\gamma,l}\,, \quad |||u^\varepsilon|||_{\alpha,\gamma,l,\beta,\tau}\le a_1||u_0||_{\alpha,l,\beta}\,,$$

$$\frac{\partial}{\partial t}u^\varepsilon\in C^0((0,\infty)\,;\ Y_{\beta-1}^{\alpha,\gamma,l-1}[0,\tau])\,.$$

(ii) *As $\varepsilon\to 0$, u^ε converges to a limit $u^0\in Y_\beta^{\alpha,\gamma,l}((0,\tau])$ strongly in $Y_\beta^{\alpha,\gamma,l}$ $([\delta,\tau])$ for any $\delta>0$.*

(iii) *For $t\in(0,\tau]$, $f^0=g+g^{1/2}u^0$ is a local Maxwellian whose hydrodynamical quantities ρ, v, T solve the compressible Euler equation (1.6) in the classical sense.*

REMARK 1.2. $f=f^\varepsilon$ converges also at $t=0$ trivially to the initial data f_0. In general, however, the convergence is not uniform on $[0,\tau]$ though so is it on $[\delta,\tau]$, $\delta>0$. And the limit $f^0=f^0(t)$ is not necessarily continuous at $t=0$. Indeed if $f^0(+0)$ exists (strongly), it should be a Maxwellian as a limit of Maxwellians $f^0(t)$, $t>0$, while $f^0(0)$ is the initial data f_0 which may be other than a Maxwellian. On the other hand the moments $\langle h_j, f^0(t)\rangle$ are continuous on $[0,\tau]$ in a topology induced from that of $X_0^{\alpha,l-1}$ for functions not depending on ξ. This follows readily from Theorem 1.1 applied to (1.5).

REMARK 1.3. Nishida [11] proved (ii) of Theorem 1.1 under the additional assumption $\rho(0,x)=\langle h_0, f_0\rangle>0$. In this case $v(0,x)$ and $T(0,x)$ are well-defined by (1.4) and analytic in x, with $\rho(0,x)$. For such smooth initial data, the uniqueness theorem is available for the Cauchy problem to the Euler equation (1.6) because (1.6) is a nonlinear symmetric hyperbolic system provided $\rho>0$ (see Kato [8]). Since his proof of the convergence relies on the compactness of $\{f^\varepsilon\}$, $0<\varepsilon\leq 1$, he needed this fact so as to ensure the uniqueness of the limit of the subsequences. In our situation $\rho(0, x)\leq 0$ may be allowed if a_0 can be found large enough. Then $v(0, x)$, $T(0, x)$ happen to be singular at those points x where $\rho(0, x)=0$, and no uniqueness theorems are known for such initial data.

As for the convergence rate, we prove the following theorem in § 3.

THEOREM 1.4. *Under the condition of Theorem* 1.1 *and for* $\sigma\in[0, 1)$, *there are positive numbers* $a_0'(\leq a_0)$, a_1' *such that if* $||u_0||_{\alpha,\gamma,l}<a_0'$, *then for all* $\varepsilon>0$ *and* $t\in(0, \tau]$,

$$||u^\varepsilon(t)-u^0(t)||_{\alpha-\gamma t, l-\sigma, \beta-\sigma}\leq a_1'\left(\frac{\varepsilon}{t^2}\right)^{\sigma/2}.$$

REMARK 1.5. It has not been able to know if we can take $a_0'=a_0$. The convergence rate found here is far short for the verification of the Hilbert expansion.

According to Remark 1.3, it is necessary, in order for the initial layer not to appear, that the initial f_0 itself is a Maxwellian. That this is also sufficient will be shown in § 4, in the form of the

THEOREM 1.6. *Under the situation of Theorem* 1.1, *suppose in addition,*

(1.11) (i) f_0 *is a local Maxwellian,*
(ii) $u_0\in\dot{X}^{\alpha,l+1}_{\beta+1}$.

Then u^0 *of Theorem* 1.1 *belongs to* $Y^{\alpha,\gamma,l}_\beta([0, \tau])$ *and* u^ε *converges to* u^0 *strongly there. Moreover,* ρ, v, T *of* f^0 *are unique classical solutions to the Cauchy problem for* (1.6) *with the initial data specified by those of* f_0.

REMARK 1.7. Caflisch [2] proved a reverse version of the above theorem in the following sense. Suppose the Cauchy problem to (1.6) have solutions ρ, v, T in an appropriate Sobolev space, on some time interval $[0, \tau]$. Construct the local Maxwellian f^0 by (1.3) from these ρ, v, T. Then (1.1) with $f_0=f^0(0)$ has a unique solution f^ε for each $\varepsilon>0$ and

$$||f^\varepsilon(t)-f^0(t)||\leq C\varepsilon,$$

on $[0, \tau]$. Theorem 1.6 indicates that the Cauchy problem to (1.6) has indeed a solution if the initial data are specified by f_0 satisfying the confitions stated in Theorem 1.6. Then $\rho(0, x)\leq 0$ may be permitted, though it should be analytic. On the other hand it seems difficult to obtain Sobolev space solutions to (1.6) without the assumption $\rho(0, x)>0$, [8].

2. Existence of solution for $\varepsilon>0$

In terms of u and by virtue of (1. 2), the Cauchy problem (1. 1) can be rewritten as

$$
\begin{aligned}
&\frac{\partial}{\partial t}u=-\xi\cdot\nabla_x u+\frac{1}{\varepsilon}Lu+\frac{1}{\varepsilon}\Gamma[u,u], && (t,x,\xi)\in(0,\infty)\times\boldsymbol{R}^n\times\boldsymbol{R}^n,\\
&u|_{t=0}=u_0, && (x,\xi)\in\boldsymbol{R}^n\times\boldsymbol{R}^n,
\end{aligned}
\tag{2. 1}
$$

where we have defined the operators L, Γ by

$$
\begin{aligned}
Lu&=2g^{-1/2}Q[g,g^{1/2}u],\\
\Gamma[u,v]&=g^{-1/2}Q[g^{1/2}u,g^{1/2}v],
\end{aligned}
$$

$Q[\cdot,\cdot]$ being understood to be a bilinear symmeric operator induced from the quadratic operator Q. L is linear and Γ is bilinear symmetric.

Define the linearized Boltzmann operator

$$
B^{\varepsilon}=-\xi\cdot\nabla_x+\frac{1}{\varepsilon}L,
\tag{2. 2}
$$

and suppose it generates a strongly continuous semigroup (shortly a semigroup) $e^{tB^{\varepsilon}}$. If $u=u^{\varepsilon}(t)$ is a solution to (2. 1), then it should solve the integral equation

$$
u^{\varepsilon}(t)=e^{tB^{\varepsilon}}u_0+\int_0^t e^{(t-s)B^{\varepsilon}}\frac{1}{\varepsilon}\Gamma[u^{\varepsilon}(s),u^{\varepsilon}(s)]\,ds.
\tag{2. 3}
$$

It is this equation to which we apply the contraction mapping principle.

To this end, we first quote from [5], [6] some necessary properties of L and Γ. The operator L has the decomposition

$$
L=-\Lambda+K,
$$

where Λ is the multiplication operator

$$
\begin{aligned}
&\Lambda=\nu(\xi)\times,\\
&\nu(\xi)\in L^{\infty}_{loc}(\boldsymbol{R}^n_{\xi}),\qquad \nu_0\le\nu(\xi)\le\nu_1\big(1+|\xi|\big),
\end{aligned}
\tag{2. 4}
$$

with some positive constants ν_0, ν_1, while K is an integral operator in ξ having nice properties, among which we mention

$$
\|Ku\|_{\beta}\le C\|u\|_{\beta-1},
\tag{2. 5}
$$

where $\beta\in\boldsymbol{R}$ and $\|u\|_{\beta}=\sup(1+|\xi|)_{\beta}|u(\xi)|$. The operator L is selfadjoint and nonpositive in $L^2(\boldsymbol{R}^n_{\xi})$ and has 0 as an isolated eigenvalue of multiplicity

$n+2$, whose eigenspace is spanned by $h_j g^{1/2}$, $0\leq j\leq n+1$, h_j being those of § 1. Denote the corresponding eigenprojection by P_0. Then

$$P_0\Gamma[u, v]=0 \tag{2.6}$$

for any u, v. Finally we note

$$\|\Lambda^{-1}\Gamma[u, v]\|_\beta\leq C\|u\|_\beta\|v\|_\beta. \tag{2.7}$$

In order to simplify the notations in the below, we choose the parameter $(\alpha, \gamma, \tau, l, \beta)$ fixed arbitrarily in the region

$$\alpha,\ \gamma,\ \tau,\ \alpha-\gamma\tau\geq 0, \qquad l,\ \beta\in\boldsymbol{R},$$

unless otherwise stated, and put

$$\dot{X}=\dot{X}_\beta^{\alpha,l}, \qquad Y(I)=Y_\beta^{\alpha,\gamma,l}(I), \qquad Z=Z_{\beta,\tau}^{\alpha,\gamma,l},$$

$$\|\ \ \|=\|\ \ \|_{\alpha,l,\beta}, \qquad |||\ \ |||=|||\ \ |||_{\alpha,\gamma,l,\beta,\tau}.$$

Further, $\|\ \ \|_\beta$ is the norm in (2.5) and C denotes various constants ≥ 0 depending only on α, l, β, but not on γ, τ.

THEOREM 2.1. (i) *B^ε is a semigroup generator in $\dot{X}$ for each $\varepsilon>0$.* (ii) *If $u_0\in\dot{X}$, then*

$$e^{tB^\varepsilon}u_0\in C^0\big((0, \infty)\times[0, \infty);\ \dot{X}\big)$$

as a function of $(\varepsilon, t)\in(0, \infty)\times[0, \infty)$.

PROOF. (i) has been stated in [11] without proof. Here we give a proof since it is needed to prove the continuity in ε in (ii). Define the operator

$$A^\varepsilon=-\xi\cdot\nabla_x+\frac{1}{\varepsilon}\Lambda,$$

where ∇_x is taken in the distribution sense. Then $\mathscr{F}_x A^\varepsilon u=\hat{A}^\varepsilon(k)\hat{u}$ where

$$\hat{A}^\varepsilon(k)=-\Big(ik\cdot\xi+\frac{1}{\varepsilon}\nu(\xi)\Big)\times.$$

Let $\hat{\dot{X}}$ denote the space of $\hat{u}\in\mathscr{S}'(\boldsymbol{R}_k^n\times\boldsymbol{R}_\xi^n)$ satisfying (1.7) and (1.8). It is a Banach space with the norm of (1.7) and $\mathscr{F}_x$ is isometric from $\dot{X}$ onto $\hat{\dot{X}}$. Therefore if A^ε is a generator in $\dot{X}$, then so is $\hat{A}^\varepsilon(k)$ in $\hat{\dot{X}}$, and vice versa. We infer that

$$e^{t\hat{A}^\varepsilon(k)}=e^{-(ik\cdot\xi\nu+\frac{1}{\varepsilon}\nu(\xi))t}\times. \tag{2.8}$$

Regard this as the definition of the operator $e^{t\hat{A}^\varepsilon(k)}$, for a while, and write as $\psi=\psi(\varepsilon, t, k, \xi)$ the function in its right hand side. By (2.4), $|\psi|\leq e^{-\nu_0 t/\varepsilon}$, so that

(2.9) $$\|e^{t\hat{A}^\varepsilon(k)}\hat{u}\|_\beta\leq e^{-\nu_0 t/\varepsilon}\|\hat{u}\|_\beta,$$

and for any compact set $\Omega\subset \boldsymbol{R}_k^n\times \boldsymbol{R}_\xi^n$,

$$\psi(\varepsilon, t)\in B^0\Big((0, \infty)\times[0, \infty)\,;\ L^\infty(\Omega)\Big).$$

which, together with (1.8), proves

(2.10) $$e^{t\hat{A}^\varepsilon(k)}\hat{u}\in B^0\Big((0, \infty)\times[0, \infty)\,;\ \hat{\dot{X}}\Big).$$

This continuity in t indicates that (2.8) indeed defines a semigroup on $\hat{\dot{X}}$. Obviously its generator is $\hat{A}^\varepsilon(k)$ restricted to the domain of the generator. Denote this generator again as $\hat{A}^\varepsilon(k)$. Hence A^ε is a generator in $\dot{X}$ if endowed with the domain $D(A^\varepsilon)$ induced from that of $\hat{A}^\varepsilon(k)$, and satisfies (2.10) with an obvious modification. In addition,

(2.11) $$\dot{X}^{\alpha, l+1}_{\beta+1}\subset D(A^\varepsilon).$$

In view of (2.5), we can now redefine B^ε as

$$B^\varepsilon=A^\varepsilon+\frac{1}{\varepsilon}K, \qquad D(B^\varepsilon)=D(A^\varepsilon),$$

and apply to it the well-known theorem on the bounded perturbation of semigroups. Then (i) follows and moreover for $u\in\dot{X}$, there holds

$$e^{tB^\varepsilon}u=\sum_{l=0}^\infty U_l, \qquad U_l=\Big(e^{tA^\varepsilon}\frac{1}{\varepsilon}K*\Big)^l e^{tA^\varepsilon}u,$$

where $*$ means the convolution in t,

$$f(t)*g(t)=\int_0^t f(t-s)\,g(s)\,ds.$$

Owing to (2.5) and (2.10) for A^ε, the convergence is uniform for $(\varepsilon, t)\in[\delta, \infty)\times[0, \tau]$, for any $\delta, \tau>0$, stongly in $\dot{X}$, and

$$U_l\in B^0\Big([\delta, \infty)\times[0, \tau]\,;\ \dot{X}\Big),$$

whence (ii) follows.

REMARK 2.2. (1.8) is essential for the proof of (2.10). Thus Theorem 2.1 is no more valid if $\dot{X}$ is replaced by X.

To proceed further we should write $\mathscr{F}_x B^\varepsilon u = \hat{B}^\varepsilon(k)\,\hat{u}$ with

$$\hat{B}^\varepsilon(k) = \hat{A}^\varepsilon(k) + \frac{1}{\varepsilon} K = -ik\cdot\xi \times + \frac{1}{\varepsilon} L,$$

and appeal to the spectral analysis for $\hat{B}^1(k)$ ($\varepsilon=1$) developed in [4, 12, 15, 16]. The following theorem contains some of those results restated for $\hat{B}^\varepsilon(k) = \frac{1}{\varepsilon}\hat{B}^1(\varepsilon k)$. This was also used in [11]. See [16] for a complete proof.

THEOREM 2.3. *There are positive numbers κ_0, σ_0, a positive integer $m\,(\leq n+2)$ and a constant $C\geq 0$ such that the followings hold for each $\varepsilon\geq 0$.*

(i) *For $|k|\leq \kappa_0/\varepsilon$,*

$$e^{t\hat{B}^\varepsilon(k)} = \sum_{j=1}^{m} e^{\lambda(\varepsilon|k|)\frac{t}{\varepsilon}} P_j(\varepsilon k) + U(\varepsilon, t, k),$$

where

(a) $\lambda_j(\kappa)\in C^\infty([-\kappa_0, \kappa_0])$, $\mathrm{Re}\,\lambda_j(\kappa)\leq 0$, *with the asymptotic expansions*

$$\lambda_j(\kappa) = i\lambda_j^{(1)}\kappa - \lambda_j^{(2)}\kappa^2 + 0\left(|\kappa|^3\right) \qquad \left(|\kappa|\to 0\right),$$

with coefficients $\lambda_j^{(1)}\in \boldsymbol{R}$ and $\lambda_j^{(2)}>0$,

(b) *$P_j(k)$ are orthogonal projections on $L^2 = L^2(\boldsymbol{R}_\xi^n)$ for each fixed k, and*

$$P_j(k) = P_j^{(0)}(\tilde{k}) + |k|P_j^{(1)}(\tilde{k}) + |k|^2 P_j^{(2)}(k), \qquad \tilde{k} = k/|k|,$$

where $P_j^{(0)}$ are also orthogonal projections on L^2, $P_0 = \sum P_j^{(0)}$ is the projection in (2. 6) *and*

$$||P_j^{(l)}u||_\beta \leq C||u||_{L^2}, \qquad l=0, 1, 2,$$

and

(c) *putting $Q(k) = I - \sum_{j=1}^{m} P_j(k)$, U is given by $U = e^{t\hat{B}^\varepsilon(k)}Q(\varepsilon k) = Q(\varepsilon k)e^{t\hat{B}^\varepsilon(k)}$ and has the decomposition*

$$U = e^{t\hat{A}^\varepsilon(k)}\,Q(\varepsilon k) + U_1(\varepsilon, t, k),$$

$$||U_1 u||_\beta \leq Ce^{-\sigma_0 t/\varepsilon}\left(||u||_{\beta-1} + ||u||_{L^2}\right).$$

(ii) *For $|k|>\kappa_0/\varepsilon$,*

$$e^{t\hat{B}^\varepsilon(k)} = e^{t\hat{A}^\varepsilon(k)} + U_2(\varepsilon, t, k),$$

$$||U_2 u||_\beta \leq Ce^{-\sigma_0 t/\varepsilon}\left(||u||_{\beta-1} + ||u||_{L^2}\right).$$

A simple consequence of this is the

PROPOSITION 2.4. *Let $\beta>\frac{n}{2}$ and $u_0\in\dot{X}$. Then*

(i) $e^{tB^\varepsilon}u_0\in Z$,

(ii) $|||e^{tB^\varepsilon}u_0|||\le C||u_0||$.

PROOF. In view of Theorem 2.1, it suffices to show (ii). Using Theorem 2.3 and (2.9), we evaluate each term of the decomposition

$$e^{tB^\varepsilon}=\sum_{j=1}^{m+2}E_j^\varepsilon(t),$$

$$\mathscr{F}_x E_j^\varepsilon(t)\,u_0=\chi\Big(|k|<\kappa_0/\varepsilon\Big)e^{\lambda_j(\varepsilon|k|)\frac{t}{\varepsilon}}P_j(\varepsilon k)\,\hat{u}_0,\qquad 1\le j\le m,$$

$$\mathscr{F}_x E_{m+1}^\varepsilon(t)u_0=\chi\Big(|k|<\kappa_0/\varepsilon\Big)U(\varepsilon,t,k)\,\hat{u}_0,\tag{2.12}$$

$$\mathscr{F}_x E_{m+2}^\varepsilon(t)u_0=\chi\Big(|k|>\kappa_0/\varepsilon\Big)e^{t\hat{B}^\varepsilon(k)}\,\hat{u}_0.$$

Then (ii) follows if we take into account that for $\beta>\frac{n}{2}$,

$$||u||_{L^2}\le C||u||_\beta.\tag{2.13}$$

The following elementary lemma is useful in what follows. The proof is omitted.

LEMMA 2.5. (i) *For any $a\in\boldsymbol{C}$, $Re\,a<0$ and $\sigma\in[0,1]$,*

$$|e^a-1|\le C|a|^\sigma.$$

(ii) *For any $t\ge\delta\ge0$, $b>0$ and $\sigma\in[0,1)$,*

$$I_0(t,\delta,b)\equiv\int_0^{t-\delta}e^{-b(t-s)}b\,ds\le e^{-b\delta},$$

$$I_1(t,\sigma,b)\equiv\int_0^t e^{-b(t-s)}bs^{-\sigma}ds\le Ct^{-\sigma},$$

$$I_2(t,\sigma,b)\equiv\int_0^t e^{-b(t-s)}b^{1+\sigma}\Big(\frac{t-s}{s}\Big)^\sigma ds\le Ct^{-\sigma}.$$

All the constants $C\ge0$ are independent of a, b, t.

We shall use (ii) of the above lemma putting $b=\gamma|k|$, $\frac{\nu(\xi)}{\varepsilon}$ and $\frac{\sigma_0}{\varepsilon}$, all of which are unbounded when $|k|$, $|\xi|\to\infty$, $\varepsilon\to0$, and on which, nevertheless, the constants C do not depend. In particular, the case $b=\gamma|k|$ is essential in the proof of the next proposition which reveals the reason why the use of the time-dependent norm (1. 10) makes possible the application of the contraction mapping principle. The importance of the case $b=\nu(\xi)$ was found by Grad [6].

Put $Q_0=I-P_0=Q(0)$ and define the operator $H=H^\varepsilon$ by

$$H^\varepsilon f^\varepsilon = e^{tB^\varepsilon} * Q_0 \frac{1}{\varepsilon} \Lambda f^\varepsilon(t) . \tag{2.14}$$

PROPOSITION 2. 6. *Suppose* $\beta>\frac{n}{2}+1$ *and* $f=f^\varepsilon\in Z$. *Then,*

(i) $H^\varepsilon f^\varepsilon \in Z$,

(ii) $|||H^\varepsilon f^\varepsilon||| \leq C\left(1+\frac{1}{\gamma}\right)|||f^\varepsilon|||$,

where the constant $C\geq 0$ *is independent of* γ *and* τ.

PROOF. We should show this for each of the operators

$$H_j^\varepsilon f^\varepsilon = E_j^\varepsilon(t) * Q_0 \frac{1}{\varepsilon} \Lambda f^\varepsilon(t) , \qquad 1\leq j\leq m+2 . \tag{2.15}$$

First we establish (ii). The conclusions as well as notations of Theorem 2. 3 will be used freely. Let $1\leq j\leq m$. Since

$$P_j(\varepsilon k)\, Q_0 = \varepsilon|k| \left(P_j^{(1)} + \varepsilon|k| P_j^{(2)}\right) Q_0 \tag{2.16}$$

and by the aid of (2. 13) with u replaced by Λf and β by $\beta-1>\frac{n}{2}$,

$$\begin{aligned} ||\mathscr{F}_x H_j^\varepsilon f^\varepsilon||_\beta &\leq C\left(1+|k|\right)^{-l} \int_0^t e^{-(\alpha-\gamma s)|k|}|k| ds |||f^\varepsilon|||_t \\ &\leq C\zeta(t, k) \frac{1}{\gamma} I_0\left(t, 0, \gamma|k|\right) |||f^\varepsilon|||_t , \end{aligned} \tag{2.17}$$

where $|||\ \ |||_t = |||\ \ |||_{\alpha,\gamma,l,\beta,t}$, and $\zeta(t, k)=(1+|k|)^{-l} e^{-(\alpha-\gamma t)|k|}$. This proves (ii) for H_j^ε, $1\leq j\leq m$. Similarly, we can get

$$\begin{aligned} ||e^{t\hat{A}^\varepsilon(k)} * Q_0 \frac{1}{\varepsilon} \Lambda \hat{f}^\varepsilon||_\beta &\leq C\zeta(t, k) \sup_\xi I_0\left(t, 0, \frac{\nu(\xi)}{\varepsilon}\right) |||f^\varepsilon|||_t , \\ ||U_j(\varepsilon, t, k) * Q^0 \frac{1}{\varepsilon} \Lambda \hat{f}^\varepsilon(t)||_\beta &\leq C\zeta(t, k)\, I_0\left(t, 0, \frac{\sigma_0}{\varepsilon}\right) |||f^\varepsilon|||_t , \end{aligned} \tag{2.18}$$

whence (ii) follows for H_{m+j}^ε, $j=1, 2$.

It remains to prove (i). If $\Lambda f\in Z$, then by virtue of Theorem 2. 1,

$$H^{\epsilon}f^{\epsilon}\in C^{0}\big((0,\infty);\ Y([0,\infty])\big). \tag{2.19}$$

The former, however, is not the case on accout of (2. 4). This can be overcome in a simple way. Note from (ii) and (1. 8) that

$$|||H^{\epsilon}\chi\big(|\xi|>R\big)f^{\epsilon}|||\to 0\ \ (R\to\infty),$$

and from (2. 4) that since $\Lambda\chi(|\xi|<R)f^{\epsilon}\in Z$, (2. 19) is true if f^{ϵ} is replaced by $\chi(|\xi|<R)f^{\epsilon}$ for each fixed R. Combine these two to see that (2. 19) holds under the mere assumption $f^{\epsilon}\in Z$. This and (ii) show (i).

Finally we need to study the operator Γ. The following proof shows implicitly that Z is a Banach algebra if $l>n$ and $\beta\geq 0$.

PROPOSITION 2. 7. *Suppose $l>n$ and $\beta\geq 0$, and let u, $v\in Z$.*

(i) $\Lambda^{-1}\Gamma[u,v]\in Z$.

(ii) $|||\Lambda^{-1}\Gamma[u,v]|||\leq C|||u|||\ |||v|||$.

PROOF. Since Γ is bilinear symmetric and acts only on ξ,

$$\mathscr{F}_{x}\Lambda^{-1}\Gamma[u,v](k)=(2\pi)^{n/2}\int_{\mathbf{R}^{n}}\Lambda^{-1}\Gamma[\hat{u}^{\epsilon}(t,k-k',\cdot),\hat{v}^{\epsilon}(t,k',\cdot)]\,dk'.$$

Use (2. 7) to deduce

$$\begin{aligned}||\mathscr{F}_{x}\Lambda^{-1}\Gamma[u,v](k)||_{\beta}&\leq C\int_{\mathbf{R}^{n}}||\hat{u}^{\epsilon}(t,k-k',\cdot)||_{\beta}||\hat{v}^{\epsilon}(t,k',\cdot)||_{\beta}dk'\\&\leq CI_{3}(k)|||u|||\ |||v|||,\end{aligned}$$

where

$$\begin{aligned}I_{3}(k)&=\int_{\mathbf{R}^{n}}\big(1+|k-k'|\big)^{-l}\big(1+|k'|\big)^{-l}e^{-(\alpha-\gamma t)\,(|k-k'|+|k'|)}\,dk'\\&\leq C\zeta(t,k), \qquad (l>n)\end{aligned}$$

ζ being that used in the proof of the previous proposition. The last inequality is obtained since $|k|\leq|k-k'|+|k'|$ and by splitting the integral over $|k-k'|\leq|k'|/2$ and $|k-k'|\geq|k'|/2$. This proves (ii). To prove (i), let V^{l} be the space defined by (1. 10) like $Z=Z_{\beta,\tau}^{\alpha,\gamma,l}$ but with $\dot{X}_{\beta}^{\alpha,l}$ replaced by $X_{\beta}^{\alpha,l}$, and put $V=\bigcap_{l'>l}V^{l'}$. We claim from (1. 8) that Z is a strong closure of V in V^{l} and from the proof of (ii) above that (i) is true if Z is replaced by V, for u, $v\in V$. These two, combined again with (ii), assure (i).

Now we are ready to proceed to the

PROOF OF THEOREM 1.1. (i). Define the nonlinear map $N=N^{\epsilon}$ by

$$N^{\epsilon}[u^{\epsilon}](t)=e^{tB^{\epsilon}}u_0+H^{\epsilon}\Lambda^{-1}\Gamma[u^{\epsilon},u^{\epsilon}](t). \tag{2.20}$$

Since $Q_0\Gamma=\Gamma$ by (2.6) and Theorem 2.2 (ic), the equation (2.3) is then equivalent to the equation $u=N[u]$. Thus we shall find a fixed point of N. Combine Propositions 2.4, 2.6 and 2.7 together. Then, whenever $u_0\in\dot{X}$ and $u\in Z$, $N[u]\in Z$ and

$$\begin{aligned}&|||N[u]|||\leq C_1||u_0||+C_2\left(1+\frac{1}{\gamma}\right)|||u|||^2,\\&|||N[u]-N[v]|||=|||H^{\epsilon}\Lambda^{-1}\Gamma[u-v,u+v]|||\\&\qquad\leq C_2\left(1+\frac{1}{\gamma}\right)|||u-v|||\;|||u+v|||.\end{aligned}\tag{2.21}$$

Since the constants C_1, $C_2\geq 0$ are independent of γ and τ, so is $a_0=(4C_1C_2)^{-1}$. Suppose $||u_0||<a_0$ and choose a γ such that

$$\gamma>||u_0||/\left(a_0-||u_0||\right).$$

If we put

$$\tau=\frac{\alpha}{\gamma},$$

$$a_1=\left(2C_2\left(1+\frac{1}{\gamma}\right)||u_0||\right)^{-1}\mu,\qquad \mu=1-\left(1-\left(1+\frac{1}{\gamma}\right)\frac{1}{a_0}||u_0||\right)^{1/2},$$

then $a_1<2C_1$ and $0<\mu<1$. Denote by Z_0 the closed ball

$$Z_0=\left\{u\in Z\,\middle|\;|||u|||\leq a_1||u_0||\right\},$$

in the space Z, and regard this as a complete metric space with the metric $d(u,v)=|||u-v|||$. Then (2.21) implies that $N[u]\in Z_0$ and $d(N[u],N[v])\leq \mu d(u,v)$ whenever u, $v\in Z_0$. In other words, N is a contraction map on Z_0 if $||u_0||<a_0$, and hence has a unique fixed point $u=u^{\epsilon}(t)\in Z_0$, which solves (2.3).

It remains to show that this u is also a unique classical solution to (2.1). Let $h>0$ and note from (2.20) the identity

$$\frac{1}{h}\Big(N[u](t+h)-N[u](t)\Big)=\frac{1}{h}(e^{hB^\varepsilon}-I)\,N[u](t)+$$
(2. 22)
$$+\frac{1}{h}\int_t^{t+h} e^{(t+h-s)B^\varepsilon}Q_0\frac{1}{\varepsilon}\Gamma[u(s),u(s)]\,ds\equiv v_1+v_2\,.$$

Since $N[u]\in Z$, $N[u]\in D(B^\varepsilon)$ by (2. 11) if B^ε is considered in the space $\dot{X}_{\beta-1}^{\alpha-\gamma t,l-1}$. Then $v_1\to B^\varepsilon N[u]$ $(h\to 0)$ strongly there, for fixed ε, t. Put $\Gamma(t)=\Gamma[u(t),u(t)]$ and write

$$v_2-\frac{1}{\varepsilon}\Gamma(t)=\frac{1}{h}\int_0^h(e^{sB^\varepsilon}-I)\,ds\frac{1}{\varepsilon}\Gamma(t)$$
(2. 23)
$$+\frac{1}{h}\int_0^h e^{sB^\varepsilon}\frac{1}{\varepsilon}\Big(\Gamma(t+h-s)-\Gamma(t)\Big)\,ds\,.$$

Evaluate this by the aid of Proposition 2. 4 and

(2. 24)
$$||u(t)\pm u(s)||_{\alpha-\gamma t,l-\sigma,\beta-\sigma}$$
$$\leq||e^{-\gamma t|k|}u(t)\pm e^{-\gamma s|k|}u(s)||_{\alpha,l-\sigma,\beta-\sigma}+C|t-s|^\sigma\,|||u|||\,,$$

which comes from lemma 2. 4 (i) with $a=-\gamma(t-s)|k|$. Since $\Gamma(t)\in Y_{\beta-1}^{\alpha,\gamma,l}([0,\tau])$ by Proposition 2. 7, then (2. 23)$\to 0$ strongly in $\dot{X}_{\beta-1}^{\alpha-\gamma t,l}$. The case $h<0$ can be dealt with similarly. All the convergences are uniform for t on $[0,\tau-\delta]$ when $h>0$ and on $[\delta,\tau]$ when $h<0$. Since $u=N[u]$, then (2. 22) implies that du/dt exists satisfying

$$\frac{du}{dt}=B^\varepsilon u(t)+\frac{1}{\varepsilon}\Gamma[u(t),u(t)]\in Y_{\beta-1}^{\alpha,\gamma,l-1}\Big([0,\tau]\Big)\,.$$

Since $u(t)$ is analytic in $x\in\boldsymbol{R}^n+i B_{\alpha-\gamma t}$, this indicates that u is a classical solution to (2. 1). Its uniqueness has been proven in [1, 7]. This completes the proof of Theorem 1. 1 (i).

3. Limit of solutions as $\varepsilon\to 0$

We shall again use the contraction mapping principle to obtain the relevant limit. A suitable space is the subset $W=W_{\beta,\tau}^{\alpha,\gamma,l}$ of Z consisting of u's having the same limit property stated in Theorem 1. 1 (ii), that is,

(3. 1)
$$u=u^\varepsilon\in W\Leftrightarrow u\in Z\,,\ \exists u^0\in Y\Big((0,\tau]\Big)\,,$$
$$\forall\delta>0,\ ||u^\varepsilon-u^0||_{Y([\delta,\tau])}\to 0\ (\varepsilon\to 0)\,.$$

Evidently W is closed in Z and so by itself a Banach space with the norm

$|||\ \ |||$. We shall show that N of (2. 20) is a contraction also in the ball $W_0=W\cap Z_0$. First, the limit of $N=N^{\varepsilon}$ must be found.

In the below we assume $\beta>\frac{n}{2}+1$, $\sigma\in[0, 1)$, and need the norm

$$[u]_{\varepsilon,\sigma}=\sup_{\substack{0\leq\varepsilon'<\varepsilon\\ 0\leq s\leq\tau}}\left(\frac{s^2}{\varepsilon'}\right)^{\sigma/2}||u^{\varepsilon'}(s)||_{\alpha-\gamma s,l-\sigma,\beta-\sigma}.$$

Define the operators $E^0(t)$, $E_j^0(t)$, $1\leq j\leq m$ by $E^0=\sum E_j^0$ with

$$\mathscr{F}_x E_j^0(t)\,u_0=e^{i\lambda_j^{(0)}|k|t}P_j^{(0)}\,\hat{u}_0(k,\,\cdot)\,.$$

Proposition 3. 1. *Suppose $u_0\in X$. Then,*

(i) *$e^{tB^{\varepsilon}}u_0\in W$ with the limit $E^0(t)u_0$.*

(ii) $[(e^{tB^{\varepsilon}}-E^0(t))u_0]_{\varepsilon,\sigma}\leq C||u_0||$.

Proof. Put $||\ \ ||_{t,\sigma}=||\ \ ||_{\alpha-\gamma t,l-\sigma,\beta-\sigma}$ and $u'=\chi(|k|>R)\,u_0$. Then we get

$$||e^{tB^{\varepsilon}}\,u'||_{t,\sigma}\leq C||u'||_{0,\sigma}\leq C(1+R)^{-\sigma}||u_0||\,,$$

by Proposition 2. 4, and similarly for $E^0(t)u'$. Put $u''=\chi(|k|<R)\,u_0$ and let $R<\kappa_0/\varepsilon$. If $|k|<R$, then

$$(3.\,2)\qquad \begin{aligned}&|e^{i\lambda_j(|\varepsilon|k|)t/\varepsilon}-e^{i\lambda_j^{(0)}|k|t}|\leq C\varepsilon R^{2-\sigma}|k|^{\sigma}\\ &||\left(P_j(\varepsilon|k|)-P_j^{(0)}\right)\hat{u}_0||_{\beta}\leq C\varepsilon R^{1-\sigma}|k|^{\sigma}||\hat{u}_0||_{\beta}\,.\end{aligned}$$

Therefore, with $E_j^{\varepsilon}(t)$ of (2. 12), we get for $1\leq j\leq m$,

$$||\left(E_j^{\varepsilon}(t)-E_j^0(t)\right)u''||_{t,\sigma}\leq C\varepsilon R^{2-\sigma}||u_0||\,.$$

Further $||E_{m+1}^{\varepsilon}(t)\,u_0||_{t,\sigma}$ can be majorized by Theorem 2. 2 (ic) by $Ce^{-\mu t/\varepsilon}||u_0||$ with $\mu=\min(\nu_0,\sigma_0)=\sigma_0$, and $E_{m+2}^{\varepsilon}u''=0$ by our choice of R. In the above estimates, we first put $\sigma=0$, $R=\kappa_0\varepsilon^{-1/4}$ to prove (i) by the aid of (1. 8) (note that $E_j^0(t)\,u_0\in Y([0,\tau])$), and then put $R=\kappa_0\varepsilon^{-1/2}$ to prove (ii).

Define the operators H_j^0, $1\leq j\leq m$ by

$$\mathscr{F}_x H_j^0 v(t)=\left(e^{i\lambda_j^{(0)}|k|t}|k|P^{(1)}(\tilde{k})*Q_0\Lambda\hat{v}(t)\right),$$

where $*$ means, as before, the convolution in t.

Lemma 3. 2. *For $f=f^{\varepsilon}\in W$ with the limit f^0 and for $1\leq j\leq m$,*

(i) *$H_j^0 f^{\varepsilon}\in W$ with the limit $H_j^0 f^0$,*

(ii) $[H_j^0(f^{\varepsilon}-f^0)]_{\varepsilon,\sigma}\leq C[f^{\varepsilon}-f^0]_{\varepsilon,\sigma}$.

Proof. For $\delta>0$, define the operators A_j^{δ}, $j=1, 2$ by

$$H_j^0 v = \int_\delta^t + \int_0^\delta \equiv A_1^\delta v + A_2^\delta v\,.$$

Similarly to (2. 17), we get for $t \geq \delta$,

$$\text{(3. 3)} \qquad \| A_1^\delta v \|_{t,0} \leq \frac{C}{\gamma} \| v \|_{Y([\delta,t])}\,.$$

Repeat the argument for Proposition 2. 6 (i) to conclude $A_1^\delta f^0 \in Y([\delta, \tau])$. Note that $|k| e^{-\gamma(t-s)|k|} \leq C/(t-s)$, $t>s$, and deduce $\| A_2^\delta v \|_{t,\sigma} \leq C\delta (t-\delta)^{-1} |||v|||$. Hence for any fixed $\delta_0 > 0$,

$$\| A_2^\delta v \|_{Y([\delta_0,\tau])} \to 0 \ (\delta \to 0)\,.$$

Putting $v=f^0$, we now have $H^0 f^0 \in Y((0, \tau])$, and putting $v=f^\varepsilon - f^0$, we can conclude (i). (ii) can be found similarly to (2. 17), with $I_0(t, 0, \gamma|k|)$ replaced by $I_1(t, \sigma, \gamma|k|)\,\varepsilon^\sigma$.

In the following three lemmas, we put $f_R^\varepsilon = \chi(|k|+|\xi|<R) f^\varepsilon$ for $f=f^\varepsilon \in W$ and assume $R<\kappa_0/\varepsilon$. All the constants $C \geq 0$ do not depend on R as well as γ, ε, τ and f.

LEMMA 3. 3. *Recall* H_j^ε *of* (2. 15). *For* $1 \leq j \leq m$,

$$[(H_j^\varepsilon - H_j^0) f_R^\varepsilon]_{\varepsilon,\sigma} \leq C \varepsilon^{1-\sigma/2} R^{2-\sigma} |||f|||\,.$$

PROOF. Note that $(P_j(\varepsilon|k|) - \varepsilon|k| P_j^{(1)}(\tilde{k}))\, Q_0 = \varepsilon^2 |k|^2 P_j^{(2)}$ and use (3. 2) to obtain

$$\| \mathscr{F}_x (H_j^\varepsilon - H_j^0) f_R^\varepsilon \|_\beta \leq C \zeta(t, k)\, I_0\big(t, 0, \gamma|k|\big)\, \varepsilon R^{2-\sigma} |k|^\sigma |||f|||\,,$$

whence the lemma follows.

For the study of H_{m+1}^ε, we need the auxisiliary operator

$$\tilde{H}_{m+1}^\varepsilon f^\varepsilon = \left(e^{tL/\varepsilon} Q_0 \frac{1}{\varepsilon} \Lambda \right) * f^\varepsilon(t)\,.$$

Since the operator L has 0 as an isolated eigenvalue, L^{-1} does not exist but $L^{-1} Q_0$ does. Put

$$H_{m+1}^0 = -L^{-1} Q_0 \Lambda\,.$$

Since $L^{-1} Q_0 \Lambda = Q_0 - L^{-1} Q_0 K$ and by (2. 5),

(3. 4) $\qquad H_{m+1}^0$ is bounded on $Y((0, \tau])$.

LEMMA 3. 4. (i) $\| \, |k \cdot \xi|^\kappa \tilde{H}_{m+1}^\varepsilon f_R^\varepsilon \|_{t,\sigma} \leq C R^{2(\kappa-\sigma)} |||f|||$, $0 \leq \sigma$, $\kappa \leq 1$.

(ii) $\|(\tilde{H}^{\varepsilon}_{m+1}-H^{0}_{m+1})f^{\varepsilon}_{R}\|_{t,\sigma}\leq C\,\{(e^{-\mu\frac{\delta}{\varepsilon}}+\delta R^{1-\sigma})|||f|||+\left(\frac{\varepsilon}{t}\right)^{\sigma}[[f^{\varepsilon}]]_{t,\delta,\sigma}\}$,

for $t\geq\delta\geq 0$, *where* $\|\quad\|_{t,\sigma}$ *is as in the proof of Proposition* 3.1, $\mu=\min(\nu_0,\sigma_0)$ *and*

$$[[v]]_{t,\delta,\sigma}=\sup_{t-\delta\leq s<t}\left(\frac{s}{t-s}\right)^{\sigma}\|e^{-\gamma s|k|}v(s)-e^{-\gamma t|k|}v(t)\|_{\alpha,l-\sigma,\beta-\sigma}\,.$$

PROOF. By Theorem 2.3 (ic), $e^{tL/\varepsilon}Q_0=U(\varepsilon,t,0)$. Then (i) comes from (2.18) since $k\cdot\xi$ commutes with $e^{t\hat{A}^{\varepsilon}(k)}$ and k with U_1. Make the decomposition

$$\tilde{H}^{\varepsilon}_{m+1}f^{\varepsilon}_{R}=\int_0^{t-\delta}+\int_{t-\delta}^{t}\equiv w_1+w_2\,,$$

and use (2.18) with $I_0(t,\delta,\cdot)$ in place of $I_0(t,0,\cdot)$ to deduce

$$\|w_1\|_{t,\sigma}\leq Ce^{-\mu\delta/\varepsilon}|||f|||\,. \tag{3.5}$$

Next, write

$$w_2=\int_{t-\delta}^{t}e^{\frac{t-s}{\varepsilon}L}Q_0\frac{1}{\varepsilon}\Lambda\left(f^{\varepsilon}_{R}(s)-f^{\varepsilon}_{R}(t)\right)ds+\int_0^{\delta/\varepsilon}e^{sL}Q_0\Lambda f^{\varepsilon}_{R}(t)\,ds$$
$$\equiv w_3+w_4\,.$$

Again by (2.18), this time with $I_2(t,\sigma,\cdot)$, and by an inequality similar to (2.24) obtained from Lemma 2.4 (i) with $\sigma=1$,

$$\|w_3\|_{t,\sigma}\leq C\left(\left(\frac{\varepsilon}{t}\right)^{\sigma}[[f^{\varepsilon}]]_{t,\delta,\sigma}+\delta R^{1-\sigma}|||f|||\right).$$

Finally we put

$$w_4=\int_0^{\infty}-\int_{\delta/\varepsilon}^{\infty}\equiv w_5+w_6\,.$$

The Laplace transform of a semigroup is the resolvent of its generator. Then $w_5=H^{0}_{m+1}f^{\varepsilon}_{R}$. And it is easy to see that (3.5) is true also for w_6. Combining w_j's yields (ii).

LEMMA 3.5. $[(H^{\varepsilon}_{m+1}-\tilde{H}^{\varepsilon}_{m+1})f^{\varepsilon}_{R}]_{\varepsilon,\sigma}\leq C\varepsilon^{1-\sigma/2}R^{2-2\sigma}|||f|||\,.$

PROOF. Notice from the perturbation theory of semigroups that

$$e^{tB^{\varepsilon}}u=\left(I+e^{tB^{\varepsilon}}(-ik\cdot\xi)*\right)e^{tL/\varepsilon}\,\hat{u}$$

holds in $\dot{X}^{\alpha,l-1}_{\beta-1}$ if $u\in\dot{X}^{\alpha,l}_{\beta}$. Therefore

$$\left(H_{m+1}-\tilde{H}_{m+1}\right)f_R^\varepsilon=\left\{\left(Q(\varepsilon k)-I\right)Q_0+U*\left(-ik\cdot\xi\right)\right\}\tilde{H}_{m+1}^\varepsilon f_R^\varepsilon,$$

where $U=\mathscr{F}_x^{-1}U(\varepsilon,t,k)\mathscr{F}_x$. By (2.18),

$$|||U*f_R^\varepsilon|||\le C\varepsilon|||f|||.$$

This and Lemma 3.4 (i), together with (2.16), prove the lemma.

PROPOSITION 3.6. *Recall H^ε of* (2.14) *and put*

$$H^0=\sum_{j=1}^{m+1}H_j^0.$$

Let $f^\varepsilon\in W$ with the limit f^0.

(i) *$H^\varepsilon f^\varepsilon\in W$ with the limit H^0f^0.*

(ii) $[H^\varepsilon f-H^0f^0]_{\varepsilon,\sigma}\le C\left(|||f|||+[f^\varepsilon-f^0]_{\varepsilon,\sigma}+[[f^\varepsilon]]_{t,\tau,\sigma}\right).$

PROOF. By virtue of Lemma 3.2 (i) and (3.4), $H^0f^0\in Y((0,\tau])$. Let f_R^ε be as above and $\tilde{f}_R^\varepsilon=f-f_R^\varepsilon=\chi(|k|+|\xi|>R)f^\varepsilon$. Write

$$H^\varepsilon f^\varepsilon-H^0f^0=(H^\varepsilon-H^0)f_R^\varepsilon+(H^\varepsilon-H^0)\tilde{f}_R^\varepsilon+H^0(f^\varepsilon-f^0)\equiv\sum_{j=1}^{3}w_j.$$

Combining Lemmas 3.3, 3.4 (ii) and 3.5 yields

$$||w_1||_{t,\sigma}\le C\left(\left(\frac{\varepsilon}{\gamma}R^{2-\sigma}+\delta R^{1-\sigma}+e^{-\mu\frac{\delta}{\varepsilon}}\right)|||f|||+\left(\frac{\varepsilon}{t}\right)^\sigma[[f^\varepsilon]]_{t,\delta,\sigma}\right).$$

By Proposition 2.6 (ii) and (3.3) (with $\delta=0$),

$$||w_2||_{t,\sigma}\le C||\tilde{f}_R^\varepsilon||_{t,\sigma}\le C(1+R)^{-\sigma}|||f|||.$$

Put $\sigma=0$, $R=\kappa_0\varepsilon^{-1/4}$ and $\delta=\varepsilon^{1/2}$, and recall (1.8). Note from (3.1) that for $f^\varepsilon\in W$ with the limit f^0, if we put $f^{\varepsilon=0}=f^0$, then,

$$e^{-\gamma t|k|}f^\varepsilon(t)\in B^0\left([0,\infty)\times[\delta_0,\tau];\dot{X}_\beta^{\alpha,l}\right) \tag{3.6}$$

for any $\delta_0>0$. Consequently $w_1+w_2\to0$ $(\varepsilon\to0)$ strongly in $Y([\delta_0,\tau])$. The same is true for w_3 due to Lemma 3.2 (i). Thus (i) follows. Put $R=\kappa_0\varepsilon^{-1/2}$ and $\delta=\varepsilon^{1/2}t$ in the above and recall Lemma 3.2 (ii) to obtain (ii) of the lemma.

PROOF OF THEOREM 1.1 (ii) (iii). Proposition 2.7 implies that $\Lambda^{-1}\Gamma[u,u]\in W$ if $u\in W$. Owing to Propositions 3.1 (i) and 3.6 (i), therefore, N maps W into itself, and so N is a contraction in $W\cap Z_0$ as well as in Z_0. Then u^ε of Theorem 1.1 (i) is in W, proving Theorem 1.1 (ii). Its limit u^0 satisfies, for $t\in(0,\tau]$,

$$u^0=N^0u^0\equiv E^0(t)u_0+H^0\Lambda^{-1}\Gamma[u^0,u^0].$$

We note $Q_0E^0(t)=0$ and $Q_0H_j^0=0$ $(1\leq j\leq m)$. Indeed $Q_0P_j^{(0)}=0$ and by $P_j(k)^2=P_j(k)$, $P_j^{(1)}Q_0=P_j^{(0)}P_j^{(1)}Q_0$. Hence $Q_0u^0=Q_0H_{m+1}^0\Lambda^{-1}\Gamma[u^0, u^0]$, or what is the same thing since $L=Q_0L=-LQ_0$ and by (2.6),

$$Lu^0=-\Gamma[u^0, u^0]. \tag{3.7}$$

This is equivalent to (1.2) if we put $f^0=g+g^{1/2}u^0$, proving Theorem 1.1 (iii).

PROOF OF THEOREM 1.4. Introduce the norm

$$[[f]]=|||f|||+\sup_{0<\varepsilon\leq 1}\left\{[f^\varepsilon-f^0]_{\varepsilon,\sigma}+[[f^\varepsilon]]_{\tau,\tau,\sigma}\right\}.$$

By virtue of Propositions 2.7, 3.1 (ii) and 3.7 (ii), we readily see that (2.21) remains true in this norm, with different constants C_1, C_2 if there hold

$$[[e^{tB^\varepsilon}u_0]]_{t,t,\sigma}\leq C\|u_0\|,$$

$$[[H^\varepsilon f^\varepsilon]]_{t,t,\sigma}\leq C\left(|||f|||+[[f^\varepsilon]]_{t,t,\sigma}\right).$$

These were proved in Propositions 2.2 and 4.1 of [11] for the case $\gamma=0$. For $\gamma>0$, it suffice to take account of (2.24). Now the proof of Theorem 1.1 (i) can be repeated to see that N is again a contraction in a closed ball $\{f\in Z\mid [[f]]\leq a_1'\}$ provided $\|u_0\|$ is small. This completes the proof of the theorem.

4. Initial layer of the solution

Introduce the space

$$V\equiv V_{\beta,\tau}^{\alpha,\gamma,l}=B^0\left([0,\infty);\ Y[0,\tau]\right).$$

V may be regarded as a subspace of W and so it is a Banach space with the norm $|||\ \ |||$. The following characterization of V comes readily from (1.10) and (3.6).

LEMMA 4.1. *Let $u=u^\varepsilon(t)\in W$ with the limit $u^0=u^0(t)$. Suppose*

$$\exists u_0\in\dot{X},\qquad \|e^{-\gamma t|k|}u^\varepsilon(t)-u_0\|\to 0\qquad (\varepsilon, t\to 0,\ \varepsilon, t>0).$$

Then $u\in V$ if $u^\varepsilon(t)$ is extended to $\varepsilon=0$ by $u^0(t)$ for $t>0$ and by u_0 for $t=0$.

Given a $u_0\in\dot{X}$, $V(u_0)$ denotes a closed subset of V defined as

$$V(u_0)=\left\{u^\varepsilon(t)\in V\mid u^0(0)=u_0\right\}.$$

The proof of Theorem 1.6 is based on the

PROPOSITION 4.2. *Suppose* $u_0 \in \dot{X}$ *fulfill the condition* (1.11). *Then* N *of* (2.20) *maps* $V(u_0)$ *into itself.*

PROOF. By (1.11)(ii), $\xi \cdot \nabla_x u_0$, $Lu_0 \in \dot{X}$ as well as $u_0 \in D(B^\varepsilon)$. Since $de^{tB^\varepsilon}/dt = e^{tB^\varepsilon} B^\varepsilon$ holds on $D(B^\varepsilon)$, we have

$$e^{tB^\varepsilon} u_0 = u_0 + \int_0^t e^{sB^\varepsilon} (-\xi \cdot \nabla_x)\, u_0 ds + \int_0^t e^{(t-s)B^\varepsilon} \frac{1}{\varepsilon} Lu_0 ds$$
$$\equiv u_0 + w_1^\varepsilon + w_2^\varepsilon .$$

Owing to Proposition 2.4 and by the aid of Lemma 4.1 ($u_0=0$), $w_1^\varepsilon \in V$ with

$$||w_1^\varepsilon||_{t,0} \leq Ct ||u_0||_{\alpha,\gamma+1,l+1} .$$

On the other hand, (1.11)(i) means that (3.7) holds for u_0, so $Lu_0 = -Q_0 \Gamma[u_0, u_0]$ due to (2.6). Hence $w_2^\varepsilon = -H^\varepsilon \Lambda^{-1} \Gamma[u_0, u_0]$ and

$$N^\varepsilon[u^\varepsilon] = u_0 + w_1^\varepsilon + H^\varepsilon \Lambda^{-1} \Gamma[u^\varepsilon - u_0, u^\varepsilon + u_0] .$$

Use Propositions 2.6, 2.7 and (2.24) to deduce

$$\begin{aligned} &||e^{-\gamma t|k|} N^\varepsilon[u^\varepsilon](t) - u_0|| \\ &\leq ||(e^{-\gamma t|k|} - 1)\, u_0|| + ||w_1^\varepsilon||_{t,0} + \\ &\qquad + C\Big(|||u||| + ||u_0||\Big) \sup_{0\leq s\leq t} ||u^\varepsilon(s) - u_0||_{t,0} \\ &\leq C\Big(1 + ||u_0|| + |||u|||\Big) \Big(t ||u_0||_{\alpha,l+1,\beta+1} + \\ &\qquad + \sup_{0\leq s\leq t} ||e^{-\gamma s|k|} u^\varepsilon(s) - u_0||\Big) . \end{aligned}$$

If $u \in V(u_0)$, the last member tends to 0 with ε, t. Then the proposition follows from Lemma 4.1.

PROOF OF THEOREM 1.6. It is now clear that N is a contraction on $V(u_0) \cap Z_0$, so that u^ε of Theorem 1.1(i) is in $V(u_0)$, which is what was desired.

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