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Linear Volterra integral equations of parabolic type

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1. Introduction

We consider the linear Volterra equation

$$u(t)+\int_{0}^{t}h(t-\tau)\,A(\tau)\,u(\tau)\,d\tau=k(t) \tag{1.1}$$

in a Banach space X, where u, k are functions with values in X, h is a scalar function and $A(t)$ is a linear closed operator which generates an analytic semigroup. In [2] Friedman and Shinbrot proved the existence and uniqueness of the solution of (1.1) assuming that the domain of $A(t)$ is independent of t, $A(t)\,A(0)^{-1}$ is Hölder continuous and h, k satisfy some smoothness conditions. In order that the integral in (1.1) exists as a Bochner integral it was assumed that $k(0)\in D(A^{\mu}(0))$ for some $0<\mu\leqq 1$.

They also constructed the fundamental solution $W(t, s)$ which is an operator valued function satisfying

$$W(t, s)+\int_{s}^{t}h(t-\tau)\,A(\tau)\,W(\tau, s)\,d\tau=I \tag{1.2}$$

in some sense. The fundamental solution constructed in [2] is not a bounded operator but has the form $W(t, s)=W_{\mu}(t, s)\,A^{\mu}(s)$ with some bounded operator $W_{\mu}(t, s)$ for any $\mu\in(0,1]$.

In this paper using the idea of Crandall-Nohel [1] we transform (1.1) to the initial value problem of the evolution equation

$$du(t)/dt+h(0)\,A(t)\,u(t)=(Gu)(t)\,, \qquad u(0)=k(0)\,, \tag{1.3}$$

where G is some mapping defined on $C([0, T]\,;\,X)$. This problem has a solution for any initial value $k(0)\in X$. In general for the solution u of (1.3) the integral of (1.1) does not exist in the sense of Bochner integral since $\|A(\tau)\,u(\tau)\|=O(\tau^{-1})$ as $\tau\rightarrow 0$. However, it will be shown that if we interpret it as the imporper integral

$$\lim_{\epsilon\downarrow 0}\int_{\epsilon}^{t}h(t-\tau)\,A(\tau)\,u(\tau)\,d\tau\,,$$

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then the integral exists and (1. 1) holds.

Analogously the equation (1. 2) is transformed to the initial value problem of some differential equation, and we define $W(t, s)$ as the solution of this problem. Then $W(t, s)$ is a bounded operator valued function, and we have

$$\|A(t)\, W(t, s)\| \leqq C/(t-s)\,, \qquad 0 \leqq s < t \leqq T\,,$$

for some constant C. It will also be shown that the integral of (1. 2) exists as an improper integral and the equality (1. 2) holds.

Under some smoothness assumptions on h, k it is shown that

$$u(t) = W(t, 0)\, k(0) + \int_0^t W(t, s)\, \dot{k}(s)\, ds$$

is the unique solution of (1. 1). Here we need not assume $k(t) \in D(A^{\mu}(t))$ for some $0 < \mu \leqq 1$ unlike Theorem 4 of [2].

In out results the domain of $A(t)$ may be dependent on t. Our fundamental assumption on $\{A(t)\}$ is that to the evolution equation

$$du(t)/dt + A(t)\, u(t) = 0\,, \qquad 0 < t \leqq T\,,$$

there exists a fundamental solution having some suitable properties.

Finally we note that the nonlinear version of the results of this paper appeared in [7].

2. Statement of results

Let X be a complex Banach space. We use the notations $C^m([0, T]\,;\, X)$, $C^m((0, T]\,;\, X)$ to denote the set of functions with values in X which are m times continuously differentiable in $[0, T]$, $(0, T]$ respectively. If X is the set of complex numbers, we simply denote them by $C^m([0, T])$, $C^m((0, T])$. For an operator A its domain and range are denoted by $D(A)$ and $R(A)$ respectively.

Let $A(t)$ be a closed, densely defined linear operator such that $-A(t)$ generates an analytic semigroup in X for each $t \in [0, T]$. We need the following assumptions:

(A) $\{A(t),\ 0 \leqq t \leqq T\}$ satisfies the conditions listed in [3] or [4] or [5]. Hence the resolvent set of $A(t)$ contains a fixed closed sector $\Sigma = \{\lambda : \theta \leqq \arg \lambda \leqq 2\pi - \theta\}$ where θ is some angle with $0 < \theta < \pi/2$, and there exists a positive constant C such that

$$\|(\lambda - A(t))^{-1}\| \leqq C/|\lambda|$$

for any $t \in [0, T]$ and $\lambda \in \Sigma$. The fundamental solution $U_0(t, s)$ to the evolution equation

$$du(t)/dt+A(t)\,u(t)=0\,, \qquad 0<t\leqq T\,,$$

exists, and has the following properties:

(i) $U_0(t, s)$ is differentiable in $t\in(s, T]$ for any $0\leqq s<T$, $R(U_0(t, s))\subset D(A(t))$ for $0\leqq s<t\leqq T$, and for some positive constant C the following inequality holds:

$$\|(\partial/\partial t)\,U_0(t, s)\|=\|A(t)\,U_0(t, s)\|\leqq C/(t-s)\,.$$

(ii) If u_0 is an arbitrary element of X and $f(t)$ is a Hölder continuous function in $[s, T]$, $0\leqq s<T$, with values in X, then

$$u(t)=U_0(t, s)\,u_0+\int_s^t U_0(t, \tau)f(\tau)\,d\tau$$

is the unique solution of the initial value problem

$$du(t)/dt+A(t)\,u(t)=f(t)\,, \qquad s<t\leqq T\,,$$
$$u(s)=u_0\,.$$

We shall consider the integral equation

$$u(t)+\int_0^t h(t-\tau)\,A(\tau)\,u(\tau)\,d\tau=k(t)\,. \qquad 0\leqq t\leqq T\,, \tag{2.1}$$

where h is a given scalar function, k is a given function with values in X. Concerning h and k we need the following assumptions which are the same as those of [2]:

(H) $h(0)>0$, $h\in C^1([0, T])$, $\dot{h}$ is absolutely continuous in $[0, T]$ and $\ddot{h}\in L^p(0, T)$ for some $p>1$.

(K) $k\in C^1([0, T]\,;\,X)$ and $\dot{k}$ is uniformly Hölder continuous, i.e.

$$\|\dot{k}(t)-\dot{k}(s)\|\leqq c|t-s|^\beta$$

for t, $s\in[0, T]$ where c, β are positive constants, and $\beta\leqq 1$.

Throughout the paper we suppose that the assumptions (A), (H), (K) hold.

We also consider the fundamental solution of (2.1) which is an operator valued function $W(t, s)$ satisfying

$$W(t, s)+\int_s^t h(t-\tau)\,A(\tau)\,W(\tau, s)\,d\tau=I\,. \tag{2.2}$$

Let r be the solution of

$$h(0)^{-1}\dot{h}(t)+h(0)\,r(t)+(\dot{h}*r)\,(t)=0 \tag{2.3}$$

where

$$(\dot{h}*r)(t)=\int_0^t \dot{h}(t-s)\, r(s)\, ds\,.$$

From (H) it follows that r is absolutely continuous and $\dot{r}\in L^p(0, T)$. As is easily seen

$$h(0)^{-1}h(t)+(h*r)(t)=1\,, \tag{2. 4}$$

$$h(0)^{-1}\dot{h}(t)+r(0)\, h(t)+(h*\dot{r})(t)=0\,. \tag{2. 5}$$

Following the argument of Crandall-Nohel [1], pp. 315–317, the equations (2. 1) and (2. 2) are transformed to

$$du(t)/dt+h(0)\, A(t)\, u(t)=(Gu)(t)\,, \qquad u(0)=k(0) \tag{2. 6}$$

where

$$\begin{aligned}(Gu)(t)=&\ \dot{k}(t)+h(0)\,(r*\dot{k})(t)+h(0)\, r(t)\, k(0)\\ &-h(0)\, r(0)\, u(t)-h(0)\,(u*\dot{r})(t)\,,\end{aligned}$$

and

$$\begin{aligned}&(\partial/\partial t)\, W(t, s)+h(0)\, A(t)\, W(t, s)\\ &\quad =h(0)\, r(t-s)-h(0)\, r(0)\, W(t, s)\\ &\quad -h(0)\int_s^t \dot{r}(t-\tau)\, W(\tau, s)\, d\tau\,, \qquad W(s, s)=I\end{aligned} \tag{2. 7}$$

respectively.

THEOREM 1. *Let u be a function belonging to $C([0, T]\,;\, X)\cap C^1((0, T]\,;\, X)$ such that $u(t)\in D(A(t))$ for $t>0$ and $A(t)\, u(t)$ is continuous in $(0, T]$. If u is the solution of* (2. 6), *then for $\varepsilon>0$ $\int_\varepsilon^t h(t-s)\, A(s)\, u(s)\, ds$ is uniformly bounded and converges to $k(t)-u(t)$ as $\varepsilon\to 0$ uniformly in any closed subset of $(0, T]$. Conversely if the last statement is true and $u(0)=k(0)$, then u is the solution of* (2. 6). *In this case $\int_\varepsilon^t \dot{h}(t-s)\, A(s)\, u(s) ds$ is uniformly bounded when $\varepsilon>0$,*

$$\int_0^t \dot{h}(t-s)\, A(s)\, u(s)\, ds=\lim_{\varepsilon\downarrow\varepsilon}\int_\varepsilon^t \dot{h}(t-s)\, A(s)\, u(s)\, ds$$

exists, and the following equality holds:

$$\dot{u}(t)+h(0)\, A(t)\, u(t)+\int_0^t \dot{h}(t-s)\, A(s)\, u(s)\, ds=\dot{k}(t)\,. \tag{2. 8}$$

THEOREM 2. *The solution $W(t, s)$ of* (2. 7) *exists under the assumptions* (A), (H). *There exists a positive constant C such that for $0\leqq s<t\leqq T$*

$$\left\| \frac{\partial}{\partial t} W(t, s) \right\| \leqq \frac{C}{t-s}, \ \left\| A(t) W(t, s) \right\| \leqq \frac{C}{t-s}. \tag{2.9}$$

For $0 \leqq s < t \leqq T$ *the imporper integral*

$$\begin{aligned} &\int_0^t h(t-\tau) A(\tau) W(\tau, s)\, d\tau \\ &\qquad = \lim_{\varepsilon \downarrow 0} \int_{s+\varepsilon}^t h(t-\tau) A(\tau) W(\tau, s)\, d\tau \end{aligned} \tag{2.10}$$

exists and (2.2) *holds.*

THEOREM 3. *Let* u *be the function defined by*

$$u(t) = W(t, 0) k(0) + \int_0^t W(t, s) \dot{k}(s)\, ds. \tag{2.11}$$

Then $u \in C([0, T]; X) \cap C^1((0, T]; X)$, $u(t) \in D(A(t))$ *for* $t > 0$ *and* $A(t) u(t)$ *is continuous in* $(0, T]$. *The improper integral*

$$\int_0^t h(t-\tau) A(\tau) u(\tau)\, d\tau = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^t h(t-\tau) A(\tau) u(\tau)\, d\tau$$

exists and (2.1) *holds.*

THEOREM 4. *If* u *is a function satisfying the conditions of Theorem* 1. *Then* (2.11) *holds. Hence the solution of* (2.1) *or* (2.6) *is unique.*

Finally we consider the regularity property as was discussed in [6].

Let $\{M_k\}$ be a sequence of positive numbers such that for some positive constants d_0, d_1, d_2

$$\begin{aligned} &M_{k+1} \leqq d_0^k M_k && \text{for} \quad k \geqq 0, \\ &\binom{k}{j} M_{k-j} M_j \leqq d_1 M_k && \text{for} \quad 0 \leqq j \leqq k, \\ &M_k \leqq M_{k+1} && \text{for} \quad k \geqq 0, \\ &M_{j+k} \leqq d_2^{j+k} M_j M_k && \text{for} \quad j \geqq 0, \ k \geqq 0. \end{aligned}$$

The set of scalar valued or X valued functions v which are infinitely differentiable in $[0, T]$ and satisfy

$$\left\| d^n v(t)/dt^n \right\| \leqq C_0 C^n M_n, \qquad 0 \leqq t \leqq T, \ n = 0, 1, 2, \cdots,$$

for some constants C_0, C are denoted by $G(\{M_k\})$ or $G(\{M_k\}, X)$ respectively.

THEOREM 5. *Suppose that the following conditions are satisfied in addition to* (A), (H), (K):

(i) $A(t)^{-1}$, *which is a bounded operator valued function in view of the assumption* (A), *is infinitely differentiable in* $[0, T]$.

(ii) *There exist constants* K_0, K_1 *such that for all* $\lambda\in\Sigma$, $t\in[0, T]$ *and non-negative integers* n

$$\left\|\left(\frac{\partial}{\partial t}\right)^n\left(\lambda-A(t)\right)^{-1}\right\|\leqq K_0K^n M_n/|\lambda| .$$

(iii) $k\in G(\{M_k\})$.

Then $W(t, s)$ *is infinitely differentiable in* $0\leqq s<t\leqq T$ *and there exist positive constants* H_0, H *such that for any non-negative integers* n, m, l

$$\begin{aligned}&\left\|\left(\frac{\partial}{\partial t}\right)^n\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^m\left(\frac{\partial}{\partial s}\right)^l W(t, s)\right\| \\ &\qquad\leqq H_0H^{n+m+l} M_{n+m+l}(t-s)^{-n-l} .\end{aligned}\tag{2. 12}$$

If furthermore $k\in G(\{M_k\}, X)$, *then the solution* $u(t)$ *of* (2. 1) *is infinitely differentiable in* $(0, T]$ *and there exist constants* F_0, F *such that for any integer* $n\geqq 0$

$$\left\|d^n u(t)/dt^n\right\|\leqq H_0H^n M_n\left\|k(0)\right\|t^{-n}+F_0F^n M_n t^{1-n} .$$

3. Proofs of Theorems 1, 2

Theorem 1 is established in essentially the same manner as Theorem 1 of [7] and the proof is omitted.

Let $U(t, s)$ be the fundamental solution of the equation

$$du(t)/dt+h(0) A(t) U(t, s)=0 .\tag{3. 1}$$

From the hypotheses (A), (H) there exists a constant C such that

$$\left\|\frac{\partial}{\partial t}U(t, s)\right\|\leqq\frac{C}{t-s}, \quad \left\|A(t) U(t, s)\right\|\leqq\frac{C}{t-s}\tag{3. 2}$$

for $0\leqq s<t\leqq T$. From (3. 2) it follows that

$$\left\|U(t, s)-U(\tau, s)\right\|\leqq\frac{C}{\rho}\left(\frac{t-\tau}{t-s}\right)^{\rho}\tag{3. 3}$$

for any $0\leqq s<\tau<t\leqq T$ and $0<\rho\leqq 1$.

The equation (2. 7) is further transformed to the integral equation

$$\begin{aligned}W(t, s)&=U(t, s)\\ &\quad+h(0)\int_s^t U(t, \tau)\left\{r(\tau-s)-r(0) W(\tau, s)-\int_s^\tau \dot{r}(\tau-\sigma) W(\sigma, s) d\sigma\right\} d\tau .\end{aligned}\tag{3. 4}$$

The equation (3. 4) can be solved by successive approximation, and we define $W(t, s)$ as the solution of this integral equation. It is clear that $W(t, s)$ is strongly continuous in $0 \leqq s \leqq t \leqq T$, and hence it is uniformly bounded. From (3. 3) and (3. 4) it follows that

$$\begin{aligned} &\| W(t, s) - W(\tau, s) \| \\ &\leqq C\left\{\frac{1}{\rho}\left(\frac{t-\tau}{t-s}\right)^{\rho} + t - \tau + \frac{1}{\rho(1-\rho)}(t-\tau)^{\rho}(t-s)^{1-\rho}\right\}. \end{aligned} \tag{3.5}$$

The following lemma will be needed in the proof of the present theorem.

LEMMA 3. 1. *For any* ρ *with* $0<\rho<1-1/p$, *there exists a constant* C_ρ *such that*

$$\begin{aligned} &\left\| \int_s^t \dot{r}(t-\sigma)\, W(\sigma, s)\, d\sigma - \int_s^\tau \dot{r}(\tau-\sigma)\, W(\sigma, s)\, d\sigma \right\| \\ &\leqq C_\rho (t-\tau)^\rho, \qquad 0 \leqq \tau < t \leqq T. \end{aligned} \tag{3.6}$$

PROOF. The left member of (3. 6)

$$\begin{aligned} &= \left\| \int_0^{t-s} \dot{r}(\sigma)\, W(t-\sigma, s)\, d\sigma - \int_0^{\tau-s} \dot{r}(\sigma)\, W(\tau-\sigma, s)\, d\sigma \right\| \\ &\leqq \int_{\tau-s}^{t-s} \| \dot{r}(\sigma)\, W(t-\sigma, s) \| d\sigma \\ &+ \int_0^{\tau-s} |\dot{r}(\sigma)| \, \| W(t-\sigma, s) - W(\tau-\sigma, s) \| d\sigma = I + II. \end{aligned}$$

Since $\dot{r} \in L^p(0, T)$ $I \leqq C(t-\tau)^{1-1/p} ||\dot{r}||_{L^p(0,T)}$. With the aid of (3. 5) and some elementary calculus it is easy to show

$$\begin{aligned} II \leqq C_\rho \Big\{ &\frac{1}{\rho}(1-\rho p')^{-1/p'}(t-\tau)^\rho (\tau-s)^{1/p'-\rho} ||\dot{r}||_{L^p(0,T)} \\ &+ (t-\tau) ||\dot{r}||_{L^1(0,T)} + \frac{1}{\rho(1-\rho)}(t-\tau)^\rho (\tau-s)^{1-\rho} ||\dot{r}||_{L^1(0,T)} \Big\}. \end{aligned}$$

Thus the proof of the lemma is complete.

Now we return to the proof of Theorem 2. Using (3. 5), Lemma 3. 1 and some well known argument on parabolic evolution equations we see that $W(t, s)$ is differentiable in t in the interval $(s, T]$, $R(W(t, s)) \subset D(A(t))$ for $t>s$, and (2. 9) holds.

Let $0 \leqq s < t \leqq T$ and $0<\varepsilon<t-s$. In view of (2. 7)

$$\begin{aligned} \int_{s+\varepsilon}^{t} h(t-\tau)\, A(\tau)\, W(\tau, s)\, d\tau = \int_{s+\varepsilon}^{t} h(t-\tau) \Big\{ &r(\tau-s) - r(0)\, W(\tau, s) \\ &- \int_s^\tau \dot{r}(\tau-\sigma)\, W(\sigma, s)\, d\sigma - \frac{1}{h(0)} \frac{\partial}{\partial\tau} W(\tau, s) \Big\} d\tau. \end{aligned} \tag{3.7}$$

By integration by parts

$$\int_{s+\varepsilon}^{t} h(t-\tau)\frac{\partial}{\partial\tau} W(\tau, s)\, d\tau$$
$$= h(0)\, W(t, s) - h(t-s-\varepsilon)\, W(s+\varepsilon, s) + \int_{s+\varepsilon}^{t} \dot{h}(t-\tau)\, W(\tau, s)\, d\tau\, .$$

Substituting this to (3. 7) and letting $\varepsilon \to 0$ we get

$$\lim_{\varepsilon\to 0}\int_{s+\varepsilon}^{t} h(t-\tau)\, A(\tau)\, W(\tau, s)\, d\tau$$
$$= (h*r)\,(t-s) - r(0)\int_{s}^{t} h(t-\tau)\, W(\tau, s)\, d\tau$$
$$-\int_{s}^{t}(h*\dot{r})\,(t-\sigma)\, W(\sigma, s)\, d\sigma - W(t, s)$$
$$+\frac{1}{h(0)}\, h(t-s) - \frac{1}{h(0)}\int_{s}^{t} \dot{h}(t-\tau)\, W(\tau, s)\, d\tau\, .$$

In view of (2. 4) and (2. 5) the right member of this equality is equal to

$$I - r(0)\int_{s}^{t} h(t-\tau)\, W(\tau, s)\, d\tau$$
$$+\int_{s}^{t}\left\{\frac{1}{h(0)}\, \dot{h}(t-\sigma) + r(0)\, h(t-\sigma)\right\} W(\sigma, s)\, d\sigma$$
$$- W(t, s) - \frac{1}{h(0)}\int_{s}^{t} \dot{h}(t-\tau)\, W(\tau, s)\, d\tau = I - W(t, s)\, .$$

Thus the proof of Theorem 2 is complete.

4. Proofs of Theorems 3, 4

PROOF OF THEOREM 3. Let $u(t)$ be the function defined by (2. 10). With the aid of (3. 4) and some elementary calculus we get

$$\int_{0}^{t} W(t, s)\, \dot{k}(s)\, ds = \int_{0}^{t} U(t, \tau) f(\tau)\, d\tau\, , \tag{4. 1}$$

where

$$\begin{aligned} f(\tau) &= \dot{k}(\tau) + h(0)\,(r*\dot{k})\,(\tau) - h(0)\, r(0)\int_{0}^{\tau} W(\tau, s)\, \dot{k}(s)\, ds \\ &\quad - h(0)\int_{0}^{\tau}\int_{s}^{\tau} \dot{r}(\tau-\sigma)\, W(\sigma, s)\, d\sigma \dot{k}(s)\, ds\, . \end{aligned} \tag{4. 2}$$

Since $f(\tau)$ is bounded and continuous, it follows from (3. 3) that the right member of (4. 1) is Hölder continuous. Combining this with the assumption

(K) we see that $f(\tau)$ is Hölder continuous. Hence in view of the assumption (A) the left member of (4. 1) is differentiable and

$$\frac{\partial}{\partial t}\int_0^t W(t, s)\, \dot{k}(s)\, ds + A(t)\int_0^t W(t, s)\, \dot{k}(s)\, ds = f(t)\,.$$

In view of (2. 7) and (4. 3) we get

$$\begin{aligned} A(t)\, u(t) = & \Big\{r(t) - r(0)\, W(t, 0) - \int_0^t \dot{r}(t-\sigma)\, W(\sigma, 0)\, d\sigma \\ & - \frac{1}{h(0)}\, \frac{\partial}{\partial t}\, W(t, 0)\Big\}\, k(0) + \frac{1}{h(0)}\, \dot{k}(t) + (r*\dot{k})\,(t) \\ & - r(0)\int_0^t W(t, s)\, \dot{k}(s)\, ds - \int_0^t\int_s^t \dot{r}(t-\sigma)\, W(\sigma, s)\, d\sigma \dot{k}(s)\, ds \\ & - \frac{1}{h(0)}\, \frac{\partial}{\partial t}\int_0^t W(t, s)\, \dot{k}(s)\, ds\,. \end{aligned}$$

Following the argument by which we derived (2. 10) from (3. 7) we can establish without difficulty

$$u(t) + \lim_{\varepsilon \downarrow 0}\int_\varepsilon^t h(t-\tau)\, A(\tau)\, u(\tau)\, d\tau = k(t)\,.$$

Proof of Theorem 4. Formally differentiating both sides of (3. 4) with respect to s we get

$$\frac{\partial}{\partial s}\, W(t, s) + \int_s^t h(t-\tau)\, A(\tau)\frac{\partial}{\partial s}\, W(\tau, s)\, d\tau = h(t-s)\, A(s)\,. \qquad (4.\,4)$$

Considering $(\partial/\partial s)\, W(t, s)$ as an unknown function in (4. 4) and taking Theorem 2 into consideration we get

$$\frac{\partial}{\partial s}\, W(t, s) = h(0)\, W(t, s)\, A(s) + \int_0^t W(t, \tau)\, \dot{h}(\tau-s)\, A(s)\, d\tau\,.$$

Thus

$$\begin{aligned} \frac{\partial}{\partial s}\Big(W(t, s)\, u(s)\Big) & = \frac{\partial}{\partial s}\, W(t, s)\cdot u(s) + W(t, s)\, \dot{u}(s) \\ & = h(0)\, W(t, s)\, A(s)\, u(s) + \int_s^t W(t, \tau)\, \dot{h}(\tau-s)\, A(s)\, u(s)\, d\tau \\ & \quad + W(t, s)\, \dot{u}(s)\,, \end{aligned} \qquad (4.\,5)$$

or equivalently

$$\begin{aligned} W(t, s')\, u(s') - W(t, s)\, u(s) = & \; h(0)\int_s^{s'} W(t, \sigma)\, A(\sigma)\, \dot{u}(\sigma)\, d\sigma \\ & + \int_s^{s'}\int_0^t W(t, \tau)\, \dot{h}(\tau-\sigma)\, A(\sigma)\, \dot{u}(\sigma)\, d\tau d\sigma + \int_s^{s'} W(t, \sigma)\, \dot{u}(\sigma)\, d\sigma \end{aligned} \qquad (4.\,6)$$

for $0 \leqq s < s' \leqq t$. This formal calculation is justified by using the Yosida approximation of $A(t)$ to establish (4. 6) and we see that (4. 5) is valid under our present assumptions. Substituting (2. 8) in the last term of (4. 5) we get

$$\begin{aligned} &\frac{\partial}{\partial t}\left(W(t, s)\, u(s)\right) \\ &= \int_s^t W(t, \tau)\, \dot{h}(\tau - s)\, A(s)\, u(s)\, ds \\ &- W(t, s) \int_0^s \dot{h}(s-\tau)\, A(\tau)\, u(\tau)\, d\tau + W(t, s)\, \dot{k}(s)\,. \end{aligned} \tag{4. 7}$$

Integrating both sides of (4. 7) over $[\varepsilon, t]$ and applying Fubini theorem to the integral of the first term of the right member we get

$$\begin{aligned} &u(t) - W(t, \varepsilon)\, u(\varepsilon) \\ &= \int_\varepsilon^t W(t, s) \left\{ \int_\varepsilon^s \dot{h}(s-\sigma)\, A(\sigma)\, u(\sigma)\, d\sigma \right. \\ &\left. - \int_0^s \dot{h}(s-\sigma)\, A(\sigma)\, u(\sigma)\, d\sigma \right\} ds + \int_\varepsilon^t W(t, s)\, \dot{k}(s)\, ds\,. \end{aligned}$$

Letting $\varepsilon \to 0$ and using Theorem 1 we obtain

$$u(t) - W(t, 0)\, u(0) = \int_0^t W(t, s)\, \dot{k}(s)\, ds$$

to complete the proof of the theorem.

Proof of Theorem 5. It follows from (3. 4) that

$$W(t, s) = W_0(t, s) - \int_s^t V(t, \tau)\, W(\tau, s)\, d\tau$$

where

$$W_0(t, s) = U(t, s) + h(0) \int_s^t U(t, \tau)\, r(\tau - s)\, d\tau\,,$$

$$V(t, s) = h(0)\, r(0)\, U(t, s) + h(0) \int_s^t U(t, \tau)\, \dot{r}(\tau - s)\, d\tau\,.$$

It is easy to show that $r \in G(\{M_k\})$. Hence using Lemma 3. 2 and Theorem 3. 2 of [6] we can show that the same type of estimates as (2. 12) hold for the derivatives of $W_0(t, s)$ and $V(t, s)$. Using the argument of the proofs of Theorems 3. 1 and 3. 2 of [6] we can verify (2. 12). The remaining part of the theorem is the same as that of Theorem 3. 3 of [6].

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