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On a second order rational systems of difference equations

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Abstract. In this paper we study the periodicity and the form of the solutions of the following systems of difference equations of order two

$$x_{n+1} = \frac{y_n x_{n-1}}{\pm x_{n-1} \pm y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{\pm x_n \pm y_{n-1}}, \quad n \in \mathbb{N}_0,$$

with nonzero real numbers initial conditions.

Key words: Periodic solutions, system of difference equations.

1. Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics and so on. So, recently there has been an increasing interest in the study of qualitative analysis of rational difference equations and systems of difference equations. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solutions. See [1]–[15], [36] and the references cited therein.

Periodic solutions of difference equations have been investigated by many researchers, and various methods have been proposed for the existence and qualitative properties of the solution. For example, the periodicity of the positive solutions of the rational difference equations system

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}},$$

was studied by Cinar in [2].

Elabbasy et al. [3] has obtained the solution of particular cases of the following general system of difference equations

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$$x_{n+1}=\frac{a_1+a_2y_n}{a_3z_n+a_4x_{n-1}z_n},\quad y_{n+1}=\frac{b_1z_{n-1}+b_2z_n}{b_3x_ny_n+b_4x_ny_{n-1}},$$
$$z_{n+1}=\frac{c_1z_{n-1}+c_2z_n}{c_3x_{n-1}y_{n-1}+c_4x_{n-1}y_n+c_5x_ny_n}.$$

In [11], Elsayed et al. studied the periodic nature and the form of the solutions of the following nonlinear difference equations systems of order three

$$x_{n+1}=\frac{x_ny_{n-2}}{y_{n-1}(\pm1\pm x_ny_{n-2})},\quad y_{n+1}=\frac{y_nx_{n-2}}{x_{n-1}(\pm1\pm y_nx_{n-2})}.$$

In [22], Kurbanli et al. dealt with the periodicity of solutions of the system of rational difference equations

$$x_{n+1}=\frac{x_{n-1}+y_n}{x_{n-1}y_n-1},\quad y_{n+1}=\frac{y_{n-1}+x_n}{y_{n-1}x_n-1}.$$

Özban [25] has investigated the positive solutions of the system of rational difference equations

$$x_{n+1}=\frac{1}{y_{n-k}},\quad y_{n+1}=\frac{y_n}{x_{n-m}y_{n-m-k}}.$$

Touafek et al. [29] investigated the periodic nature and gave the form of the solutions of the following systems of rational difference equations

$$x_{n+1}=\frac{x_{n-3}}{\pm1\pm x_{n-3}y_{n-1}},\quad y_{n+1}=\frac{y_{n-3}}{\pm1\pm y_{n-3}x_{n-1}}.$$

In [30] Yalçınkaya investigated the sufficient condition for the global asymptotic stability of the following system of difference equations

$$z_{n+1}=\frac{t_nz_{n-1}+a}{t_n+z_{n-1}},\quad t_{n+1}=\frac{z_nt_{n-1}+a}{z_n+t_{n-1}}.$$

Also, Yalçınkaya [31] has obtained the sufficient conditions for the global asymptotic stability of the system of two nonlinear difference equations

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1}, \quad y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}.$$

Yang et al. [35] has investigated the behavior of the positive solutions of the systems

$$x_n = \frac{a}{y_{n-p}}, \quad y_n = \frac{b y_{n-p}}{x_{n-q} y_{n-q}}.$$

Similar nonlinear systems of rational difference equations were investigated see [16]–[37].

Our aim in this paper is to consider the following systems of difference equations

$$x_{n+1} = \frac{y_n x_{n-1}}{\pm x_{n-1} \pm y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{\pm x_n \pm y_{n-1}}, \quad n \in \mathbb{N}_0$$

with nonzero real numbers initial conditions.

Definition 1 (Periodicity) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 2 (Fibonacci Sequence) The sequence $\{F_m\}_{m=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$ i.e., $F_m = F_{m-1} + F_{m-2}$, $m \geq 2$, $F_0 = 0$, $F_1 = 1$ is called Fibonacci Sequence.

2. Main Results

2.1. The system: $x_{n+1} = x_{n-1}y_n/(x_{n-1} + y_n)$, $y_{n+1} = x_n y_{n-1}/(x_n + y_{n-1})$

In this section, we study the solutions of the system of the difference equations

$$x_{n+1} = \frac{x_{n-1} y_n}{x_{n-1} + y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{x_n + y_{n-1}}, \tag{1}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary nonzero real numbers such that y_0/x_{-1}, $x_0/y_{-1} \notin \{-(F_{n+1}/F_n), n = 1, 2, \dots, \}$.

The following theorem is devoted to the form of the solutions of system (1).

Theorem 1 *Suppose that $\{x_n, y_n\}$ are solutions of system (1). Then for $n = 0, 1, 2, \ldots,$ we have*

$$x_{2n-1} = \frac{x_{-1}y_0}{x_{-1}F_{2n} + y_0F_{2n-1}}, \quad x_{2n} = \frac{x_0y_{-1}}{x_0F_{2n} + y_{-1}F_{2n+1}},$$
$$y_{2n-1} = \frac{x_0y_{-1}}{y_{-1}F_{2n} + x_0F_{2n-1}}, \quad y_{2n} = \frac{x_{-1}y_0}{y_0F_{2n} + x_{-1}F_{2n+1}},$$

where $\{F_n\}_{n=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$, $F_{-1} = 1$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$x_{2n-3} = \frac{x_{-1}y_0}{x_{-1}F_{2n-2} + y_0F_{2n-3}}, \quad x_{2n-2} = \frac{x_0y_{-1}}{x_0F_{2n-2} + y_{-1}F_{2n-1}},$$
$$y_{2n-3} = \frac{y_{-1}x_0}{y_{-1}F_{2n-2} + x_0F_{2n-3}}, \quad y_{2n-2} = \frac{y_0x_{-1}}{y_0F_{2n-2} + x_{-1}F_{2n-1}}.$$

It is concluded from Eq.(1) that

$$x_{2n} = \frac{y_{2n-1}x_{2n-2}}{y_{2n-1} + x_{2n-2}} = \frac{\left(\frac{x_{2n-2}y_{2n-3}}{x_{2n-2} + y_{2n-3}}\right)x_{2n-2}}{\left(\frac{x_{2n-2}y_{2n-3}}{x_{2n-2} + y_{2n-3}}\right) + x_{2n-2}} = \frac{x_{2n-2}y_{2n-3}}{2y_{2n-3} + x_{2n-2}}$$

$$= \frac{\left(\frac{x_0y_{-1}}{x_0F_{2n-2} + y_{-1}F_{2n-1}}\right)\left(\frac{y_{-1}x_0}{y_{-1}F_{2n-2} + x_0F_{2n-3}}\right)}{\left(\frac{2y_{-1}x_0}{y_{-1}F_{2n-2} + x_0F_{2n-3}}\right) + \left(\frac{x_0y_{-1}}{x_0F_{2n-2} + y_{-1}F_{2n-1}}\right)}$$

$$= \frac{\left(\frac{(x_0y_{-1})^2}{(x_0F_{2n-2} + y_{-1}F_{2n-1})(y_{-1}F_{2n-2} + x_0F_{2n-3})}\right)}{\left(\frac{y_{-1}x_0(y_{-1}F_{2n-2} + x_0F_{2n-3} + 2x_0F_{2n-2} + 2y_{-1}F_{2n-1})}{(y_{-1}F_{2n-2} + x_0F_{2n-3})(x_0F_{2n-2} + y_{-1}F_{2n-1})}\right)}$$

$$= \frac{x_0y_{-1}}{y_{-1}F_{2n-2} + x_0F_{2n-3} + 2x_0F_{2n-2} + 2y_{-1}F_{2n-1}}$$

$$= \frac{x_0y_{-1}}{y_{-1}(F_{2n-2} + F_{2n-1}) + y_{-1}F_{2n-1} + x_0(F_{2n-3} + F_{2n-2}) + x_0F_{2n-2}}$$

$$= \frac{x_0 y_{-1}}{y_{-1}F_{2n} + y_{-1}F_{2n-1} + x_0 F_{2n-1} + x_0 F_{2n-2}}$$
$$= \frac{x_0 y_{-1}}{y_{-1}F_{2n+1} + x_0 F_{2n}},$$

and

$$y_{2n} = \frac{x_{2n-1}y_{2n-2}}{x_{2n-1} + y_{2n-2}} = \frac{\left(\frac{y_{2n-2}x_{2n-3}}{y_{2n-2} + x_{2n-3}}\right) y_{2n-2}}{\left(\frac{y_{2n-2}x_{2n-3}}{y_{2n-2} + x_{2n-3}}\right) + y_{2n-2}} = \frac{y_{2n-2}x_{2n-3}}{2x_{2n-3} + y_{2n-2}}$$
$$= \frac{\left(\frac{y_0 x_{-1}}{y_0 F_{2n-2} + x_{-1}F_{2n-1}}\right)\left(\frac{x_{-1}y_0}{x_{-1}F_{2n-2} + y_0 F_{2n-3}}\right)}{\left(\frac{2x_{-1}y_0}{x_{-1}F_{2n-2} + y_0 F_{2n-3}}\right) + \left(\frac{y_0 x_{-1}}{y_0 F_{2n-2} + x_{-1}F_{2n-1}}\right)}$$
$$= \frac{\left(\frac{(y_0 x_{-1})^2}{(y_0 F_{2n-2} + x_{-1}F_{2n-1})(x_{-1}F_{2n-2} + y_0 F_{2n-3})}\right)}{\left(\frac{x_{-1}y_0(x_{-1}F_{2n-2} + y_0 F_{2n-3} + 2y_0 F_{2n-2} + 2x_{-1}F_{2n-1})}{(x_{-1}F_{2n-2} + y_0 F_{2n-3})(y_0 F_{2n-2} + x_{-1}F_{2n-1})}\right)}$$
$$= \frac{y_0 x_{-1}}{x_{-1}F_{2n-2} + y_0 F_{2n-3} + 2y_0 F_{2n-2} + 2x_{-1}F_{2n-1}}$$
$$= \frac{y_0 x_{-1}}{x_{-1}(F_{2n-2} + F_{2n-1}) + x_{-1}F_{2n-1} + y_0(F_{2n-3} + F_{2n-2}) + y_0 F_{2n-2}}$$
$$= \frac{y_0 x_{-1}}{x_{-1}F_{2n} + x_{-1}F_{2n-1} + y_0 F_{2n-1} + y_0 F_{2n-2}}$$
$$= \frac{y_0 x_{-1}}{x_{-1}F_{2n+1} + y_0 F_{2n}}.$$

Similarly, one can prove the other relations. The proof is complete. □

Lemma 1 *Every positive solution of system* (1) *is bounded and* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.

Proof. Eq.(1) shows that

$$x_{n+1} = \frac{x_{n-1}y_n}{x_{n-1} + y_n} < \frac{x_{n-1}y_n}{y_n} = x_{n-1},$$

$$y_{n+1} = \frac{x_n y_{n-1}}{x_n + y_{n-1}} < \frac{x_n y_{n-1}}{x_n} = y_{n-1},$$

or

$$x_{n+1} < x_{n-1}, \quad y_{n+1} < y_{n-1}.$$

Then, the subsequences $\{x_{2n-1}\}_{n=0}^{\infty}$, $\{x_{2n}\}_{n=0}^{\infty}$, $\{y_{2n-1}\}_{n=0}^{\infty}$, $\{y_{2n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by M, N respectively since $M = \max\{x_{-1}, x_0\}$, $N = \max\{y_{-1}, y_0\}$. □

Example 1 In order to illustrate the results of this section and to support our theoretical discussion, we consider an interesting numerical example for the difference system (1) with the initial conditions $x_{-1} = 0.8$, $x_0 = 3$, $y_{-1} = 2$ and $y_0 = 0.7$. (See Fig. 1).

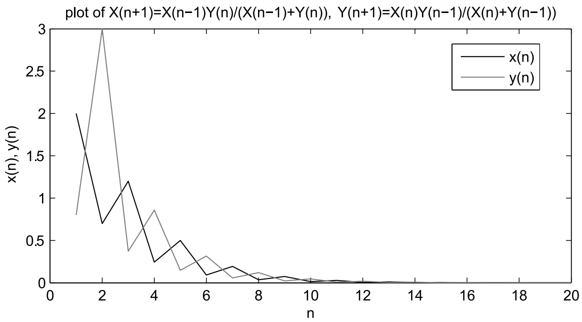

Figure 1. This figure shows the behavior of the solution of the system (1) with the initial values as in example (1).

Similar to the previous theorem, we can prove the following theorem:

Theorem 2 *The solutions of the following system of difference equations*

$$x_{n+1} = \frac{x_{n-1} y_n}{x_{n-1} - y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{-x_n - y_{n-1}},$$

where $n \in \mathbb{N}_0$ *and the initial conditions are arbitrary nonzero real numbers such that* $y_0/x_{-1}, \notin \{F_{n+1}/F_n, n = 1, 2, \ldots,\}$ *and* $x_0/y_{-1} \notin \{-(F_{n+1}/F_n), n = 1, 2, \ldots,\}$, *are given by the following expressions for* $n = 0, 1, 2, \ldots$

$$x_{2n-1} = \frac{(-1)^{n+1} x_{-1} y_0}{x_{-1} F_{2n} - y_0 F_{2n-1}}, \quad x_{2n} = \frac{(-1)^n x_0 y_{-1}}{x_0 F_{2n} + y_{-1} F_{2n+1}},$$

$$y_{2n-1} = \frac{(-1)^n x_0 y_{-1}}{x_0 F_{2n-1} + y_{-1} F_{2n}}, \quad y_{2n} = \frac{(-1)^n x_{-1} y_0}{x_{-1} F_{2n+1} - y_0 F_{2n}}.$$

2.2. The system: $x_{n+1} = x_{n-1} y_n / (x_{n-1} + y_n)$, $y_{n+1} = x_n y_{n-1} / (x_n - y_{n-1})$

In this section, we study the solutions of the system of the difference equations

$$x_{n+1} = \frac{x_{n-1} y_n}{x_{n-1} + y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{x_n - y_{n-1}}, \tag{2}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary nonzero real numbers with $x_0 \neq y_{-1}$ and $x_{-1} \neq -y_0$.

Theorem 3 *Suppose that $\{x_n, y_n\}$ are solutions of system (2). Also, assume that x_{-1}, x_0, y_{-1} and y_0 are arbitrary nonzero real numbers with $x_0 \neq y_{-1}$ and $x_{-1} \neq -y_0$. Then, every solution of Eq.(2) is a periodic solution with period twelve and given by the following formulas for $n = 0, 1, 2, \ldots$*

$$x_{12n-1} = x_{-1}, \quad x_{12n} = x_0, \quad x_{12n+1} = \frac{x_{-1} y_0}{x_{-1} + y_0},$$

$$x_{12n+2} = y_{-1}, \quad x_{12n+3} = y_0, \quad x_{12n+4} = \frac{x_0 y_{-1}}{x_0 - y_{-1}},$$

$$x_{12n+5} = -x_{-1}, \quad x_{12n+6} = -x_0, \quad x_{12n+7} = \frac{-x_{-1} y_0}{x_{-1} + y_0},$$

$$x_{12n+8} = -y_{-1}, \quad x_{12n+9} = -y_0, \quad x_{12n+10} = \frac{-x_0 y_{-1}}{x_0 - y_{-1}},$$

$$y_{12n-1} = y_{-1}, \quad y_{12n} = y_0, \quad y_{12n+1} = \frac{x_0 y_{-1}}{x_0 - y_{-1}},$$

$$y_{12n+2} = -x_{-1}, \quad y_{12n+3} = -x_0, \quad y_{12n+4} = -\frac{x_{-1} y_0}{x_{-1} + y_0},$$

$$y_{12n+5} = -y_{-1}, \quad y_{12n+6} = -y_0, \quad y_{12n+7} = \frac{-x_0 y_{-1}}{x_0 - y_{-1}},$$

$$y_{12n+8} = x_{-1}, \quad y_{12n+9} = x_0, \quad y_{12n+10} = \frac{x_{-1} y_0}{x_{-1} + y_0}.$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$, that is,

$$x_{12n-13} = x_{-1}, \quad x_{12n-12} = x_0, \quad x_{12n-11} = \frac{x_{-1}y_0}{x_{-1}+y_0},$$
$$x_{12n-10} = y_{-1}, \quad x_{12n-9} = y_0, \quad x_{12n-8} = \frac{x_0y_{-1}}{x_0-y_{-1}},$$
$$x_{12n-7} = -x_{-1}, \quad x_{12n-6} = -x_0, \quad x_{12n-5} = \frac{-x_{-1}y_0}{x_{-1}+y_0},$$
$$x_{12n-4} = -y_{-1}, \quad x_{12n-3} = -y_0, \quad x_{12n-2} = \frac{-x_0y_{-1}}{x_0-y_{-1}},$$
$$y_{12n-13} = y_{-1}, \quad y_{12n-12} = y_0, \quad y_{12n-11} = \frac{x_0y_{-1}}{x_0-y_{-1}},$$
$$y_{12n-10} = -x_{-1}, \quad y_{12n-9} = -x_0, \quad y_{12n-8} = -\frac{x_{-1}y_0}{x_{-1}+y_0},$$
$$y_{12n-7} = -y_{-1}, \quad y_{12n-6} = -y_0, \quad y_{12n-5} = \frac{-x_0y_{-1}}{x_0-y_{-1}},$$
$$y_{12n-4} = x_{-1}, \quad y_{12n-3} = x_0, \quad y_{12n-2} = \frac{x_{-1}y_0}{x_{-1}+y_0}.$$

From Eq.(2), we see that

$$\begin{aligned}
x_{12n-1} &= \frac{x_{12n-3}y_{12n-2}}{x_{12n-3}+y_{12n-2}} = \frac{-y_0\left(\frac{x_{-1}y_0}{x_{-1}+y_0}\right)}{-y_0+\left(\frac{x_{-1}y_0}{x_{-1}+y_0}\right)} \\
&= \frac{-x_{-1}y_0}{(x_{-1}+y_0)\left(-1+\left(\frac{x_{-1}}{x_{-1}+y_0}\right)\right)} \\
&= \frac{-x_{-1}y_0}{(-x_{-1}-y_0+x_{-1})} = x_{-1},
\end{aligned}$$

$$y_{12n-1} = \frac{x_{12n-2}y_{12n-3}}{x_{12n-2}-y_{12n-3}} = \frac{\left(-\frac{x_0y_{-1}}{x_0-y_{-1}}\right)x_0}{\left(-\frac{x_0y_{-1}}{x_0-y_{-1}}\right)-x_0}$$

$$= \frac{x_0 y_{-1}}{(x_0 - y_{-1})\left(\left(\frac{y_{-1}}{x_0 - y_{-1}}\right) + 1\right)}$$

$$= \frac{x_0 y_{-1}}{(y_{-1} + x_0 - y_{-1})} = y_{-1},$$

$$x_{12n} = \frac{y_{12n-1} x_{12n-2}}{x_{12n-2} + y_{12n-1}} = \frac{y_{-1}\left(-\frac{x_0 y_{-1}}{x_0 - y_{-1}}\right)}{\left(-\frac{x_0 y_{-1}}{x_0 - y_{-1}}\right) + y_{-1}}$$

$$= \frac{(-x_0 y_{-1})}{(x_0 - y_{-1})\left(\left(-\frac{x_0}{x_0 - y_{-1}}\right) + 1\right)}$$

$$= \frac{(-x_0 y_{-1})}{(-x_0 + x_0 - y_{-1})} = x_0,$$

$$y_{12n} = \frac{x_{12n-1} y_{12n-2}}{x_{12n-1} - y_{12n-2}} = \frac{x_{-1}\left(\frac{x_{-1} y_0}{x_{-1} + y_0}\right)}{x_{-1} - \left(\frac{x_{-1} y_0}{x_{-1} + y_0}\right)}$$

$$= \frac{x_{-1} y_0}{(x_{-1} + y_0)\left(1 - \left(\frac{y_0}{x_{-1} + y_0}\right)\right)}$$

$$= \frac{x_{-1} y_0}{(x_{-1} + y_0 - y_0)} = y_0.$$

Similarly, one can prove the other relations. The proof is complete. □

Example 2 We consider a numerical example for the difference equations system (2) with the initial conditions $x_{-1} = 0.8$, $x_0 = 3$, $y_{-1} = 2$ and $y_0 = 0.7$. (See Fig. 2).

The following cases can be proved similarly.

Theorem 4 *Assume that $\{x_n, y_n\}$ are solutions of the system*

$$x_{n+1} = \frac{x_{n-1} y_n}{x_{n-1} - y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{-x_n + y_{n-1}},$$

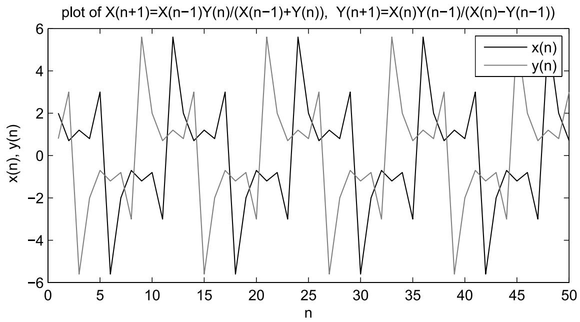

Figure 2. This figure shows the periodicity of the solution of the system (2) with the initial values as in example (2).

with the initial conditions which are arbitrary nonzero real numbers. Then, every solution of this system is periodic with period six and

$$x_{6n-1} = x_{-1}, \qquad x_{6n} = x_0, \qquad x_{6n+1} = \frac{x_{-1}y_0}{x_{-1} - y_0},$$
$$x_{6n+2} = -y_{-1}, \quad x_{6n+3} = -y_0, \quad x_{6n+4} = \frac{x_0y_{-1}}{x_0 - y_{-1}},$$
$$y_{6n-1} = y_{-1}, \qquad y_{6n} = y_0, \qquad y_{6n+1} = \frac{x_0y_{-1}}{-x_0 + y_{-1}},$$
$$y_{6n+2} = -x_{-1}, \quad y_{6n+3} = -x_0, \quad y_{6n+4} = \frac{x_{-1}y_0}{-x_{-1} + y_0},$$

Or equivalently,

$$\{x_n\}_{n=0}^{\infty} = \left\{x_{-1},\, x_0,\, \frac{x_{-1}y_0}{x_{-1} - y_0},\, -y_{-1},\, -y_0,\, \frac{x_0y_{-1}}{x_0 - y_{-1}},\, x_{-1},\, x_0, \ldots\right\},$$
$$\{y_n\}_{n=0}^{\infty} = \left\{y_{-1},\, y_0,\, \frac{x_0y_{-1}}{-x_0 + y_{-1}},\, -x_{-1},\, -x_0,\, \frac{x_{-1}y_0}{-x_{-1} + y_0},\, y_{-1},\, y_0, \ldots\right\},$$

where $x_0 \neq y_{-1}$ *and* $x_{-1} \neq y_0$.

2.3. The system: $x_{n+1} = x_{n-1}y_n/(x_{n-1} - y_n)$, $y_{n+1} = x_ny_{n-1}/(x_n + y_{n-1})$

In this section, we study the solutions of the system of the difference equations

$$x_{n+1} = \frac{x_{n-1}y_n}{x_{n-1} - y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{x_n + y_{n-1}}, \tag{3}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary nonzero real numbers with $y_{-1}/x_0 \notin \{-(F_{n+1}/F_n),\ n = 1, 2, \ldots,\}$ and $x_{-1}/y_0 \notin \{F_n/F_{n+2},\ n = 1, 2, \ldots,\} \cup \{1\}$.

Theorem 5 *If $\{x_n, y_n\}$ are solutions of system (3). Then the solutions of system (3) are given by the following formulas for $n = 0, 1, 2, \ldots$*

$$x_{2n} = \frac{x_0 y_{-1}}{x_0 F_n + y_{-1} F_{n-1}}, \quad x_{2n-1} = \frac{x_{-1} y_0}{x_{-1} F_n - y_0 F_{n-2}},$$
$$y_{2n} = \frac{y_0 x_{-1}}{x_{-1} F_{n+2} - y_0 F_n}, \quad y_{2n-1} = \frac{y_{-1} x_0}{y_{-1} F_n + x_0 F_{n+1}},$$

where $\{F_n\}_{n=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$, $F_{-2} = -1$, $F_{-1} = 1$.

Lemma 2 *Every positive solution of the equation $y_{n+1} = x_n y_{n-1}/(x_n + y_{n-1})$ is bounded and $\lim_{n\to\infty} y_n = 0$.*

Example 3 See Figure 3, for the initial conditions $x_{-1} = 5$, $x_0 = 0.11$, $y_{-1} = 0.4$ and $y_0 = 3$ when we consider system (3).

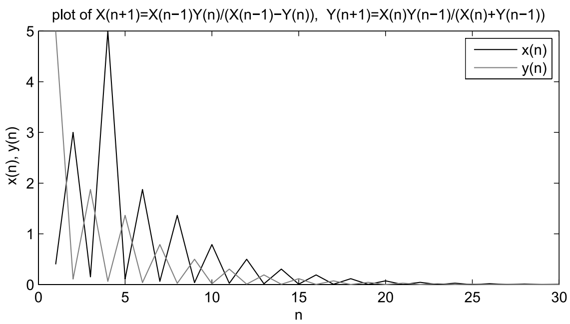

Figure 3. This figure shows the solution of the system (3) with the initial values as in example (3).

Theorem 6 *The solutions of the system*

$$x_{n+1} = \frac{x_{n-1}y_n}{x_{n-1} + y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{-x_n + y_{n-1}},$$

where $n \in \mathbb{N}_0$ *and the initial conditions are arbitrary nonzero real numbers with* $x_{-1}/y_0 \notin \{-(F_{n+1}/F_n),\ n = 1, 2, \ldots,\}$ *and* $y_{-1}/x_0 \notin \{F_n/F_{n+2},\ n = 1, 2, \ldots,\} \cup \{1\}$ *are given for* $n = 0, 1, 2, \ldots,$ *by*

$$x_{2n} = \frac{x_0 y_{-1}}{y_{-1}F_{n+2} - x_0 F_n}, \quad x_{2n-1} = \frac{x_{-1}y_0}{x_{-1}F_n + y_0 F_{n+1}},$$
$$y_{2n} = \frac{y_0 x_{-1}}{x_{-1}F_{n-1} + y_0 F_n}, \quad y_{2n-1} = \frac{y_{-1}x_0}{y_{-1}F_n - x_0 F_{n-2}}.$$

2.4. The system: $x_{n+1} = (x_{n-1}y_n)/(x_{n-1} - y_n)$, $y_{n+1} = x_n y_{n-1}/(x_n - y_{n-1})$

In this section, we investigate the solutions of the following system of the difference equations

$$x_{n+1} = \frac{x_{n-1}y_n}{x_{n-1} - y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{x_n - y_{n-1}}, \tag{4}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary nonzero real numbers with $x_{-1}/y_0 \notin \{F_{n+1}/F_n,\ n = 1, 2, \ldots,\}$ and $y_{-1}/x_0 \notin \{F_n/F_{n+2},\ n = 1, 2, \ldots,\} \cup \{1\}$.

Theorem 7 *Assume that* $\{x_n, y_n\}$ *are solutions of system* (4)*. Then for* $n = 0, 1, 2, \ldots$

$$x_{2n} = \frac{(-1)^{n+1}x_0 y_{-1}}{x_0 F_n - y_{-1}F_{n+2}}, \quad x_{2n-1} = \frac{(-1)^{n+1}x_{-1}y_0}{x_{-1}F_n - y_0 F_{n+1}},$$
$$y_{2n} = \frac{(-1)^n y_0 x_{-1}}{x_{-1}F_{n-1} - y_0 F_n}, \quad y_{2n-1} = \frac{(-1)^{n+1}y_{-1}x_0}{x_0 F_{n-2} - y_{-1}F_n}, \quad \text{where } F_{-2} = -1.$$

Example 4 Consider the difference system equation (4) with the initial conditions $x_{-1} = 0.5$, $x_0 = 0.13$, $y_{-1} = 0.7$ and $y_0 = -0.3$. (See Fig. 4).

Theorem 8 *Suppose that* $\{x_n, y_n\}$ *are solutions of the following difference equations system*

$$x_{n+1} = \frac{x_{n-1}y_n}{x_{n-1} + y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{-x_n - y_{n-1}},$$

where $n \in \mathbb{N}_0$ *and the initial conditions are arbitrary nonzero real*

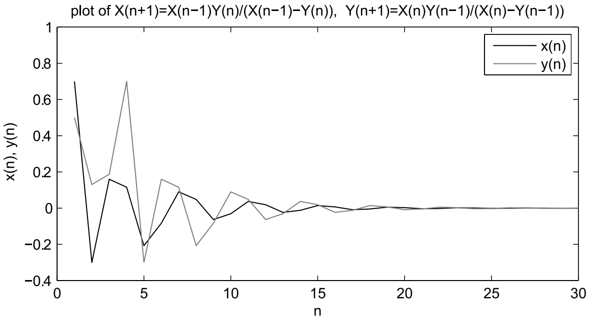

Figure 4. This figure shows the solution of the difference equations system (4) with the initial values as given in example (4).

numbers with $y_{-1}/x_0 \notin \{-(F_{n+1}/F_n), n = 1, 2, \ldots, \}$ *and* $x_{-1}/y_0 \notin \{-(F_n/F_{n+2}), n = 1, 2, \ldots, \} \cup \{-1\}$. *Then for* $n = 0, 1, 2, \ldots$

$$x_{2n} = \frac{(-1)^n x_0 y_{-1}}{x_0 F_n + y_{-1} F_{n-1}}, \quad x_{2n-1} = \frac{(-1)^{n+1} x_{-1} y_0}{x_{-1} F_n + y_0 F_{n-2}},$$

$$y_{2n} = \frac{(-1)^n y_0 x_{-1}}{x_{-1} F_{n+2} + y_0 F_n}, \quad y_{2n-1} = \frac{(-1)^n y_{-1} x_0}{x_0 F_{n+1} + y_{-1} F_n}.$$

Remark 1 The solutions of the following systems can be also obtained.

$$x_{n+1} = \frac{x_{n-1} y_n}{-x_{n-1} + y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{x_n + y_{n-1}},$$

$$x_{n+1} = \frac{x_{n-1} y_n}{-x_{n-1} + y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{x_n - y_{n-1}},$$

$$x_{n+1} = \frac{x_{n-1} y_n}{-x_{n-1} + y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{-x_n + y_{n-1}},$$

$$x_{n+1} = \frac{x_{n-1} y_n}{-x_{n-1} + y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{-x_n - y_{n-1}},$$

$$x_{n+1} = \frac{x_{n-1} y_n}{-x_{n-1} - y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{x_n + y_{n-1}},$$

$$x_{n+1} = \frac{x_{n-1} y_n}{-x_{n-1} - y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{x_n - y_{n-1}},$$

$$x_{n+1} = \frac{x_{n-1} y_n}{-x_{n-1} - y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{-x_n + y_{n-1}},$$

$$x_{n+1}=\frac{x_{n-1}y_n}{-x_{n-1}-y_n},\quad y_{n+1}=\frac{x_n y_{n-1}}{-x_n-y_{n-1}}.$$

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