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K-theory for the group C^*-algebras of certain solvable discrete groups

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Abstract. We compute the K-theory groups for the group C^*-algebras of certain solvable discrete groups. The solvable discrete groups considered are the discrete elementary $ax + b$ group and the generalized discrete elementary $ax + b$ groups and their proper versions, and also the generalized discrete elementary Mautner groups and products of the generalized discrete elementary $ax + b$ groups and their proper versions.

Key words: group C^*-algebra, K-theory, solvable discrete group.

1. Introduction

The K-theory groups for the group C^*-algebra of the discrete Heisenberg group of rank three are computed in [1] of Anderson and Paschke. Based on their result, the K-theory groups for the group C^*-algebra of the generalized discrete Heisenberg group of higher rank are computed in [10].

On the other hand, the structure, i.e., composition series of closed ideals, of the group C^*-algebras of certain solvable discrete groups is considered in [11]. The groups contain the discrete (elementary) $ax + b$ group and the generalized discrete (elementary) $ax + b$ groups and the generalized discrete (elementary) Mautner groups defined in [11].

In this paper we compute the K-theory groups for the group C^*-algebras of certain solvable discrete groups. The solvable discrete groups considered are the discrete elementary $ax+b$ group and the generalized discrete elementary $ax + b$ groups and their proper versions defined as their quotients, and also the generalized discrete elementary Mautner groups and products of the generalized discrete elementary $ax + b$ groups and their proper versions defined as their quotients.

For computation on K-theory groups, we use the six-term exact sequence of the K-theory groups for extensions of C^*-algebras and the Pimsner-

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Voiculescu six-term exact sequence of the K-theory groups for crossed products of C^*-algebras by the group $\mathbb{Z}$ of integers, and also the Künneth theorem on the K-theory groups for tensor products of C^*-algebras (see [2] and also [13]). In particular, the torsion product in the Künneth theorem is computed in those cases considered.

The computation process performed and the results obtained in this paper should be useful for further study in this topic.

Furthermore, as an application, we compute the topological stable rank and the connected stable rank for C^*-algebras (Rieffel [7]) in the case of the group C^*-algebras of those proper solvable discrete groups. The case of the non-proper is considered in [11].

In addition, we consider the case of inductive limits of those groups and group C^*-algebras.

After Introduction, this paper is organized of the following sections:

2 The discrete elementary $ax+b$ group
3 The generalized discrete elementary $ax+b$ groups
4 The generalized discrete elementary Mautner groups
5 Products of the generalized discrete elementary $ax+b$ groups
6 Their inductive limits

Their proper versions are also considered in the Sections 2 to 5 respectively.

Notation Let $\mathbb{C}$ be the C^*-algebra of all complex numbers. We denote by $C(X)$ the C^*-algebra of all continuous functions on a compact Hausdorff space X and by $C_0(X)$ the C^*-algebra of all continuous functions on a locally compact Hausdorff space X vanishing at infinity. Denote by $C^*(G)$ the full group C^*-algebra of a discrete group G. Denote by $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$ the K_0-group and the K_1-group of a C^*-algebra $\mathfrak{A}$ respectively, both of which are abelian (see [2] or [13]).

We denote by $\mathrm{sr}(\mathfrak{A})$ and $\mathrm{csr}(\mathfrak{A})$ the topological stable rank and the connected stable rank of a unital C^*-algebra $\mathfrak{A}$, respectively (see [7] of Rieffel). If $\mathfrak{A}$ is a non-unital C^*-algebra, then its topological and connected stable ranks are defined to be the topological and connected stable ranks of the unitization $\mathfrak{A}^+$ of $\mathfrak{A}$.

2. The discrete elementary $ax + b$ group

Recall from [11] that the discrete (elementary) $ax + b$ group is defined to be the semi-direct product $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$ with the action α defined by $\alpha_t(n) = (-1)^t n$ for $t, n \in \mathbb{Z}$, where $\alpha_1 = \alpha$ is the only non-trivial automorphism of $\mathbb{Z}$. Note that there is a quotient map:

$$\mathbb{Z} \rtimes_\alpha \mathbb{Z} \ni (n, t) \mapsto \begin{pmatrix} e^{\pi i t} & n \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z}),$$

where the quotient group is isomorphic to $\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$ with $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, which may be better to be called the discrete elementary $ax + b$ group, instead, so that we call it the proper discrete elementary $ax + b$ group.

Proposition 2.1 *Let $G = \mathbb{Z} \rtimes_\alpha \mathbb{Z}$ be the discrete elementary $ax+b$ group. Then*

$$K_0(C^*(G)) \cong \mathbb{Z}, \quad \text{but} \quad K_1(C^*(G)) \cong \mathbb{Z}_2 \times \mathbb{Z}.$$

Proof. Since G is the semi-direct product $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$ we have $C^*(G)$ isomorphic to the crossed product C^*-algebra $C^*(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$ with the action α (by the same symbol) corresponding to that of G, which is defined to be the C^*-algebra generated by the images $\pi(C^*(\mathbb{Z}))$ and $u(\mathbb{Z})$ under π a representation (i.e., a $*$-homomorphism) of $C^*(\mathbb{Z})$ and u a unitary representation of $\mathbb{Z}$ both acting on the same Hilbert space associated to the (faithful) covariant representation $\pi \times u$ acting on the same Hilbert space defined by

$$\pi \times u(g) = \sum_{s \in \mathbb{Z}} \pi(g(s)) u_s$$

such that $u_s \pi(g(s)) u_s^* = \pi(\alpha_t(g(s)))$ for $g \in l^1(\mathbb{Z}, C^*(\mathbb{Z}))$ (or $c_c(\mathbb{Z}, C^*(\mathbb{Z}))$) the algebra of all summable (or finitely supported) $C^*(\mathbb{Z})$-valued functions on $\mathbb{Z}$ with α-convolution product, (both of) which is dense in the crossed product (see [6]), where the action α on $C^*(\mathbb{Z})$ by $\mathbb{Z}$ is given by $\alpha_t(g(s)) = g(s) \circ \alpha_{-t}$ the composition. And the crossed product is isomorphic to $C(\mathbb{T}) \rtimes_{\alpha^\wedge} \mathbb{Z}$ by the Fourier transform, where the action $\alpha^\wedge = \alpha_1^\wedge$ (by the same symbol) on the one-torus $\mathbb{T}$ associated to the action $\alpha^\wedge = \alpha_1^\wedge$ on $C(\mathbb{T})$ is the reflection on $\mathbb{T}$ given by $\alpha_1^\wedge(z) = \bar{z}$ the complex conjugate for $z \in \mathbb{T}$.

Indeed, check that for $f \in C(\mathbb{T})$, we have (as in [14])

$$\alpha_t^\wedge f(z) = f(\varphi_z \circ \mathrm{Ad}(t))$$

for $t \in \mathbb{Z}$ with $\mathrm{Ad}(t)(n) = tnt^{-1} \in G$, where $z \in \mathbb{T}$ is identified with the homomorphism φ_z from $\mathbb{Z}$ to $\mathbb{T}$ defined by $\varphi_z(n) = z^n$ for $n \in \mathbb{Z}$, and we compute

$$\varphi_z \circ \mathrm{Ad}(t)(n) = \varphi_z(\alpha_t(n)) = \varphi_z((-1)^t n) = z^{(-1)^t n} = \varphi_{z^{(-1)^t}}(n)$$

for $n \in \mathbb{Z}$, and hence $\alpha_1^\wedge f(z) = f(\bar{z})$.

Using the Pimsner-Voiculescu six-term exact sequence for crossed product C^*-algebras by $\mathbb{Z}$ (see [2]), we get the following diagram:

$$\begin{array}{ccccc} K_0(C(\mathbb{T})) & \xrightarrow{(\mathrm{id}-\alpha^\wedge)_*} & K_0(C(\mathbb{T})) & \xrightarrow{i_*} & K_0(C(\mathbb{T}) \rtimes_{\alpha^\wedge} \mathbb{Z}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(C(\mathbb{T}) \rtimes_{\alpha^\wedge} \mathbb{Z}) & \xleftarrow{i_*} & K_1(C(\mathbb{T})) & \xleftarrow{(\mathrm{id}-\alpha^\wedge)_*} & K_1(C(\mathbb{T})) \end{array}$$

where id means the identity map on $C(\mathbb{T})$ and i means the inclusion map from $C(\mathbb{T})$ to $C(\mathbb{T}) \rtimes_{\alpha^\wedge} \mathbb{Z}$. We have $K_0(C(\mathbb{T})) \cong \mathbb{Z}$ generated by the class $[1]$ of the identity 1 of $C(\mathbb{T})$ and $K_1(C(\mathbb{T})) \cong \mathbb{Z}$ generated by the class $[z]$ of the unitary z of the coordinate function $z \mapsto z \in \mathbb{T}$ (cf. [13]). Then we compute

$$(\mathrm{id} - \alpha^\wedge)_*[1] = [\mathrm{id}(1)] - [\alpha^\wedge(1)] = [1] - [1] = 0,$$
$$(\mathrm{id} - \alpha^\wedge)_*[z] = [\mathrm{id}(z)][\alpha^\wedge(z)]^{-1} = [z][\bar{z}]^{-1} = [z]^2.$$

Hence the map $(\mathrm{id} - \alpha^\wedge)_*$ on K_0 is zero but the map $(\mathrm{id} - \alpha^\wedge)_*$ on K_1 is injective and surjective onto $2\mathbb{Z}$ in $\mathbb{Z}$. Therefore, the map i_* on K_0 is injective, and the map ∂ from K_0 to K_1 is zero because the image of this ∂ is equal to the kernel of $(\mathrm{id} - \alpha^\wedge)_*$ on K_1 by exactness of the diagram, which is zero, so that we obtain $K_0(C(\mathbb{T}) \rtimes_{\alpha^\wedge} \mathbb{Z}) \cong \mathbb{Z}$. Moreover, we get the following short exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 \to K_1(C^*(G)) \to \mathbb{Z} \to 0,$$

and hence $K_1(C^*(G)) \cong \mathbb{Z}_2 \times \mathbb{Z}$. □

Remark The group C^*-algebra of the discrete elementary $ax+b$ group is the easiest example to have torsion in K-theory groups among the group C^*-algebras of non-nilpotent, solvable discrete groups. On the other hand, it is shown in [12] that the group C^*-algebras of nilpotent discrete groups without torsion have K-theory groups torsion free. This result in the nilpotent case, in fact, is a motivation to study the case of non-nilpotent, solvable discrete groups in this paper.

Proposition 2.2 *Let $G = \mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$ be the proper discrete elementary $ax+b$ group. Then*

$$K_0(C^*(G)) \cong \mathbb{Z}^3, \quad \textit{but} \quad K_1(C^*(G)) \cong \mathbb{Z}_2.$$

Also, the group C^*-algebra $C^*(G)$ has the following short exact sequence:

$$0 \to C_0(\mathbb{R}) \otimes M_2(\mathbb{C}) \to C^*(G) \to \mathbb{C}^4 \to 0,$$

where $M_2(\mathbb{C})$ is the C^*-algebra of all 2×2 matrices over $\mathbb{C}$.

Proof. We have $C^*(G) \cong C(\mathbb{T}) \rtimes_{\alpha^\wedge} \mathbb{Z}_2$ by the Fourier transform. Since the points $\{\pm 1\}$ in $\mathbb{T}$ is fixed under the action $\alpha^\wedge$, we have the following short exact sequence:

$$0 \to C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_2 \xrightarrow{i} C^*(G) \xrightarrow{q} C(\{\pm 1\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_2 \to 0,$$

where i is the inclusion map and q is the quotient map and $C(\{\pm 1\})$ means the C^*-algebra of all continuous functions on two points $\{\pm 1\}$, which is isomorphic to $\mathbb{C}^2$. Moreover, the quotient C^*-algebra viewed as the crossed product $\mathbb{C}^2 \rtimes_{\alpha^\wedge} \mathbb{Z}_2$ is isomorphic to the direct sum $\oplus^2 \mathbb{C}^2 \cong \mathbb{C}^4$ since the action $\alpha^\wedge$ is trivial on $C(\{\pm 1\})$ and $C^*(\mathbb{Z}_2) \cong C(\{0,1\}) \cong \mathbb{C}^2$ by the Fourier transform, and the closed ideal splits into the tensor product

$$C_0((0,\pi)) \otimes (C(\{\pm i\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_2) \cong C_0(\mathbb{R}) \otimes M_2(\mathbb{C}),$$

where this i means $\sqrt{-1}$ and the action $\alpha^\wedge$ of $\mathbb{Z}_2$ on the set $\{\pm i\}$ is the shift and $C(\{\pm i\})$ means the C^*-algebra of all continuous functions on two points $\{\pm i\}$, which is isomorphic to $\mathbb{C}^2$. Thus the six-term exact sequence for the short exact sequence of C^*-algebras (see [2]) becomes the following diagram:

$$\begin{array}{ccccc} 0 & \xrightarrow{i_*} & K_0(C^*(G)) & \xrightarrow{q_*} & \mathbb{Z}^4 \\ \partial \uparrow & & & & \downarrow \partial \\ 0 & \xleftarrow{q_*} & K_1(C^*(G)) & \xleftarrow{i_*} & \mathbb{Z} \end{array}$$

where $K_0(\mathbb{C}^4) \cong \mathbb{Z}^4$ and $K_1(\mathbb{C}^4) \cong 0$ and $K_j(C_0(\mathbb{R}) \otimes M_2(\mathbb{C})) \cong K_{j+1}(M_2(\mathbb{C})) \cong K_{j+1}(\mathbb{C})$ by the Bott periodicity and the stability of K-theory groups, where $j+1 \pmod 2$. Note that if we denote by u the unitary implementing the action $\alpha^\wedge$ of $\mathbb{Z}_2$ in the crossed product $C(\mathbb{T}) \rtimes_{\alpha^\wedge} \mathbb{Z}_2$ such that $ufu^* = \alpha_1^\wedge(f)$ for $f \in C(\mathbb{T})$, then $u^2 = 1$ and

$$K_1(C^*(G)) \ni [\alpha_1^\wedge(z)] = [uzu^*] = [u][z][u] = [z]$$

where $z \in C^*(G)$ corresponds to the coordinate function $z \mapsto z \in \mathbb{T}$ in $C(\mathbb{T})$ and thus $[z]^{-1} = [z]$, i.e., $[z]^2 = 1$ in $K_1(C^*(G))$. It follows from exactness of the diagram that $K_1(C^*(G)) \cong \mathbb{Z}_2$, and thus the map ∂ from K_0 to K_1 is surjective onto $2\mathbb{Z}$ in $\mathbb{Z}$. Therefore, the kernel of ∂ from K_0 to K_1 is isomorphic to $\mathbb{Z}^3$ since the group extension by $\mathbb{Z}$ always splits, so that the image q_* from $K_0(C^*(G))$ is $\mathbb{Z}^3$ by exactness of the diagram and $K_0(C^*(G)) \cong \mathbb{Z}^3$. □

Remark Note that there is the following short exact sequence:

$$0 \to C_0(\mathbb{R}) \otimes C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \xrightarrow{i} C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}) \xrightarrow{q} C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \to 0$$

by viewing the extension as the mapping torus on the quotient (see [2]). Then the six-term exact sequence of K-theory groups associated to this extension becomes, by our computation:

$$\begin{array}{ccccc} \mathbb{Z}_2 & \xrightarrow{i_*} & \mathbb{Z} & \xrightarrow{q_*} & \mathbb{Z}^3 \\ \partial \uparrow & & & & \downarrow \partial \\ \mathbb{Z}_2 & \xleftarrow{q_*} & \mathbb{Z}_2 \times \mathbb{Z} & \xleftarrow{i_*} & \mathbb{Z}^3 \end{array}$$

where q_* on the first line is injective, so that i_* on the same line is zero, and thus ∂ from K_1 to K_0 is an isomorphism, and so that q_* on the second line is zero and i_* on the same line is onto, and that the image under ∂ from K_0 to K_1, which is isomorphic to $\mathbb{Z}^2$ by injectiveness of q_* on the first line, is $2\mathbb{Z} \times \mathbb{Z}$ in $\mathbb{Z}^3$ which is mapped to zero under i_* to make perfect sense, i.e., no contradiction.

Proposition 2.3 *The group C^*-algebra $C^*(G)$ of G the proper discrete elementary $ax+b$ group $\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$ has topological stable rank one and connected stable rank two.*

Proof. Applying the following topological stable rank formulae:

$$\max\{\mathrm{sr}(\mathfrak{I}), \mathrm{sr}(\mathfrak{A}/\mathfrak{I})\} \leq \mathrm{sr}(\mathfrak{A}) \leq \max\{\mathrm{sr}(\mathfrak{I}), \mathrm{sr}(\mathfrak{A}/\mathfrak{I}), \mathrm{csr}(\mathfrak{A}/\mathfrak{I})\}$$

for a short exact sequence $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ of C^*-algebras (Theorems 4.3, 4.4, and 4.11 of Rieffel [7]), and

$$\mathrm{sr}(\mathfrak{A} \otimes M_m(\mathbb{C})) = \lceil (\mathrm{sr}(\mathfrak{A}) - 1)m^{-1} \rceil + 1$$

for a C^*-algebra $\mathfrak{A}$ where $\lceil x \rceil$ means the least integer $\geq x$, to the short exact sequence of $C^*(G)$ obtained in Proposition 2.2 above, we obtain

$$\begin{aligned} &\max\{\mathrm{sr}(C_0(\mathbb{R}) \otimes M_2(\mathbb{C})), \mathrm{sr}(\mathbb{C}^4)\} = 1 \\ &\leq \mathrm{sr}(C^*(G)) \leq \\ &\max\{1, 1, \mathrm{csr}(\mathbb{C}^4)\} = 1, \end{aligned}$$

where $\mathrm{sr}(C_0(\mathbb{R})) = \mathrm{sr}(C_0(\mathbb{R})^+) = \mathrm{sr}(C(\mathbb{T})) = 1$ ([7, Proposition 1.7]).

Using the following connected stable rank formulae:

$$\mathrm{csr}(\mathfrak{A}) \leq \max\{\mathrm{csr}(\mathfrak{I}), \mathrm{csr}(\mathfrak{A}/\mathfrak{I})\}$$

for the short exact sequence of C^*-algebras (Theorem 3.9 of Sheu [9]), and

$$\mathrm{csr}(\mathfrak{A} \otimes M_m(\mathbb{C})) \leq \lceil (\mathrm{csr}(\mathfrak{A}) - 1)m^{-1} \rceil + 1$$

for a C^*-algebra $\mathfrak{A}$ ([8, Theorem 4.7]), we obtain

$$\begin{aligned} \mathrm{csr}(C^*(G)) &\le \max\{\mathrm{csr}(C_0(\mathbb{R}) \otimes M_2(\mathbb{C})), \mathrm{csr}(\mathbb{C}^4)\} \\ &\le \max\{\lceil(\mathrm{csr}(C_0(\mathbb{R})) - 1)2^{-1}\rceil + 1, 1\} \\ &= \lceil(2-1)2^{-1}\rceil + 1 = 2, \end{aligned}$$

where $\mathrm{csr}(C_0(\mathbb{R})) = \mathrm{csr}(C_0(\mathbb{R})^+) = \mathrm{csr}(C(\mathbb{T})) = 2$ (see [9, Page 381]). On the other hand, since $K_1(C^*(G)) \ne 0$ as obtained in Proposition 2.2, we get $\mathrm{csr}(C^*(G)) \ge 2$ by [3, Corollary 1.6] of Elhage Hassan. □

Remark Similarly, we can show that if G is the discrete elementary $ax+b$ group $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$, then $C^*(G)$ has topological stable rank two and connected stable rank two, by applying those formulae to the short exact sequence of $C^*(G)$ viewed as the mapping torus, as mentioned in [11].

3. The generalized discrete elementary $ax + b$ groups

Recall from [11] that the generalized discrete (elementary) $ax+b$ group of rank $m+1$ is defined to be the semi-direct product $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$ with the action α defined by $\alpha_t(n) = (-1)^t n = ((-1)^t n_j)$ for $t \in \mathbb{Z}$ and $n = (n_j) \in \mathbb{Z}^m$. Note that there is a quotient map:

$$\mathbb{Z}^m \rtimes_\alpha \mathbb{Z} \ni (n,t) \mapsto \begin{pmatrix} e^{\pi i t} & & 0 & n_1 \\ & \ddots & & \vdots \\ & & e^{\pi i t} & n_m \\ 0 & & & 1 \end{pmatrix} \in GL_{m+1}(\mathbb{Z}),$$

where the quotient group is isomorphic to $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$, which may be better to be called the generalized discrete elementary $ax+b$ group, instead, so that we call it the proper generalized discrete elementary $ax+b$ group.

Remark Recall from [10] (or [1] originally) that the Bott projection P in $M_2(C(\mathbb{T}^2))$ is defined as a projection-valued function from $\mathbb{T}^2$ to $M_2(\mathbb{C})$:

$$\begin{aligned} P(w,z) &= \mathrm{Ad}(U(w,z)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= U(w,z) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U(w,z)^* \in M_2(\mathbb{C}), \quad (w,z) \in \mathbb{T}^2, \end{aligned}$$

where $U(w,z) = Y(t,z)^*$ with $w = e^{2\pi i t} \in \mathbb{T}$ for $t \in [0,1]$ and

$$Y(t,z) = \exp\left(\frac{i\pi t}{2}K(z)\right)\exp\left(\frac{i\pi t}{2}S\right)$$

$$K(z) = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}, \quad S = K(1).$$

Moreover, the generalized Bott projection Q_k in $M_2(C(\mathbb{T}^{2k}))$ is defined in [10] by a projection-valued function from $\mathbb{T}^{2k}$ to $M_2(\mathbb{C})$:

$$\begin{aligned} &Q_k(z_1,\dots,z_{2k}) \\ &\quad = \mathrm{Ad}(U_1(z_1,z_2))\mathrm{Ad}(U_2(z_3,z_4))\dots\mathrm{Ad}(U_k(z_{2k-1},z_{2k}))\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

where $U_j(\cdot,\cdot) = U(\cdot,\cdot)$ for $1 \le j \le k$. Furthermore, the unitary V_k in $M_2(C(\mathbb{T}^{2k+1}))$ obtained from the generalized Bott projection Q_k and a unitary generator u of $C^*(G)$ corresponding to a generator of $G = \mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$ is defined in [10] by

$$V_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (u-1)\otimes Q_k \in M_2(C(\mathbb{T}^{2k+1})).$$

Theorem 3.1 *Let $G = \mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$ be the generalized discrete elementary $ax+b$ group. Then*

$$K_0(C^*(G)) \cong \mathbb{Z}^{2^{m-1}}, \quad \textit{but} \quad K_1(C^*(G)) \cong (\Pi^{2^{m-1}}\mathbb{Z}_2)\times\mathbb{Z}^{2^{m-1}}.$$

Proof. As shown in the proof of Proposition 2.1, we have $C^*(G) \cong C^*(\mathbb{Z}^m)\rtimes_\alpha \mathbb{Z} \cong C(\mathbb{T}^m)\rtimes_{\alpha^\wedge}\mathbb{Z}$ by the Fourier transform, where the action $\alpha^\wedge = \alpha_1^\wedge$ on $\mathbb{T}^m$ by the same symbol associated to the action $\alpha^\wedge = \alpha_1^\wedge$ on $C(\mathbb{T}^m)$ is the reflection on $\mathbb{T}^m$ given by $\alpha_1^\wedge(z_j) = (\bar{z}_j) \in \mathbb{T}^m$. Using the Pimsner-Voiculescu six-term exact sequence for crossed product C^*-algebras by $\mathbb{Z}$, we get the following diagram:

$$\begin{array}{ccccc} K_0(C(\mathbb{T}^m)) & \xrightarrow{(\mathrm{id}-\alpha^\wedge)_*} & K_0(C(\mathbb{T}^m)) & \xrightarrow{i_*} & K_0(C(\mathbb{T}^m)\rtimes_{\alpha^\wedge}\mathbb{Z}) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(C(\mathbb{T}^m)\rtimes_{\alpha^\wedge}\mathbb{Z}) & \xleftarrow{i_*} & K_1(C(\mathbb{T}^m)) & \xleftarrow{(\mathrm{id}-\alpha^\wedge)_*} & K_1(C(\mathbb{T}^m)) \end{array}$$

where id means the identity map on $C(\mathbb{T}^m)$ and i means the inclusion map from $C(\mathbb{T}^m)$ to $C(\mathbb{T}^m) \rtimes_{\alpha^\wedge} \mathbb{Z}$. We have $K_0(C(\mathbb{T}^m)) \cong \mathbb{Z}^{2^{m-1}}$ generated by the class $[1]$ of the identity 1 of $C(\mathbb{T})$ and the classes of generalized Bott projections Q_k in $M_2(C(\mathbb{T}^{2k}))$ combinatoricly in $M_2(C(\mathbb{T}^m))$, where each $\mathbb{T}^{2k}$ is identified with a direct factor in $\mathbb{T}^m$ coordinate-wise and is taken combinatoricly in $\mathbb{T}^m$, and $K_1(C(\mathbb{T}^m)) \cong \mathbb{Z}^{2^{m-1}}$ generated by the class $[z_j]$ of the unitary z_j of the j-th coordinate function $z_j \mapsto z_j \in \mathbb{T}$ in $C(\mathbb{T}^m)$ and the unitaries $V_k \in M_2(C(\mathbb{T}^{2k+1}))$ combinatoricly in $M_2(C(\mathbb{T}^m))$ associated to Q_k and z_j, where each $\mathbb{T}^{2k+1}$ is identified with a direct factor in $\mathbb{T}^m$ coordinate-wise and is taken combinatoricly in $\mathbb{T}^m$ (see [10] or the remark above and also [13]). Then we compute

$$
\begin{aligned}
(\mathrm{id} - \alpha^\wedge)_*[1] &= [\mathrm{id}(1)] - [\alpha^\wedge(1)] = [1] - [1] = 0, \\
(\mathrm{id} - \alpha^\wedge)_*[Q_k] &= [Q_k] - [\alpha^\wedge(Q_k)] = [Q_k] - [Q_k^*] = 0, \\
(\mathrm{id} - \alpha^\wedge)_*[z_j] &= [\mathrm{id}(z_j)][\alpha^\wedge(z_j)]^{-1} = [z_j][\bar{z}_j]^{-1} = [z_j]^2, \\
(\mathrm{id} - \alpha^\wedge)_*[V_k] &= [\mathrm{id}(V_k)][\alpha^\wedge(V_k)]^{-1} = [V_k][V_k^*]^{-1} = [V_k]^2
\end{aligned}
$$

since $Q_k = Q_k^*$. Hence the map $(\mathrm{id} - \alpha^\wedge)_*$ on K_0 is zero but the map $(\mathrm{id} - \alpha^\wedge)_*$ on K_1 is injective and surjective to the direct product $\Pi^{2^{m-1}} 2\mathbb{Z}$ in $\mathbb{Z}^{2^{m-1}}$. Therefore, the map i_* on K_0 is injective, and the map ∂ from K_0 to K_1 is zero because the image of this ∂ is equal to the kernel of $(\mathrm{id} - \alpha^\wedge)_*$ on K_1 by exactness of the diagram, which is zero, so that we obtain

$$K_0(C(\mathbb{T}^m) \rtimes_{\alpha^\wedge} \mathbb{Z}) \cong K_0(C(\mathbb{T}^m)) \cong \mathbb{Z}^{2^{m-1}}.$$

Moreover, we get the following short exact sequence:

$$0 \to \mathbb{Z}^{2^{m-1}} / \Pi^{2^{m-1}} 2\mathbb{Z} = \Pi^{2^{m-1}} \mathbb{Z}_2 \to K_1(C^*(G)) \to \mathbb{Z}^{2^{m-1}} \to 0,$$

and hence $K_1(C^*(G)) \cong (\Pi^{2^{m-1}} \mathbb{Z}_2) \times \mathbb{Z}^{2^{m-1}}$. □

Proposition 3.2 *Let $G = \mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$ be the proper generalized discrete elementary $ax + b$ group. Then the group C^*-algebra $C^*(G)$ has a finite composition series $\{\mathfrak{I}_k\}_{k=0}^{m+1}$ of closed ideals with $\mathfrak{I}_0 = \{0\}$ and $\mathfrak{I}_{m+1} = C^*(G)$ such that the subquotients $\mathfrak{I}_k / \mathfrak{I}_{k-1}$ for $1 \leq k \leq m$ are isomorphic to*

$$\oplus^{\binom{m}{k-1}2^{m-1}}[C_0(\mathbb{R}^{m-k+1}) \otimes M_2(\mathbb{C})]$$

and

$$C^*(G)/\mathfrak{I}_m \cong \oplus^{2^m} C^*(\mathbb{Z}_2) \cong \mathbb{C}^{2^{m+1}}.$$

Proof. We have $C^*(G) \cong C(\mathbb{T}^m) \rtimes_{\alpha^\wedge} \mathbb{Z}_2$ by the Fourier transform. Since the 2^m points $\Pi^m\{\pm1\}$ in $\mathbb{T}^m$ are fixed under the action $\alpha^\wedge$, we have the following short exact sequence:

$$0 \to C_0(\mathbb{T}^m \setminus \Pi^m\{\pm1\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_2 \xrightarrow{i} C^*(G)$$
$$\xrightarrow{q} C(\Pi^m\{\pm1\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_2 \to 0,$$

where i is the inclusion map and q is the quotient map and $C(\Pi^m\{\pm1\})$ means the C^*-algebra of all continuous functions on the 2^m points, which is isomorphic to $\mathbb{C}^{2^m}$. Moreover, the quotient C^*-algebra is isomorphic to the direct sum $\oplus^{2^m}\mathbb{C}^2 \cong \mathbb{C}^{2^{m+1}}$, with $C^*(\mathbb{Z}_2) \cong \mathbb{C}^2$. And the above closed ideal, which we now denote by $\mathfrak{I}_m$, has the following short exact sequence:

$$0 \to \mathfrak{I}_{m-1} \xrightarrow{i} \mathfrak{I}_m \xrightarrow{q} C_0(\sqcup^m[(\mathbb{T} \setminus \{\pm1\}) \times \Pi^{m-1}\{\pm1\}]) \rtimes_{\alpha^\wedge} \mathbb{Z}_2 \to 0,$$

since the disjoint union $\sqcup^m[\cdots]$ in the quotient is closed in $\mathbb{T}^m\setminus\Pi^m\{\pm1\}$ and invariant under $\alpha^\wedge$, where the components $(\mathbb{T} \setminus \{\pm1\}) \times \Pi^{m-1}\{\pm1\}$ of the disjoint union are taken combinatoricly from $\mathbb{T}^m \setminus \Pi^m\{\pm1\}$ coordinate-wise, each of which is denoted by the same symbol, and the closed ideal $\mathfrak{I}_{m-1}$ has the spectrum that corresponds to the complement of the spectrum of the quotient in the spectrum of $\mathfrak{I}_m$, more precisely, which is the crossed product by $\mathbb{Z}_2$ of the C^*-algebra of all continuous functions on the complement of the disjoint union in $\mathbb{T}^m \setminus \Pi^m\{\pm1\}$ vanishing at infinity. Moreover, the quotient is isomorphic to

$$\oplus^{m2^{m-1}}[C_0(\mathbb{R}) \otimes C(\{\pm i\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_2] \cong \oplus^{m2^{m-1}}[C_0(\mathbb{R}) \otimes M_2(\mathbb{C})].$$

Inductively, we can construct a finite composition series of closed ideals $\mathfrak{I}_k$ of $C^*(G)$ such that

$$\mathfrak{I}_k/\mathfrak{I}_{k-1} \cong C_0(\sqcup^{\binom{m}{k-1}}[(\mathbb{T}\setminus\{\pm 1\})^{m-k+1}\times \Pi^{k-1}\{\pm 1\}]) \rtimes_{\alpha^\wedge} \mathbb{Z}_2$$

for $1 \leq k \leq m$ with $\mathfrak{I}_0 = \{0\}$, where $\binom{m}{k-1}$ means the combination, and $\binom{m}{0} = 1$. Moreover, the subquotient $\mathfrak{I}_k/\mathfrak{I}_{k-1}$ of $C^*(G)$ is isomorphic to the direct sum

$$\oplus^{\binom{m}{k-1}2^{k-1}}[C_0((\mathbb{T}\setminus\{\pm 1\})^{m-k+1}) \rtimes_{\alpha^\wedge} \mathbb{Z}_2]$$

and furthermore,

$$C_0((\mathbb{T}\setminus\{\pm 1\})^{m-k+1}) \rtimes_{\alpha^\wedge} \mathbb{Z}_2 \cong C_0(\mathbb{R}^{m-k+1}) \otimes [C(\Pi^{m-k+1}\{\pm i\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_2]$$

and then

$$\begin{aligned} C(\Pi^{m-k+1}\{\pm i\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_2 &\cong \mathbb{C}^{2^{m-k+1}} \rtimes_{\alpha^\wedge} \mathbb{Z}_2 \\ &\cong \oplus^{2^{m-k}} M_2(\mathbb{C}) \end{aligned}$$

because the space $\Pi^{m-k+1}\{\pm i\}$ is viewed as the disjoint union of orbits that consists of two points in the space. □

Remark The similar composition series of $C^*(G)$ of G the generalized (elementary) $ax+b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$ is given in [11], for which it is found out that there is a mistake in counting direct sums which should be corrected as in this proposition.

Theorem 3.3 *Let $G = \mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$ be the proper generalized discrete elementary $ax+b$ group. Then*

$$K_0(C^*(G)) \cong \mathbb{Z}^{2^m+1}, \quad \textit{but} \quad K_1(C^*(G)) \cong \Pi^{2^{m-1}}\mathbb{Z}_2.$$

Proof. We have $C^*(G) \cong C(\mathbb{T}^m) \rtimes_{\alpha^\wedge} \mathbb{Z}_2$ by the Fourier transform. We use the following short exact sequence:

$$0 \to C_0(\mathbb{R}) \otimes C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2) \xrightarrow{i} C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}) \xrightarrow{q} C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2) \to 0$$

by viewing the extension as the mapping torus on the quotient (see [2]). Then the six-term exact sequence of K-theory groups associated to this extension becomes, by our computation:

$$\begin{array}{ccccc} K_1(C^*(G)) & \xrightarrow{i_*} & \mathbb{Z}^{2^{m-1}} & \xrightarrow{q_*} & K_0(C^*(G)) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(C^*(G)) & \xleftarrow{q_*} & (\Pi^{2^{m-1}}\mathbb{Z}_2) \times \mathbb{Z}^{2^{m-1}} & \xleftarrow{i_*} & K_0(C^*(G)) \end{array}$$

so that, as in the case where $m = 1$, we have

$$K_0(C^*(G)) \cong \mathbb{Z}^{2^m} \times \mathbb{Z} \cong \mathbb{Z}^{2^m+1},$$

and $K_1(C^*(G)) \cong \Pi^{2^{m-1}}\mathbb{Z}_2$. Indeed, note that the map from $K_0(C^*(G))$ to $K_0(\oplus^{2^m}\mathbb{C}^2)$ induced from the structure of $C^*(G)$ obtained in Proposition 3.2 is injective as in the case of $m = 1$, and also that each factor of the form $\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$ in G plays the same role as the proper discrete elementary $ax + b$ group, to produce non-equivalent K-theory classes. □

Proposition 3.4 *The group C^*-algebra $C^*(G)$ of G the proper generalized discrete elementary $ax+b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$ has the following topological stable rank estimate*:

$$\lceil(\lfloor m2^{-1}\rfloor)2^{-1}\rceil + 1 \leq \mathrm{sr}(C^*(G)) \leq \lceil(\lfloor(m+1)2^{-1}\rfloor)2^{-1}\rceil + 1$$

and the following connected stable rank estimate:

$$2 \leq \mathrm{csr}(C^*(G)) \leq \lceil(\lfloor(m+1)2^{-1}\rfloor)2^{-1}\rceil + 1,$$

where $\lfloor x\rfloor$ means the maximum integer $\leq x$.

In particular, if $m \geq 2$, then $\mathrm{sr}(C^*(G)) \geq 2$.

Proof. Applying those stable rank formulae in the proof of Proposition 2.3 to the composition series $\{\mathfrak{I}_k\}_{k=0}^{m+1}$ of $C^*(G)$ obtained in Proposition 3.2, repeatedly, we obtain

$$\begin{aligned} \mathrm{sr}(C^*(G)) &\geq \max_{1\leq k\leq m} \mathrm{sr}(C_0(\mathbb{R}^{m-k+1}) \otimes M_2(\mathbb{C})) \\ &= \mathrm{sr}(C_0(\mathbb{R}^m) \otimes M_2(\mathbb{C})) \\ &= \lceil(\mathrm{sr}(C_0(\mathbb{R}^m)) - 1)2^{-1}\rceil + 1 \\ &= \lceil(\lfloor m2^{-1}\rfloor)2^{-1}\rceil + 1, \end{aligned}$$

where $\mathrm{sr}(C(X)) = \lfloor 2^{-1}\dim X\rfloor + 1$ with $\dim X$ the covering dimension of a compact Hausdorff space X ([7, Proposition 1.7]), and $\mathrm{sr}(C_0(\mathbb{R}^m)) = \mathrm{sr}(C_0(\mathbb{R}^m)^+) = \mathrm{sr}(C(S^m))$ with S^m the m-dimensional sphere, and also

$$\mathrm{sr}(C^*(G)) \leq \max_{1\leq k\leq m}\{\mathrm{sr}(C_0(\mathbb{R}^{m-k+1})\otimes M_2(\mathbb{C})), \mathrm{csr}(C_0(\mathbb{R}^{m-k+1})\otimes M_2(\mathbb{C}))\}$$

and

$$\begin{aligned}\mathrm{csr}(C_0(\mathbb{R}^{m-k+1})\otimes M_2(\mathbb{C})) &\leq \lceil(\mathrm{csr}(C_0(\mathbb{R}^{m-k+1})) - 1)2^{-1}\rceil + 1\\ &= \lceil(\lfloor(m-k+2)2^{-1}\rfloor)2^{-1}\rceil + 1,\end{aligned}$$

where $\mathrm{csr}(C(X)) \leq \lfloor(\dim X + 1)2^{-1}\rfloor + 1$ by Nistor [5, Corollary 2.5].

Moreover, we get

$$\begin{aligned}2 \leq \mathrm{csr}(C^*(G)) &\leq \max_{1\leq k\leq m}\mathrm{csr}(C_0(\mathbb{R}^{m-k+1})\otimes M_2(\mathbb{C}))\\ &\leq \max_{1\leq k\leq m}\lceil(\lfloor(m-k+2)2^{-1}\rfloor)2^{-1}\rceil + 1,\end{aligned}$$

where the lower bound is obtained from that $K_1(C^*(G)) \neq 0$ by Theorem 3.3 and [3, Corollary 1.16]. □

Remark Similarly, we can show the similar topological stable rank and connected stable rank estimats of $C^*(G)$ of G the generalized discrete (elementary) $ax+b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$, as given in [11]. But the estimates given in [11] need to be slightly corrected as their n to be replaced with $n+1$, as $m+1$ given in the statemtent.

4. The generalized discrete elementary Mautner groups

Recall from [11] that the generalized discrete (elementary) Mautner group of rank $2m$ is defined to be the semi-direct product $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m$ with the action α given by

$$\alpha_{1_j}(n) = (n_1, \ldots, n_{j-1}, -n_j, n_{j+1}, \ldots, n_m)$$

for $n = (n_j) \in \mathbb{Z}^m$ and $1_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^m$ (1 at the j-th position). The discrete (elementary) Mautner group defined in [11] is just $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}^2$ the case where $m = 2$ and the discrete elementary $ax+b$ group is just $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$ the case where $m = 1$. Note that there is a quotient map:

$$\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m \ni (n,t) \mapsto \begin{pmatrix} e^{\pi i t_1} & 0 & \dots & 0 & n_1 \\ 0 & e^{\pi i t_2} & & & n_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & e^{\pi i t_m} & n_m \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in GL_{m+1}(\mathbb{Z})$$

for $n = (n_1, \dots, n_m)$, $t = (t_1, \dots, t_m) \in \mathbb{Z}^m$, where the quotient group is isomorphic to $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2^m$. As before, we call it the proper generalized Mautner group. The proper discrete elementary $ax+b$ group is just $\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$ the case where $m = 1$.

Recall from [2] that **the bootstrap category** is the smallest class N of separable nuclear C^*-algebras with the following properties:

(1) N contains $\mathbb{C}$.
(2) N is closed under countable inductive limits.
(3) For a short exact sequence of C^*-algebras, if non-zero two terms of the sequence are in N, then so is the non-zero third term.
(4) N is closed under KK-equivalence (and in particular, closed under stable isomorphism, and hence, taking tensor products with matrix algebras over $\mathbb{C}$).

In particular, N contains commutative C^*-algebras and their tensor products with matrix algebras over $\mathbb{C}$ and the type I C^*-algebras which have finite composition series of closed ideals with subquotients given by direct sums of such tensor products.

Recall also from [2] **the Künneth theorem** that states that if $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras, with $\mathfrak{A}$ in the bootstrap category N, then there is the following short exact sequence of groups:

$$0 \to K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \xrightarrow{\beta} K_*(\mathfrak{A} \otimes \mathfrak{B}) \xrightarrow{\sigma} \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0,$$

where $K_*(\cdot) = K_0(\cdot) \oplus K_1(\cdot)$, and the map β has degree 0 and the map σ has degree 1. The short exact sequence splits unnaturally. If $K_*(\mathfrak{A})$ or $K_*(\mathfrak{B})$ is torsion free, then β is an isomorphism.

Note that $K_p(\mathfrak{A}) \otimes K_q(\mathfrak{B})$, $K_p(\mathfrak{A} \otimes \mathfrak{B}) \oplus K_q(\mathfrak{A} \otimes \mathfrak{B})$, and $\mathrm{Tor}_1^{\mathbb{Z}}(K_p(\mathfrak{A}),$

$K_q(\mathfrak{B})$) for $p, q \in \mathbb{Z}_2$ all have degree $p+q$ (mod 2).

Lemma 4.1 *The group C^*-algebras of the generalized discrete elementary Mautner group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m$ and the proper $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2^m$ are in the bootstrap category N.*

Proof. It follows from the short exact sequence stated in Proposition 2.2 that the group C^*-algebra of the proper discrete elementary $ax+b$ group $\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$ is in the bootstrap category N. Since the group C^*-algebra of the discrete elementary $ax+b$ group $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$ is viewed as the mapping torus on $C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2)$ as in Remark after Proposition 2.2, it also follows that $C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z})$ is in N.

Since $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m$ is isomorphic to $\Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z})$ the m-fold direct product of the discrete elementary $ax+b$ group $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$, we have

$$C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m) \cong \otimes^m C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z})$$

the m-fold tensor product of the group C^*-algebra of $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$. Therefore, it follows that $C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m)$ is in N. Indeed, $C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m)$ has a finite composition series of closed ideals with subquotients in N, by using the structure of $C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z})$ in N shown above.

Similarly, one can show that $C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2^m)$ is in N since $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2^m$ is isomorphic to $\Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2)$. Indeed, see Proposition 4.4 below. □

Theorem 4.2 *Let $G = \mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m$ be the generalized discrete elementary Mautner group. Then*

$$K_0(C^*(G)) \cong \mathbb{Z}^{s(m)} \oplus \mathbb{Z}_2^{t(m)},$$

$$K_1(C^*(G)) \cong \mathbb{Z}^{u(m)} \oplus \mathbb{Z}_2^{v(m)},$$

where the indexes $s(m), t(m), u(m), v(m) \in \mathbb{N}$ with $s(1) = 1$, $t(1) = 0$, $u(1) = 1$, and $v(1) = 1$ are determined inductively by

$$s(m+1) = s(m) + u(m),$$

$$t(m+1) = 2\,t(m) + u(m) + 2\,v(m),$$

$$u(m+1) = s(m) + u(m),$$

$$v(m+1) = 2\,t(m) + s(m) + 2\,v(m)$$

$(m \geq 1)$. *In other words, letting* $G_m = \mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m$, *we have*

$$\begin{aligned}
\mathrm{rank}_{\mathbb{Z}} K_j(C^*(G_{m+1})) &= \mathrm{rank}_{\mathbb{Z}} K_0(C^*(G_m)) + \mathrm{rank}_{\mathbb{Z}} K_1(C^*(G_m)), \\
\mathrm{rank}_{\mathbb{Z}_2} K_0(C^*(G_{m+1})) &= \mathrm{rank}_{\mathbb{Z}} K_1(C^*(G_m)) \\
&\quad + 2\,\mathrm{rank}_{\mathbb{Z}_2} K_0(C^*(G_m)) + 2\,\mathrm{rank}_{\mathbb{Z}_2} K_1(C^*(G_m)), \\
\mathrm{rank}_{\mathbb{Z}_2} K_1(C^*(G_{m+1})) &= \mathrm{rank}_{\mathbb{Z}} K_0(C^*(G_m)) \\
&\quad + 2\,\mathrm{rank}_{\mathbb{Z}_2} K_0(C^*(G_m)) + 2\,\mathrm{rank}_{\mathbb{Z}_2} K_1(C^*(G_m))
\end{aligned}$$

$(j = 0, 1)$, *where* $\mathrm{rank}_{\mathbb{Z}}(\cdot)$ *and* $\mathrm{rank}_{\mathbb{Z}_2}(\cdot)$ *mean the free rank and the torsion rank with respect to* $\mathbb{Z}$ *and* $\mathbb{Z}_2$ *respectively.*

In particular,

$$K_0(C^*(G_2)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2^3, \quad K_1(C^*(G_2)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2^3,$$

and also

$$K_0(C^*(G_3)) \cong \mathbb{Z}^4 \oplus \mathbb{Z}_2^{14}, \quad K_1(C^*(G_3)) \cong \mathbb{Z}^4 \oplus \mathbb{Z}_2^{14}.$$

In addition, it follows from the inductive equations above that

$$K_0(C^*(G_m)) \cong K_1(C^*(G_m))$$

if $m \geq 2$.

Moreover, it does follow that if $m \geq 2$, *then*

$$K_j(C^*(G_m)) \cong \mathbb{Z}^{2^{m-1}} \oplus \mathbb{Z}_2^{2^{2m-2} - 2^{m-2}}$$

for $j = 0, 1$.

Remark The statement is somewhat long, but we would like to reveal the process to achieve the last general complicated formula, probably, from which it would be difficult to see the process. As a note, it is the ingenious referee who suggested that the last general formula would hold and encouraged the author to prove it as well as such formulae in Theorems 4.3, 5.2, and 5.3 given below.

Proof. Since $G \cong \Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z})$ the m-fold direct product of the discrete $ax + b$ group $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$, we have

$$C^*(G) \cong \otimes^m C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z})$$

the m-fold tensor product of the group C^*-algebra of $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$.

We first consider the case where $m = 2$. Let $H = \mathbb{Z} \rtimes_\alpha \mathbb{Z}$. The Künneth theorem (see [2]) implies that since $C^*(H)$ is in N by Lemma 4.1, we have the following short exact sequence of abelian groups:

$$\begin{aligned} &0 \to K_*(C^*(H)) \otimes K_*(C^*(H)) \\ &\quad \xrightarrow{\beta} K_*(C^*(H) \otimes C^*(H)) \xrightarrow{\sigma} \mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(H)), K_*(C^*(H))) \to 0 \end{aligned}$$

where $K_*(\cdot) = K_0(\cdot) \oplus K_1(\cdot)$ and the map β has degree 0 and the map σ has degree 1 and the short exact sequence splits unnaturally. As obtained in Proposition 2.1, we have

$$K_*(C^*(H)) = K_0(C^*(H)) \oplus K_1(C^*(H)) \cong \mathbb{Z} \oplus (\mathbb{Z} \times \mathbb{Z}_2).$$

By using several facts in homology theory as in [4], we compute the torsion product as follows:

$$\begin{aligned} &\mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(H)), K_*(C^*(H))) \\ &\quad \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z} \oplus (\mathbb{Z} \times \mathbb{Z}_2), K_*(C^*(H))) \\ &\quad \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, K_*(C^*(H))) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, K_*(C^*(H))) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, K_*(C^*(H))) \\ &\quad \cong 0 \oplus 0 \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}^2 \times \mathbb{Z}_2) \\ &\quad \cong \mathbb{Z}_2 \end{aligned}$$

where note that this consequence comes from the pair $(K_1(C^*(H)), K_1(C^*(H)))$, so that the torsion product is in $K_1(C^*(H \times H))$.

Therefore, it follows that

$$K_*(C^*(H) \otimes C^*(H)) \cong [K_*(C^*(H)) \otimes K_*(C^*(H))] \oplus \mathbb{Z}_2$$

(unnaturally). Moreover, we obtain

$$\begin{aligned} &K_0(C^*(H \times H)) \\ &\quad \cong (K_0(C^*(H)) \otimes K_0(C^*(H))) \oplus (K_1(C^*(H)) \otimes K_1(C^*(H))) \end{aligned}$$

$$\cong (\mathbb{Z} \otimes \mathbb{Z}) \oplus [(\mathbb{Z} \times \mathbb{Z}_2) \otimes (\mathbb{Z} \times \mathbb{Z}_2)]$$
$$\cong \mathbb{Z} \oplus [\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2]$$
$$= (\oplus^2 \mathbb{Z}) \oplus (\oplus^3 \mathbb{Z}_2)$$

where note that $(1 \otimes 1) + (1 \otimes 1) = 2 \otimes 1 = 1 \otimes 2$ in $\mathbb{Z} \otimes \mathbb{Z}$, and $(1 \otimes 1) + (1 \otimes 1) = 2 \otimes 1 = 1 \otimes 2 = 0$ in $\mathbb{Z} \otimes \mathbb{Z}_2$ and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$. Furthermore,

$$\begin{aligned} &K_1(C^*(H \times H))/\mathbb{Z}_2 \\ &\quad \cong (K_0(C^*(H)) \otimes K_1(C^*(H))) \oplus (K_1(C^*(H)) \otimes K_0(C^*(H))) \\ &\quad \cong \oplus^2 (\mathbb{Z} \otimes (\mathbb{Z} \times \mathbb{Z}_2)) \\ &\quad \cong \oplus^2 (\mathbb{Z} \oplus \mathbb{Z}_2) \\ &\quad \cong (\oplus^2 \mathbb{Z}) \oplus (\oplus^2 \mathbb{Z}_2). \end{aligned}$$

Hence we get $K_1(C^*(H \times H)) \cong \mathbb{Z}^2 \times \mathbb{Z}_2^3$.

Repeating the same argument for $C^*(\Pi^3 H) \cong C^*(H \times H) \otimes C^*(H)$, we compute

$$\begin{aligned} &\mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(H \times H)), K_*(C^*(H))) \\ &\quad \cong \mathrm{Tor}_1^{\mathbb{Z}}((\mathbb{Z}^2 \times \mathbb{Z}_2^3) \oplus (\mathbb{Z}^2 \times \mathbb{Z}_2^3), \mathbb{Z} \oplus (\mathbb{Z} \times \mathbb{Z}_2)) \\ &\quad \cong (\oplus^3 \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z} \oplus (\mathbb{Z} \times \mathbb{Z}_2)) \oplus (\oplus^3 \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z} \oplus (\mathbb{Z} \times \mathbb{Z}_2)) \\ &\quad \cong (\oplus^3 \mathbb{Z}_2) \oplus (\oplus^3 \mathbb{Z}_2) \end{aligned}$$

where the first summand corresponds to the pair $(K_0(C^*(H \times H)), K_1(C^*(H)))$ and the second summand corresponds to the pair $(K_1(C^*(H \times H)), K_1(C^*(H)))$. Therefore, we obtain

$$\begin{aligned} K_0(C^*(\Pi^3 H))/\mathbb{Z}_2^3 &\cong [(\mathbb{Z}^2 \oplus \mathbb{Z}_2^3) \otimes \mathbb{Z}] \oplus [(\mathbb{Z}^2 \oplus \mathbb{Z}_2^3) \otimes (\mathbb{Z} \times \mathbb{Z}_2)] \\ &\cong (\mathbb{Z}^2 \oplus \mathbb{Z}_2^3) \oplus (\mathbb{Z}^2 \oplus \mathbb{Z}_2^3) \oplus (\mathbb{Z}_2^2 \oplus \mathbb{Z}_2^3) \\ &\cong \mathbb{Z}^4 \oplus \mathbb{Z}_2^{11}, \end{aligned}$$

and thus, $K_0(C^*(\Pi^3 H)) \cong \mathbb{Z}^4 \oplus \mathbb{Z}_2^{14}$, and also

$$\begin{aligned} K_1(C^*(\Pi^3 H))/\mathbb{Z}_2^3 &\cong [(\mathbb{Z}^2 \oplus \mathbb{Z}_2^3) \otimes (\mathbb{Z} \times \mathbb{Z}_2)] \oplus [(\mathbb{Z}^2 \oplus \mathbb{Z}_2^3) \otimes \mathbb{Z}] \\ &\cong (\mathbb{Z}^2 \oplus \mathbb{Z}_2^3) \oplus (\mathbb{Z}_2^2 \oplus \mathbb{Z}_2^3) \oplus (\mathbb{Z}^2 \oplus \mathbb{Z}_2^3) \\ &\cong \mathbb{Z}^4 \oplus \mathbb{Z}_2^{11} \end{aligned}$$

and hence, $K_1(C^*(\Pi^3 H)) \cong \mathbb{Z}^4 \oplus \mathbb{Z}_2^{14}$.

By induction, for $G = \mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m$, we may assume that

$$K_0(C^*(G)) \cong \mathbb{Z}^{s(m)} \oplus \mathbb{Z}_2^{t(m)}, \quad K_1(C^*(G)) \cong \mathbb{Z}^{u(m)} \oplus \mathbb{Z}_2^{v(m)},$$

for some $s(m), t(m), u(m), v(m) \in \mathbb{N}$. Then

$$\mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(G)), K_*(C^*(H))) = \mathbb{Z}_2^{t(m)} \oplus \mathbb{Z}_2^{v(m)},$$

where the first summand corresponds to the pair $(K_0(C^*(G)), K_1(C^*(H)))$ and the second summand corresponds to the pair $(K_1(C^*(G)), K_1(C^*(H)))$. Therefore, we have

$$\begin{aligned} &K_0(C^*(G \times H))/\mathbb{Z}_2^{t(m)} \\ &\quad\cong [(\mathbb{Z}^{s(m)} \oplus \mathbb{Z}_2^{t(m)}) \otimes \mathbb{Z}] \oplus [(\mathbb{Z}^{u(m)} \oplus \mathbb{Z}_2^{v(m)}) \otimes (\mathbb{Z} \times \mathbb{Z}_2)] \\ &\quad\cong \mathbb{Z}^{s(m)+u(m)} \oplus \mathbb{Z}_2^{t(m)+u(m)+2\,v(m)}, \end{aligned}$$

and also

$$\begin{aligned} &K_1(C^*(G \times H))/\mathbb{Z}_2^{v(m)} \\ &\quad\cong [(\mathbb{Z}^{u(m)} \oplus \mathbb{Z}_2^{v(m)}) \otimes \mathbb{Z}] \oplus [(\mathbb{Z}^{s(m)} \oplus \mathbb{Z}_2^{t(m)}) \otimes (\mathbb{Z} \times \mathbb{Z}_2)] \\ &\quad\cong \mathbb{Z}^{s(m)+u(m)} \oplus \mathbb{Z}_2^{2\,t(m)+s(m)+v(m)}, \end{aligned}$$

Hence we get

$$\begin{aligned} K_0(C^*(\mathbb{Z}^{m+1} \rtimes_\alpha \mathbb{Z}^{m+1})) &\cong \mathbb{Z}^{s(m)+u(m)} \oplus \mathbb{Z}_2^{2\,t(m)+u(m)+2\,v(m)}, \\ K_1(C^*(\mathbb{Z}^{m+1} \rtimes_\alpha \mathbb{Z}^{m+1})) &\cong \mathbb{Z}^{s(m)+u(m)} \oplus \mathbb{Z}_2^{2\,t(m)+s(m)+2\,v(m)}. \end{aligned}$$

It then follows that

$$s(m+1) = u(m+1) = s(m) + u(m) \quad \text{for } m \geq 1 \text{ and}$$
$$t(m+1) = v(m+1) = 2\,t(m) + s(m) + 2\,v(m) \quad \text{for } m \geq 1$$

since $s(1) = u(1) = 1$. Therefore, we obtain that $K_0(C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m)) \cong K_1(C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m))$ for $m \geq 2$.

Moreover, since $s(1) = u(1) = 1$, it follows from the first inductive equation that $s(m) = u(m) = 2^{m-1}$ for $m \geq 1$. It then follows from the second inductive equation that $t(m+1) = 4\,t(m) + 2^{m-1}$ for $m \geq 2$. Dividing both sides by the power $2^{(m+1)-1}$ of 2 yields the following:

$$\frac{t(m+1)}{2^{(m+1)-1}} = 2 \cdot \left(\frac{t(m)}{2^{m-1}}\right) + \frac{1}{2}.$$

Put $b(m) = t(m)/2^{m-1}$ for $m \geq 2$. Then $b(m+1) = 2\,b(m) + 1/2$. This equation is transposed to the following: $b(m+1) + 1/2 = 2(b(m) + 1/2)$. Therefore, the general term is given by $b(m) + 1/2 = 2^{m-2}(b(2) + 1/2)$ with

$$b(2) = \frac{t(2)}{2} = \frac{2\,t(1) + u(1) + 2\,v(1)}{2} = \frac{3}{2}.$$

Thus, we obtain $t(m) = 2^{2m-2} - 2^{m-2}$ for $m \geq 2$. □

Theorem 4.3 *Let $G = \mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2^m$ be the proper generalized discrete elementary Mautner group. Then*

$$K_0(C^*(G)) \cong \mathbb{Z}^{s(m)} \oplus \mathbb{Z}_2^{t(m)},$$
$$K_1(C^*(G)) \cong \mathbb{Z}_2^{v(m)},$$

where the indexes $s(m), t(m), v(m) \in \mathbb{N}$ with $s(1) = 3$, $t(1) = 0$, and $v(1) = 1$ are determined inductively by

$$s(m+1) = 3\,s(m),$$
$$t(m+1) = 4\,t(m) + v(m),$$
$$v(m+1) = s(m) + t(m) + 4\,v(m)$$

$(m \geq 1)$. *In other words, letting $G_m = \mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2^m$, we have*

$$\begin{aligned}
\mathrm{rank}_{\mathbb{Z}} K_0(C^*(G_{m+1})) &= 3\,\mathrm{rank}_{\mathbb{Z}} K_0(C^*(G_m)),\\
\mathrm{rank}_{\mathbb{Z}_2} K_0(C^*(G_{m+1})) &= 4\,\mathrm{rank}_{\mathbb{Z}_2} K_0(C^*(G_m)) + \mathrm{rank}_{\mathbb{Z}_2} K_1(C^*(G_m)),\\
\mathrm{rank}_{\mathbb{Z}_2} K_1(C^*(G_{m+1})) &= \mathrm{rank}_{\mathbb{Z}} K_0(C^*(G_m))\\
&\quad + \mathrm{rank}_{\mathbb{Z}_2} K_0(C^*(G_m)) + 4\,\mathrm{rank}_{\mathbb{Z}_2} K_1(C^*(G_m)).
\end{aligned}$$

In particular,

$$K_0(C^*(G_2)) \cong \mathbb{Z}^9 \oplus \mathbb{Z}_2, \quad K_1(C^*(G_2)) \cong \mathbb{Z}_2^7,$$

and also

$$K_0(C^*(G_3)) \cong \mathbb{Z}^{27} \oplus \mathbb{Z}_2^{11}, \quad K_1(C^*(G_3)) \cong \mathbb{Z}_2^{38}.$$

Moreover, it does follow that if $m \geq 2$, then

$$\begin{aligned}
K_0(C^*(G_m)) &\cong \mathbb{Z}^{3^m} \oplus \mathbb{Z}_2^{2^{-2}[5^m-(2m+3)3^{m-1}]},\\
K_1(C^*(G_m)) &\cong \mathbb{Z}_2^{2^{-2}[5^m+(2m-3)3^{m-1}]}
\end{aligned}$$

and this also holds for the case $m = 1$, where $\mathbb{Z}_2^0$ is assumed to be trivial.

Proof. Since $G \cong \Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2)$ the m-fold direct product of the proper discrete $ax + b$ group $\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$, we have

$$C^*(G) \cong \otimes^m C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2)$$

the m-fold tensor product of the group C^*-algebra of $\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$.

We first consider the case where $m = 2$. Let $H = \mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$. The Künneth theorem (see [2]) implies that since $C^*(H)$ is in N by Lemma 4.1, we have the following short exact sequence of abelian groups:

$$\begin{aligned}
&0 \to K_*(C^*(H)) \otimes K_*(C^*(H))\\
&\quad \xrightarrow{\beta} K_*(C^*(H) \otimes C^*(H)) \xrightarrow{\sigma} \mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(H)), K_*(C^*(H))) \to 0
\end{aligned}$$

where $K_*(\cdot) = K_0(\cdot) \oplus K_1(\cdot)$ and the map β has degree 0 and the map σ has degree 1 and the short exact sequence splits unnaturally. As obtained in Proposition 2.2, we have

$$K_*(C^*(H)) = K_0(C^*(H)) \oplus K_1(C^*(H)) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_2.$$

By using several facts in homology theory, we compute the torsion product as follows:

$$\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(H)), K_*(C^*(H))) \\
&\quad \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^3 \oplus \mathbb{Z}_2, K_*(C^*(H))) \\
&\quad \cong [\oplus^3 \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, K_*(C^*(H)))] \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, K_*(C^*(H))) \\
&\quad \cong [\oplus^3 0] \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}^3 \times \mathbb{Z}_2) \\
&\quad \cong \mathbb{Z}_2
\end{aligned}$$

where note that this consequence comes from the pair $(K_1(C^*(H)), K_1(C^*(H)))$, so that the torsion product is in $K_1(C^*(H \times H))$.

Therefore, it follows that

$$K_*(C^*(H) \otimes C^*(H)) \cong [K_*(C^*(H)) \otimes K_*(C^*(H))] \oplus \mathbb{Z}_2$$

(unnaturally). Moreover, we obtain

$$\begin{aligned}
&K_0(C^*(H \times H)) \\
&\quad \cong (K_0(C^*(H)) \otimes K_0(C^*(H))) \oplus (K_1(C^*(H)) \otimes K_1(C^*(H))) \\
&\quad \cong (\mathbb{Z}^3 \otimes \mathbb{Z}^3) \oplus (\mathbb{Z}_2 \otimes \mathbb{Z}_2) \\
&\quad = (\oplus^9 \mathbb{Z}) \oplus \mathbb{Z}_2.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&K_1(C^*(H \times H))/\mathbb{Z}_2 \\
&\quad \cong (K_0(C^*(H)) \otimes K_1(C^*(H))) \oplus (K_1(C^*(H)) \otimes K_0(C^*(H))) \\
&\quad \cong \oplus^2(\mathbb{Z}^3 \otimes \mathbb{Z}_2) \\
&\quad \cong \oplus^2(\oplus^3 \mathbb{Z}_2) \\
&\quad \cong \oplus^6 \mathbb{Z}_2
\end{aligned}$$

and hence, $K_1(C^*(H \times H)) \cong \mathbb{Z}_2^7$.

Repeating the same argument for $C^*(\Pi^3 H) \cong C^*(H \times H) \otimes C^*(H)$, we compute

$$\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(H \times H)), K_*(C^*(H))) \\
&\quad \cong \mathrm{Tor}_1^{\mathbb{Z}}((\mathbb{Z}^9 \times \mathbb{Z}_2) \oplus \mathbb{Z}_2^7, \mathbb{Z}^3 \oplus \mathbb{Z}_2) \\
&\quad \cong (\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}^3 \oplus \mathbb{Z}_2)) \oplus (\oplus^7 \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}^3 \oplus \mathbb{Z}_2)) \\
&\quad \cong (\mathbb{Z}_2) \oplus (\oplus^7 \mathbb{Z}_2)
\end{aligned}$$

where the first summand corresponds to the pair $(K_0(C^*(H \times H)), K_1(C^*(H)))$ and the second summand corresponds to the pair $(K_1(C^*(H \times H)), K_1(C^*(H)))$. Therefore, we obtain

$$\begin{aligned}
K_0(C^*(\Pi^3 H))/\mathbb{Z}_2 &\cong [(\mathbb{Z}^9 \oplus \mathbb{Z}_2) \otimes \mathbb{Z}^3] \oplus [\mathbb{Z}_2^7 \otimes \mathbb{Z}_2] \\
&\cong (\mathbb{Z}^{27} \oplus \mathbb{Z}_2^3) \oplus \mathbb{Z}_2^7 \\
&\cong \mathbb{Z}^{27} \oplus \mathbb{Z}_2^{10},
\end{aligned}$$

and thus, $K_0(C^*(\Pi^3 H)) \cong \mathbb{Z}^{27} \oplus \mathbb{Z}_2^{11}$, and also

$$\begin{aligned}
K_1(C^*(\Pi^3 H))/\mathbb{Z}_2^7 &\cong [(\mathbb{Z}^9 \oplus \mathbb{Z}_2) \otimes \mathbb{Z}_2] \oplus [\mathbb{Z}_2^7 \otimes \mathbb{Z}^3] \\
&\cong (\mathbb{Z}_2^9 \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2^{21} \\
&\cong \mathbb{Z}_2^{31}
\end{aligned}$$

and hence, $K_1(C^*(\Pi^3 H)) \cong \mathbb{Z}_2^{38}$.

By induction, for $G = \mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2^m$, we may assume that

$$K_0(C^*(G)) \cong \mathbb{Z}^{s(m)} \oplus \mathbb{Z}_2^{t(m)}, \quad K_1(C^*(G)) \cong \mathbb{Z}_2^{v(m)},$$

for some $s(m), t(m), v(m) \in \mathbb{N}$. Then

$$\mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(G)), K_*(C^*(H))) = \mathbb{Z}_2^{t(m)} \oplus \mathbb{Z}_2^{v(m)},$$

where the first summand corresponds to the pair $(K_0(C^*(G)), K_1(C^*(H)))$ and the second summand corresponds to the pair $(K_1(C^*(G)), K_1(C^*(H)))$.

Therefore, we have

$$K_0(C^*(G \times H))/\mathbb{Z}_2^{t(m)} \cong [(\mathbb{Z}^{s(m)} \oplus \mathbb{Z}_2^{t(m)}) \otimes \mathbb{Z}^3] \oplus [\mathbb{Z}_2^{v(m)} \otimes \mathbb{Z}_2]$$
$$\cong \mathbb{Z}^{3\,s(m)} \oplus \mathbb{Z}_2^{3\,t(m)+v(m)},$$

and also

$$K_1(C^*(G \times H))/\mathbb{Z}_2^{v(m)} \cong [\mathbb{Z}_2^{v(m)} \otimes \mathbb{Z}^3] \oplus [(\mathbb{Z}^{s(m)} \oplus \mathbb{Z}_2^{t(m)}) \otimes \mathbb{Z}_2]$$
$$\cong \mathbb{Z}_2^{s(m)+t(m)+3\,v(m)},$$

Hence we get

$$K_0(C^*(\mathbb{Z}^{m+1} \rtimes_\alpha \mathbb{Z}_2^{m+1})) \cong \mathbb{Z}^{3\,s(m)} \oplus \mathbb{Z}_2^{4\,t(m)+v(m)},$$
$$K_1(C^*(\mathbb{Z}^{m+1} \rtimes_\alpha \mathbb{Z}_2^{m+1})) \cong \mathbb{Z}_2^{s(m)+t(m)+4\,v(m)}.$$

Moreover, it follows from the inductive equation $s(m+1) = 3\,s(m)$ $(m \geq 1)$ with $s(1) = 3$ that $s(m) = 3^m$ $(m \geq 1)$. The indexes $t(m+1)$ and $v(m+1)$ are viewed as the vector $X(m+1)$ in the following equation with matrix multiplication:

$$X(m+1) \equiv \begin{pmatrix} t(m+1) \\ v(m+1) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} t(m) \\ v(m) \end{pmatrix} + \begin{pmatrix} 0 \\ s(m) \end{pmatrix}$$
$$\equiv MX(m) + Y(m)$$

$(m \geq 1)$. Inductively, it follows that

$$\begin{aligned} X(m) &= MX(m-1) + Y(m-1) \\ &= M(MX(m-2) + Y(m-2)) + Y(m-1) \\ &= M^2X(m-2) + MY(m-2) + Y(m-1) \\ &= \cdots\cdots\cdots \\ &= M^{m-1}X(1) + M^{m-2}Y(1) + \cdots + MY(m-2) + Y(m-1). \end{aligned}$$

Since the matrices $D = \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right)$ and $F = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ commute, we compute the matrix product M^m by binary expansion:

$$\begin{aligned} M^m &= (D+F)^m \\ &= D^m + \binom{m}{1} D^{m-1}F + \cdots + \binom{m}{k} D^{m-k}F^k + \cdots + \binom{m}{m} F^m. \end{aligned}$$

Since F^2 is the identity matrix, if m is even, then $M^m =$

$$\begin{pmatrix} 4^m + \binom{m}{2}4^{m-2} + \cdots + \binom{m}{m}4^0 & \binom{m}{1}4^{m-1} + \binom{m}{3}4^{m-3} + \cdots + \binom{m}{m-1}4 \\ \binom{m}{1}4^{m-1} + \binom{m}{3}4^{m-3} + \cdots + \binom{m}{m-1}4 & 4^m + \binom{m}{2}4^{m-2} + \cdots + \binom{m}{m}4^0 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{-1}(5^m + 3^m) & 2^{-1}(5^m - 3^m) \\ 2^{-1}(5^m - 3^m) & 2^{-1}(5^m + 3^m) \end{pmatrix}$$

where we use binary expansion of $5^m = (4+1)^m$ and $3^m = (4-1)^m$. Similarly, if m is odd, then $M^m =$

$$\begin{pmatrix} 4^m + \binom{m}{2}4^{m-2} + \cdots + \binom{m}{m-1}4 & \binom{m}{1}4^{m-1} + \binom{m}{3}4^{m-3} + \cdots + \binom{m}{m}4^0 \\ \binom{m}{1}4^{m-1} + \binom{m}{3}4^{m-3} + \cdots + \binom{m}{m}4^0 & 4^m + \binom{m}{2}4^{m-2} + \cdots + \binom{m}{m-1}4 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{-1}(5^m + 3^m) & 2^{-1}(5^m - 3^m) \\ 2^{-1}(5^m - 3^m) & 2^{-1}(5^m + 3^m) \end{pmatrix}.$$

Note that as suggested by the referee, one may use linear algebra theory to compute the product M^m to be diagonalized as Jordan normal form by an invertible matrix. Indeed, the eigenvalues λ of the 2×2 matrix M are given by 3 and 5 by computing the determinant $\left|\begin{smallmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{smallmatrix}\right| = 0$, and the corresponding eigenvectors are given by $\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ respectively. It follows that

$$P^{-1}MP \equiv \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} M \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix},$$

i.e., M is diagonalizable, and hence, we have

$$M^m = P \begin{pmatrix} 3^m & 0 \\ 0 & 5^m \end{pmatrix} P^{-1} = \frac{1}{2}\begin{pmatrix} 5^m + 3^m & 5^m - 3^m \\ 5^m - 3^m & 5^m + 3^m \end{pmatrix}.$$

Therefore, we obtain that for $m \geq 2$,

$$\begin{aligned} t(m) &= 2^{-1}(5^{m-1} - 3^{m-1}) + 2^{-1}(5^{m-2} - 3^{m-2})3 + \cdots + 2^{-1}(5-3)3^{m-2} \\ &= 2^{-1} \sum_{k=1}^{m-1} (5^{m-k} - 3^{m-k})3^{k-1}, \\ &= 2^{-1} \sum_{k=1}^{m-1} 3^{k-1}5^{m-k} - 2^{-1} \sum_{k=1}^{m-1} 3^{m-1} \\ &= 2^{-1} \sum_{k=1}^{m-1} 3^{k-1}5^{m-k} - 2^{-1}(m-1)3^{m-1}, \end{aligned}$$

and

$$\begin{aligned} v(m) &= 2^{-1}(5^{m-1} + 3^{m-1}) + 2^{-1}(5^{m-2} + 3^{m-2})3 \\ &\quad + \cdots + 2^{-1}(5+3)3^{m-2} + 3^{m-1} \\ &= 2^{-1} \sum_{k=1}^{m-1} (5^{m-k} + 3^{m-k})3^{k-1} + 3^{m-1} \\ &= 2^{-1} \sum_{k=1}^{m-1} 3^{k-1}5^{m-k} + 2^{-1} \sum_{k=1}^{m-1} 3^{m-1} + 3^{m-1} \\ &= 2^{-1} \sum_{k=1}^{m-1} 3^{k-1}5^{m-k} + 2^{-1}(m-1)3^{m-1} + 3^{m-1} \\ &= 2^{-1} \sum_{k=1}^{m-1} 3^{k-1}5^{m-k} + 2^{-1}(m+1)3^{m-1}. \end{aligned}$$

Furthermore, we now put $S_m = \sum_{k=1}^{m-1} 3^{k-1}5^{m-k}$. Then we have

$$S_m - \frac{3}{5} S_m = 3^0 5^{m-1} - 3^{m-1}5^0.$$

Hence, $S_m = (5/2)(5^{m-1} - 3^{m-1})$. Therefore, we get

$$\begin{aligned} t(m) &= 2^{-2}5(5^{m-1} - 3^{m-1}) - 2^{-1}(m-1)3^{m-1} \\ &= 2^{-2}[5^m - (2m+3)3^{m-1}], \end{aligned}$$

$$\begin{aligned} v(m) &= 2^{-2}5(5^{m-1} - 3^{m-1}) + 2^{-1}(m+1)3^{m-1} \\ &= 2^{-2}[5^m + (2m-3)3^{m-1}]. \end{aligned}$$ □

Proposition 4.4 *The group C^*-algebra $C^*(G)$ of G the proper generalized discrete elementary Mautner group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2^m$ has a composition series $\{\mathfrak{I}_j\}_{j=1}^{2^m}$ of closed ideals such that $\mathfrak{I}_0 = \{0\}$ and $\mathfrak{I}_{2^m} = C^*(G)$ and*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \mathfrak{L}_{1_j} \otimes \cdots \otimes \mathfrak{L}_{l_j} \otimes \cdots \otimes \mathfrak{L}_{m_j}$$

combinatoricly, for some $l_j = 1, 2$ and $l_{j-1} \le l_j$ with $(1_j, \ldots, m_j)$ totally ordered properly in the sense as

$$\begin{aligned} (1,1,\ldots,1) &\le (2,1,\ldots,1) \le (1,2,\ldots,1) \le \\ &\le (2,2,1,\ldots,1) \le (1,1,2,\ldots,1) \le \cdots\cdots \le (2,2,\ldots,2), \end{aligned}$$

and

$$\mathfrak{L}_1 = C_0(\mathbb{R}) \otimes M_2(\mathbb{C}), \quad \text{and} \quad \mathfrak{L}_2 = \mathbb{C}^4.$$

Proof. This is obtained from that $C^*(G) \cong \otimes^m C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2)$ and the short exact sequence of $C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2)$ obtained in Proposition 2.2. Indeed, the closed ideals $\mathfrak{I}_j$ are defined inductively as in the following:

$$\begin{aligned} \mathfrak{I}_1 &= \mathfrak{L}^1 \otimes \mathfrak{L}^1 \otimes \cdots \otimes \mathfrak{L}^1, \\ \mathfrak{I}_2 &= C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes \mathfrak{L}^1 \otimes \cdots \otimes \mathfrak{L}^1, \end{aligned}$$

$\mathfrak{I}_3$ is generated by $\mathfrak{I}_2$ and $\mathfrak{L}_1 \otimes C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes \mathfrak{L}^1 \otimes \cdots \otimes \mathfrak{L}^1$;

$$\mathfrak{I}_4 = C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes \mathfrak{L}^1 \otimes \cdots \otimes \mathfrak{L}^1,$$

$\mathfrak{I}_5$ is generated by $\mathfrak{I}_4$ and $\mathfrak{L}_1 \otimes \mathfrak{L}_1 \otimes C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes \mathfrak{L}^1 \otimes \cdots \otimes \mathfrak{L}^1$; and similarly, ..., and finally, $\mathfrak{I}_{2^m-1}$ is generated by

$$\begin{aligned} &C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes \cdots \otimes C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes \mathfrak{L}^1, \\ &\mathfrak{L}_1 \otimes C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes \cdots \otimes C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2), \\ &C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes \mathfrak{L}_1 \otimes C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes \cdots \otimes C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2), \end{aligned}$$

$$\ldots\ldots\ldots\ldots, \quad \text{and}$$
$$C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes \cdots \otimes C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \otimes \mathfrak{L}_1 \otimes C^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2). \qquad \square$$

Proposition 4.5 *The group C^*-algebra $C^*(G)$ of G the proper generalized discrete elementary Mautner group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2^m$ $(m \geq 2)$ has topological stable rank two and connected stable rank two.*

Proof. Using the stable rank formulae as used in Propositions 2.3 and 3.4 to the composition series obtained in Proposition 4.4, we estimate

$$\begin{aligned}
\mathrm{sr}(C^*(G)) &\geq \max_{1\leq k\leq m} \mathrm{sr}(\otimes^k (C_0(\mathbb{R}) \otimes M_2(\mathbb{C}))) \\
&= \max_{1\leq k\leq m} \mathrm{sr}(C_0(\mathbb{R}^k) \otimes M_{2^k}(\mathbb{C}))) \\
&= \max_{1\leq k\leq m} \lceil (\mathrm{sr}(C_0(\mathbb{R}^k)) - 1)2^{-k} \rceil + 1 \\
&= \max_{1\leq k\leq m} \lceil (\lfloor k/2 \rfloor) 2^{-k} \rceil + 1
\end{aligned}$$

and each component in the last maximum is equal to

$$\begin{cases} \lceil l/2^{2l} \rceil + 1 = 2 & \text{if } k = 2l, \\ \lceil l/2^{2l+1} \rceil + 1 = 2 & \text{if } k = 2l+1 \neq 1 \end{cases}$$

and is equal to 1 if $k = 1$. Also, we estimate

$$\begin{aligned}
\mathrm{sr}(C^*(G)) &\leq \max_{1\leq k\leq m} \{\mathrm{sr}(\otimes^k (C_0(\mathbb{R}) \otimes M_2(\mathbb{C}))), \mathrm{csr}(\otimes^k (C_0(\mathbb{R}) \otimes M_2(\mathbb{C})))\} \\
&\leq \max_{1\leq k\leq m} \{2, \mathrm{csr}(C_0(\mathbb{R}^k) \otimes M_{2^k}(\mathbb{C}))\} \\
&\leq \max_{1\leq k\leq m} \{2, \lceil (\mathrm{csr}(C_0(\mathbb{R}^k)) - 1)/2^k \rceil + 1\} \\
&\leq \max_{1\leq k\leq m} \{2, \lceil (\lfloor (k+1)/2 \rfloor)/2^k \rceil + 1\}
\end{aligned}$$

and each component in the last maximum is equal to

$$\begin{cases} \lceil l/2^{2l} \rceil + 1 = 2 & \text{if } k = 2l, \\ \lceil (l+1)/2^{2l+1} \rceil + 1 = 2 & \text{if } k = 2l+1. \end{cases}$$

Moreover, we obtain

$$\mathrm{csr}(C^*(G)) \leq \max_{1\leq k\leq m} \mathrm{csr}(\otimes^k(C_0(\mathbb{R}) \otimes M_2(\mathbb{C}))) \leq 2.$$

Since we have $K_1(C^*(G)) \neq 0$ by Theorem 4.3, we obtain $\mathrm{csr}(C^*(G)) \geq 2$. □

Remark it is shown in [11] that if G is the (non-proper) generalized discrete (elementary) Mautner group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m$, then

$$\lfloor m/2 \rfloor \leq \mathrm{sr}(C^*(G)) \leq \lfloor (m+1)/2 \rfloor + 1$$

and $\mathrm{csr}(C^*(G)) \leq \lfloor (m+1)/2 \rfloor + 1$ by the same way as given in the proof above. In addition, we obtain $\mathrm{csr}(C^*(G)) \geq 2$ since $K_1(C^*(G)) \neq 0$ as shown in Theorem 4.2. Consequently, we find out the difference in the topological stable rank for $C^*(G)$ of G the proper and the non-proper G, to have the topological stable rank lower or higher. Possibly, we can find out the same thing in the connected stable rank. As a fact, it is shown by [7, Corollary 4. 10] that

$$\mathrm{csr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1$$

for a C^*-algebra $\mathfrak{A}$, but what we need to show that thing is a sort of reverse inequality if any.

5. Products of the generalized discrete elementary $ax+b$ groups

Lemma 5.1 *The group C^*-algebras of finite direct products $\Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$ of the generalized discrete elementary $ax+b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$ and those of the proper $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$ are in the bootstrap category N.*

Proof. It follows from the finite composition series of closed ideals of the group C^*-algebra of the proper generalized discrete elementary $ax+b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$ stated in Proposition 3.2 that $C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$ is in the bootstrap category N. Since the group C^*-algebra of the generalized discrete elementary $ax+b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$ is viewed as the mapping torus on $C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$ as in Remark after Proposition 2.2, it also follows that $C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$ is in N.

Since we have

$$C^*(\Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})) \cong \otimes^l C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$$

the l-fold tensor product of the group C^*-algebra of $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$, it follows that $C^*(\Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}))$ is in N, indeed, which has a finite composition series of closed ideals with subquotients in N, by using the structure of $C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$ in N shown above.

Similarly, one can show that $C^*(\Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2))$ is in N. Indeed, see Proposition 5.4 below. □

Theorem 5.2 *Let $G = \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$ be a finite direct product of the generalized discrete elementary $ax+b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$. Then*

$$K_0(C^*(G)) \cong \mathbb{Z}^{s(l)} \oplus \mathbb{Z}_2^{t(l)},$$

$$K_1(C^*(G)) \cong \mathbb{Z}^{u(l)} \oplus \mathbb{Z}_2^{v(l)},$$

where the indexes $s(l), t(l), u(l), v(l) \in \mathbb{N}$ with $s(1) = 2^{m-1}$, $t(1) = 0$, $u(1) = 2^{m-1}$, and $v(1) = 2^{m-1}$ are determined inductively by

$$\begin{aligned} s(l+1) &= 2^{m-1}(s(l) + u(l)), \\ t(l+1) &= 2^{m-1}(2\,t(l) + u(l) + 2\,v(l)), \\ u(l+1) &= 2^{m-1}(s(l) + u(l)), \\ v(l+1) &= 2^{m-1}(2\,t(l) + s(l) + 2\,v(l)) \end{aligned}$$

$(m \geq 1)$. In other words, letting $G_l = \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$, we have

$$\begin{aligned} \mathrm{rank}_{\mathbb{Z}} K_j(C^*(G_{l+1})) &= 2^{m-1}[\mathrm{rank}_{\mathbb{Z}} K_0(C^*(G_l)) + \mathrm{rank}_{\mathbb{Z}} K_1(C^*(G_l))], \\ \mathrm{rank}_{\mathbb{Z}_2} K_0(C^*(G_{l+1})) &= 2^{m-1}[\mathrm{rank}_{\mathbb{Z}} K_1(C^*(G_l)) \\ &\quad + 2\,\mathrm{rank}_{\mathbb{Z}_2} K_0(C^*(G_l)) + 2\,\mathrm{rank}_{\mathbb{Z}_2} K_1(C^*(G_l))], \\ \mathrm{rank}_{\mathbb{Z}_2} K_1(C^*(G_{l+1})) &= 2^{m-1}[\mathrm{rank}_{\mathbb{Z}} K_0(C^*(G_l)) \\ &\quad + 2\,\mathrm{rank}_{\mathbb{Z}_2} K_0(C^*(G_l)) + 2\,\mathrm{rank}_{\mathbb{Z}_2} K_1(C^*(G_l))] \end{aligned}$$

$(j = 0, 1)$.

In particular,

$$K_0(C^*(G_2)) \cong \mathbb{Z}^{2^{2m-1}} \oplus \mathbb{Z}_2^{2^{2m-2}3}, \quad K_1(C^*(G_2)) \cong \mathbb{Z}^{2^{2m-1}} \oplus \mathbb{Z}_2^{2^{2m-2}3},$$

and also

$$K_0(C^*(G_3)) \cong \mathbb{Z}^{2^{3m-1}} \oplus \mathbb{Z}_2^{2^{3m-2}7}, \quad K_1(C^*(G_3)) \cong \mathbb{Z}^{2^{3m-1}} \oplus \mathbb{Z}_2^{2^{3m-2}7}.$$

In addition, it follows from the inductive equations above that

$$K_0(C^*(G_l)) \cong K_1(C^*(G_l))$$

if $l \geq 2$.

Moreover, it does follow that for $l \geq 2$,

$$K_j(C^*(G_l)) \cong \mathbb{Z}^{2^{lm-1}} \oplus \mathbb{Z}_2^{2^{(m+1)l-2}-2^{lm-2}}$$

for $j = 0, 1$.

Proof. Since $G \cong \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$, we have

$$C^*(G) \cong \otimes^l C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$$

the l-fold tensor product of the group C^*-algebra of $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$.

We first consider the case where $l = 2$. Let $H = \mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$. The Künneth theorem (see [2]) implies that since $C^*(H)$ is in N by Lemma 5.1, we have the following short exact sequence of abelian groups:

$$\begin{aligned} &0 \to K_*(C^*(H)) \otimes K_*(C^*(H)) \\ &\quad \xrightarrow{\beta} K_*(C^*(H) \otimes C^*(H)) \xrightarrow{\sigma} \mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(H)), K_*(C^*(H))) \to 0 \end{aligned}$$

where $K_*(\cdot) = K_0(\cdot) \oplus K_1(\cdot)$ and the map β has degree 0 and the map σ has degree 1 and the short exact sequence splits unnaturally. As obtained in Theorem 3.1, we have

$$K_*(C^*(H)) = K_0(C^*(H)) \oplus K_1(C^*(H)) \cong \mathbb{Z}^{2^{m-1}} \oplus (\mathbb{Z}^{2^{m-1}} \times \mathbb{Z}_2^{2^{m-1}}).$$

By using several facts in homology theory as in [4], we compute the torsion product as follows:

$$\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(H)), K_*(C^*(H))) \\
&\quad \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{2^{m-1}} \oplus (\mathbb{Z}^{2^{m-1}} \times \mathbb{Z}_2^{2^{m-1}}), K_*(C^*(H))) \\
&\quad \cong [\oplus^{2^m} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, K_*(C^*(H)))] \oplus [\oplus^{2^{m-1}} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, K_*(C^*(H)))] \\
&\quad \cong \oplus^{2^{m-1}} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}^{2^{m-1}} \oplus (\mathbb{Z}^{2^{m-1}} \times \mathbb{Z}_2^{2^{m-1}})) \\
&\quad \cong \oplus^{2^{m-1}}(\oplus^{2^{m-1}} \mathbb{Z}_2) \\
&\quad \cong \oplus^{2^{2m-2}} \mathbb{Z}_2
\end{aligned}$$

where note that this consequence comes from the pair $(K_1(C^*(H)), K_1(C^*(H)))$, so that the torsion product is in $K_1(C^*(H \times H))$.

Therefore, it follows that

$$K_*(C^*(H) \otimes C^*(H)) \cong [K_*(C^*(H)) \otimes K_*(C^*(H))] \oplus (\oplus^{2^{2m-2}} \mathbb{Z}_2)$$

(unnaturally). Moreover, we obtain

$$\begin{aligned}
&K_0(C^*(H \times H)) \\
&\quad \cong (K_0(C^*(H)) \otimes K_0(C^*(H))) \oplus (K_1(C^*(H)) \otimes K_1(C^*(H))) \\
&\quad \cong (\mathbb{Z}^{2^{m-1}} \otimes \mathbb{Z}^{2^{m-1}}) \oplus ((\mathbb{Z}^{2^{m-1}} \times \mathbb{Z}_2^{2^{m-1}}) \otimes (\mathbb{Z}^{2^{m-1}} \times \mathbb{Z}_2^{2^{m-1}})) \\
&\quad \cong (\oplus^{2^{2m-2}} \mathbb{Z}) \oplus (\oplus^{2^{2m-2}} \mathbb{Z}) \oplus (\oplus^3(\oplus^{2^{2m-2}} \mathbb{Z}_2)) \\
&\quad \cong (\oplus^{2^{2m-1}} \mathbb{Z}) \oplus (\oplus^{2^{2m-2}3} \mathbb{Z}_2).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&K_1(C^*(H \times H))/(\oplus^{2^{2m-2}} \mathbb{Z}_2) \\
&\quad \cong (K_0(C^*(H)) \otimes K_1(C^*(H))) \oplus (K_1(C^*(H)) \otimes K_0(C^*(H))) \\
&\quad \cong \oplus^2(\mathbb{Z}^{2^{m-1}} \otimes (\mathbb{Z}^{2^{m-1}} \times \mathbb{Z}_2^{2^{m-1}})) \\
&\quad \cong \oplus^2(\mathbb{Z}^{2^{2m-2}} \oplus \mathbb{Z}_2^{2^{2m-2}}) \\
&\quad \cong (\oplus^{2^{2m-1}} \mathbb{Z}) \oplus (\oplus^{2^{2m-1}} \mathbb{Z}_2)
\end{aligned}$$

and hence, $K_1(C^*(H\times H))\cong \mathbb{Z}^{2^{2m-1}}\oplus \mathbb{Z}_2^{2^{2m-2}3}$.

Repeating the same argument for $C^*(\Pi^3 H)\cong C^*(H\times H)\otimes C^*(H)$, we compute

$$\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(H\times H)), K_*(C^*(H)))\\
&\quad\cong \mathrm{Tor}_1^{\mathbb{Z}}((\mathbb{Z}^{2^{2m-1}}\times \mathbb{Z}_2^{2^{2m-2}3})\oplus(\mathbb{Z}^{2^{2m-1}}\times \mathbb{Z}_2^{2^{2m-2}3}), K_*(C^*(H)))\\
&\quad\cong [\oplus^{2^{2m-2}3}\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}^{2^{m-1}}\oplus(\mathbb{Z}^{2^{m-1}}\times \mathbb{Z}_2^{2^{m-1}}))]\\
&\qquad\oplus [\oplus^{2^{2m-2}3}\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}^{2^{m-1}}\oplus(\mathbb{Z}^{2^{m-1}}\times \mathbb{Z}_2^{2^{m-1}}))]\\
&\quad\cong [\oplus^{2^{2m-2}3}(\oplus^{2^{m-1}}\mathbb{Z}_2)]\oplus[\oplus^{2^{2m-2}3}(\oplus^{2^{m-1}}\mathbb{Z}_2)]\\
&\quad\cong [\oplus^{2^{3m-3}3}\mathbb{Z}_2]\oplus[\oplus^{2^{3m-3}3}\mathbb{Z}_2]
\end{aligned}$$

where the first summand corresponds to the pair $(K_0(C^*(H\times H)), K_1(C^*(H)))$ and the second summand corresponds to the pair $(K_1(C^*(H\times H)), K_1(C^*(H)))$. Therefore, we obtain

$$\begin{aligned}
&K_0(C^*(\Pi^3 H))/\mathbb{Z}_2^{2^{3m-3}3}\\
&\quad\cong [(\mathbb{Z}^{2^{2m-1}}\oplus \mathbb{Z}_2^{2^{2m-2}3})\otimes \mathbb{Z}^{2^{m-1}}]\\
&\qquad\oplus [(\mathbb{Z}^{2^{2m-1}}\oplus \mathbb{Z}_2^{2^{2m-2}3})\otimes(\mathbb{Z}^{2^{m-1}}\times \mathbb{Z}_2^{2^{m-1}})]\\
&\quad\cong [\mathbb{Z}^{2^{3m-2}}\oplus \mathbb{Z}_2^{2^{3m-3}3}]\oplus[\mathbb{Z}^{2^{3m-2}}\oplus \mathbb{Z}_2^{2^{3m-2}}\oplus \mathbb{Z}_2^{2^{3m-3}3}\oplus \mathbb{Z}_2^{2^{3m-3}3}]\\
&\quad\cong \mathbb{Z}^{2^{3m-1}}\oplus \mathbb{Z}_2^{2^{3m-3}11},
\end{aligned}$$

and thus, $K_0(C^*(\Pi^3 H))\cong \mathbb{Z}^{2^{3m-1}}\oplus \mathbb{Z}_2^{2^{3m-2}7}$, and also

$$\begin{aligned}
&K_1(C^*(\Pi^3 H))/\mathbb{Z}_2^{2^{3m-3}3}\\
&\quad\cong [(\mathbb{Z}^{2^{2m-1}}\oplus \mathbb{Z}_2^{2^{2m-2}3})\otimes(\mathbb{Z}^{2^{m-1}}\times \mathbb{Z}_2^{2^{m-1}})]\\
&\qquad\oplus [(\mathbb{Z}^{2^{2m-1}}\oplus \mathbb{Z}_2^{2^{2m-2}3})\otimes \mathbb{Z}^{2^{m-1}}]\\
&\quad\cong [\mathbb{Z}^{2^{3m-2}}\oplus \mathbb{Z}_2^{2^{3m-2}}\oplus \mathbb{Z}_2^{2^{3m-3}3}\oplus \mathbb{Z}_2^{2^{3m-3}3}]\oplus[\mathbb{Z}^{2^{3m-2}}\oplus \mathbb{Z}_2^{2^{3m-3}3}]\\
&\quad\cong \mathbb{Z}^{2^{3m-1}}\oplus \mathbb{Z}_2^{2^{3m-3}11},
\end{aligned}$$

and thus, $K_1(C^*(\Pi^3 H)) \cong \mathbb{Z}^{2^{3m-1}} \oplus \mathbb{Z}_2^{2^{3m-2}7}$.

By induction, for $G = \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$, we may assume that

$$K_0(C^*(G)) \cong \mathbb{Z}^{s(l)} \oplus \mathbb{Z}_2^{t(l)}, \quad K_1(C^*(G)) \cong \mathbb{Z}^{u(l)} \oplus \mathbb{Z}_2^{v(l)},$$

for some $s(l), t(l), u(l), v(l) \in \mathbb{N}$. Then

$$\mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(G)), K_*(C^*(H))) = \mathbb{Z}_2^{2^{m-1}t(l)} \oplus \mathbb{Z}_2^{2^{m-1}v(l)},$$

where the first summand corresponds to the pair $(K_0(C^*(G)), K_1(C^*(H)))$ and the second summand corresponds to the pair $(K_1(C^*(G)), K_1(C^*(H)))$. Therefore,

$$\begin{aligned} &K_0(C^*(G \times H))/\mathbb{Z}_2^{2^{m-1}t(l)} \\ &\quad \cong [(\mathbb{Z}^{s(l)} \oplus \mathbb{Z}_2^{t(l)}) \otimes \mathbb{Z}^{2^{m-1}}] \oplus [(\mathbb{Z}^{u(l)} \oplus \mathbb{Z}_2^{v(l)}) \otimes (\mathbb{Z}^{2^{m-1}} \times \mathbb{Z}_2^{2^{m-1}})] \\ &\quad \cong \mathbb{Z}^{2^{m-1}(s(l)+u(l))} \oplus \mathbb{Z}_2^{2^{m-1}(t(l)+u(l)+2\,v(l))}, \end{aligned}$$

and

$$\begin{aligned} &K_1(C^*(G \times H))/\mathbb{Z}_2^{2^{m-1}v(l)} \\ &\quad \cong [(\mathbb{Z}^{u(l)} \oplus \mathbb{Z}_2^{v(l)}) \otimes \mathbb{Z}^{2^{m-1}}] \oplus [(\mathbb{Z}^{s(l)} \oplus \mathbb{Z}_2^{t(l)}) \otimes (\mathbb{Z}^{2^{m-1}} \times \mathbb{Z}_2^{2^{m-1}})] \\ &\quad \cong \mathbb{Z}^{2^{m-1}(s(l)+u(l))} \oplus \mathbb{Z}_2^{2^{m-1}(2\,t(l)+s(l)+v(l))}. \end{aligned}$$

Hence we get

$$K_0(C^*(\Pi^{l+1}(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}))) \cong \mathbb{Z}^{2^{m-1}(s(l)+u(l))} \oplus \mathbb{Z}_2^{2^{m-1}(2\,t(l)+u(l)+2\,v(l))},$$

$$K_1(C^*(\Pi^{l+1}(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}))) \cong \mathbb{Z}^{2^{m-1}(s(l)+u(l))} \oplus \mathbb{Z}_2^{2^{m-1}(2\,t(l)+s(l)+2\,v(l))}.$$

It then follows that

$$s(l+1) = u(l+1) = 2^{m-1}(s(l) + u(l)) \quad \text{for } l \geq 1 \text{ and}$$
$$t(l+1) = v(l+1) = 2^{m-1}(2\,t(l) + s(l) + 2\,v(l)) \quad \text{for } l \geq 1$$

since $s(1) = u(1) = 2^{m-1}$. Therefore, we obtain that $K_0(C^*(\Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}))) \cong K_1(C^*(\Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})))$ for $l \geq 2$.

Moreover, it follows from the first inductive equation that $s(l) = u(l) = 2^{lm-1}$ for $l \geq 1$. It then follows from the second inductive equation that $t(l+1) = 2^{m-1}(4\,t(l) + 2^{lm-1})$ for $l \geq 2$. Dividing both sides by the power $2^{(l+1)m-1}$ of 2 yields the following:

$$\frac{t(l+1)}{2^{(l+1)m-1}} = 2 \cdot \left(\frac{t(l)}{2^{lm-1}}\right) + \frac{1}{2}.$$

Now put $c(l) = t(l)/2^{lm-1}$ for $l \geq 2$. Then $c(l+1) = 2\,c(l) + 1/2$. This equation is transposed to the following: $c(l+1)+1/2 = 2(c(l)+1/2)$. Thus, the general term is given by $c(l) + 1/2 = 2^{l-2}(c(2) + 1/2)$ with

$$c(2) = \frac{t(2)}{2^{2m-1}} = \frac{2^{m-1}(2\,t(1) + u(1) + 2\,v(1))}{2^{2m-1}} = \frac{3}{2}.$$

Hence $c(l) = 2^{l-1} - 1/2$. Therefore, we get $t(l) = v(l) = 2^{(m+1)l-2} - 2^{lm-2}$ for $l \geq 2$. □

Theorem 5.3 *Let $G = \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$ be a finite direct product of the proper generalized discrete elementary $ax + b$ group. Then*

$$K_0(C^*(G)) \cong \mathbb{Z}^{s(l)} \oplus \mathbb{Z}_2^{t(l)},$$
$$K_1(C^*(G)) \cong \mathbb{Z}_2^{v(l)},$$

where the indexes $s(l), t(l), v(l) \in \mathbb{N}$ with $s(1) = 2^m + 1$, $t(1) = 0$, and $v(1) = 2^{m-1}$ are determined inductively by

$$s(l+1) = (2^m + 1)s(l),$$
$$t(l+1) = (2^{m-1}3 + 1)t(l) + 2^{m-1}v(l),$$
$$v(l+1) = (2^{m-1}3 + 1)v(l) + 2^{m-1}(s(l) + t(l))$$

$(l \geq 1)$. In other words, letting $G_l = \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$, we have

$$\mathrm{rank}_{\mathbb{Z}} K_0(C^*(G_{l+1})) = (2^m + 1)\,\mathrm{rank}_{\mathbb{Z}} K_0(C^*(G_l)),$$

$$\begin{aligned}\operatorname{rank}_{\mathbb{Z}_2} K_0(C^*(G_{l+1})) &= (2^{m-1}3+1)\operatorname{rank}_{\mathbb{Z}_2} K_0(C^*(G_l)) \\ &\quad + 2^{m-1}\operatorname{rank}_{\mathbb{Z}_2} K_1(C^*(G_l)), \\ \operatorname{rank}_{\mathbb{Z}_2} K_1(C^*(G_{l+1})) &= (2^{m-1}3+1)\operatorname{rank}_{\mathbb{Z}_2} K_1(C^*(G_l)) \\ &\quad + 2^{m-1}[\operatorname{rank}_{\mathbb{Z}} K_0(C^*(G_l)) + \operatorname{rank}_{\mathbb{Z}_2} K_0(C^*(G_l))].\end{aligned}$$

In particular,

$$K_0(C^*(G_2)) \cong \mathbb{Z}^{2^{2m}+2^{m+1}+1} \oplus \mathbb{Z}_2^{2^{2m-2}}, \quad K_1(C^*(G_2)) \cong \mathbb{Z}_2^{2^{2m-2}5+2^m},$$

and also

$$\begin{aligned} K_0(C^*(G_3)) &\cong \mathbb{Z}^{2^{3m}+2^{2m}3+2^m3+1} \oplus \mathbb{Z}_2^{2^{3m}+2^{2m-2}3}, \\ K_1(C^*(G_3)) &\cong \mathbb{Z}_2^{2^{3m-1}5+2^{2m-2}15+2^{m-1}3}. \end{aligned}$$

Moreover, it does follow that if $l \geq 2$,

$$\begin{aligned} K_0(C^*(G_l)) &\cong \mathbb{Z}^{(2^m+1)^l} \oplus \mathbb{Z}_2^{2^{-2}[(2^{m+1}+1)^l-(2^m+1)^{l-1}(2^m(l+1)+1)]}, \\ K_1(C^*(G_l)) &\cong \mathbb{Z}_2^{2^{-2}[(2^{m+1}+1)^l+(2^m+1)^{l-1}(2^m(l-1)-1)]} \end{aligned}$$

and this also holds for the case where $l = 1$.

Proof. Since $G \cong \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$, we have

$$C^*(G) \cong \otimes^l C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$$

the l-fold tensor product of the group C^*-algebra of $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$.

We first consider the case where $l = 2$. Let $H = \mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$. The Künneth theorem (see [2]) implies that since $C^*(H)$ is in N by Lemma 5.1, we have the following short exact sequence of abelian groups:

$$\begin{aligned} 0 \to K_*(C^*(H)) \otimes K_*(C^*(H)) \\ \xrightarrow{\beta} K_*(C^*(H) \otimes C^*(H)) \xrightarrow{\sigma} \operatorname{Tor}_1^{\mathbb{Z}}(K_*(C^*(H)), K_*(C^*(H))) \to 0 \end{aligned}$$

where $K_*(\cdot) = K_0(\cdot) \oplus K_1(\cdot)$ and the map β has degree 0 and the map σ

has degree 1 and the short exact sequence splits unnaturally. As obtained in Theorem 3.3, we have

$$K_*(C^*(H)) = K_0(C^*(H)) \oplus K_1(C^*(H)) \cong \mathbb{Z}^{2^m+1} \oplus \mathbb{Z}_2^{2^{m-1}}.$$

By using several facts in homology theory as in [4], we compute the torsion product as follows:

$$\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(H)), K_*(C^*(H))) \\
&\quad \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{2^m+1} \oplus \mathbb{Z}_2^{2^{m-1}}, K_*(C^*(H))) \\
&\quad \cong [\oplus^{2^m+1}\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, K_*(C^*(H)))] \oplus [\oplus^{2^{m-1}}\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, K_*(C^*(H)))] \\
&\quad \cong \oplus^{2^{m-1}}\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}^{2^m+1} \oplus \mathbb{Z}_2^{2^{m-1}}) \\
&\quad \cong \oplus^{2^{m-1}}(\oplus^{2^{m-1}}\mathbb{Z}_2) \\
&\quad \cong \oplus^{2^{2m-2}}\mathbb{Z}_2
\end{aligned}$$

where note that this consequence comes from the pair $(K_1(C^*(H)), K_1(C^*(H)))$, so that the torsion product is in $K_1(C^*(H \times H))$.

Therefore, it follows that

$$K_*(C^*(H) \otimes C^*(H)) \cong [K_*(C^*(H)) \otimes K_*(C^*(H))] \oplus (\oplus^{2^{2m-2}}\mathbb{Z}_2)$$

(unnaturally). Moreover, we obtain

$$\begin{aligned}
&K_0(C^*(H \times H)) \\
&\quad \cong (K_0(C^*(H)) \otimes K_0(C^*(H))) \oplus (K_1(C^*(H)) \otimes K_1(C^*(H))) \\
&\quad \cong (\mathbb{Z}^{2^m+1} \otimes \mathbb{Z}^{2^m+1}) \oplus (\mathbb{Z}_2^{2^{m-1}} \otimes \mathbb{Z}_2^{2^{m-1}}) \\
&\quad \cong (\oplus^{2^{2m}+2^{m+1}+1}\mathbb{Z}) \oplus (\oplus^{2^{2m-2}}\mathbb{Z}_2).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&K_1(C^*(H \times H))/(\oplus^{2^{2m-2}}\mathbb{Z}_2) \\
&\quad \cong (K_0(C^*(H)) \otimes K_1(C^*(H))) \oplus (K_1(C^*(H)) \otimes K_0(C^*(H)))
\end{aligned}$$

$$\cong \oplus^2(\mathbb{Z}^{2^m+1} \otimes \mathbb{Z}_2^{2^{m-1}})$$
$$\cong \oplus^2(\mathbb{Z}_2^{2^{2m-1}+2^{m-1}})$$
$$\cong \oplus^{2^{2m}+2^m}\mathbb{Z}_2$$

and hence, $K_1(C^*(H \times H)) \cong \mathbb{Z}_2^{2^{2m-2}5+2^m}$.

Repeating the same argument for $C^*(\Pi^3 H) \cong C^*(H \times H) \otimes C^*(H)$, we compute

$$\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(H \times H)), K_*(C^*(H))) \\
&\quad \cong \mathrm{Tor}_1^{\mathbb{Z}}((\mathbb{Z}^{2^{2m}+2^{m+1}+1} \times \mathbb{Z}_2^{2^{2m-2}}) \oplus \mathbb{Z}_2^{2^{2m-2}5+2^m}), K_*(C^*(H))) \\
&\quad \cong [\oplus^{2^{2m-2}}\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}^{2^m+1} \oplus \mathbb{Z}_2^{2^{m-1}})] \\
&\qquad \oplus [\oplus^{2^{2m-2}5+2^m}\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}^{2^m+1} \oplus \mathbb{Z}_2^{2^{m-1}})] \\
&\quad \cong [\oplus^{2^{2m-2}}(\oplus^{2^{m-1}}\mathbb{Z}_2)] \oplus [\oplus^{2^{2m-2}5+2^m}(\oplus^{2^{m-1}}\mathbb{Z}_2)] \\
&\quad \cong [\oplus^{2^{3m-3}}\mathbb{Z}_2] \oplus [\oplus^{2^{3m-3}5+2^{2m-1}}\mathbb{Z}_2]
\end{aligned}$$

where the first summand corresponds to the pair $(K_0(C^*(H \times H)), K_1(C^*(H)))$ and the second summand corresponds to the pair $(K_1(C^*(H \times H)), K_1(C^*(H)))$. Therefore, we obtain

$$\begin{aligned}
&K_0(C^*(\Pi^3 H))/\mathbb{Z}_2^{2^{3m-3}} \\
&\quad \cong [(\mathbb{Z}^{2^{2m}+2^{m+1}+1} \oplus \mathbb{Z}_2^{2^{2m-2}}) \otimes \mathbb{Z}^{2^m+1}] \oplus [\mathbb{Z}_2^{2^{2m-2}5+2^m} \otimes \mathbb{Z}_2^{2^{m-1}}] \\
&\quad \cong [\mathbb{Z}^{2^{3m}+2^{2m}3+2^m3+1} \oplus \mathbb{Z}_2^{2^{3m-2}+2^{2m-2}}] \oplus [\mathbb{Z}_2^{2^{3m-3}5+2^{2m-1}}] \\
&\quad \cong \mathbb{Z}^{2^{3m}+2^{2m}3+2^m3+1} \oplus \mathbb{Z}_2^{2^{3m-3}7+2^{2m-2}3}
\end{aligned}$$

and thus, $K_0(C^*(\Pi^3 H)) \cong \mathbb{Z}^{2^{3m}+2^{2m}3+2^m3+1} \oplus \mathbb{Z}_2^{2^{3m}+2^{2m-2}3}$, and also

$$\begin{aligned}
&K_1(C^*(\Pi^3 H))/\mathbb{Z}_2^{2^{3m-3}5+2^{2m-1}} \\
&\quad \cong [(\mathbb{Z}^{2^{2m}+2^{m+1}+1} \oplus \mathbb{Z}_2^{2^{2m-2}}) \otimes \ \mathbb{Z}_2^{2^{m-1}}] \oplus [\mathbb{Z}_2^{2^{2m-2}5+2^m} \otimes \mathbb{Z}^{2^m+1}] \\
&\quad \cong [\mathbb{Z}_2^{2^{3m-1}+2^{2m}+2^{m-1}} \oplus \mathbb{Z}_2^{2^{3m-3}}] \oplus [\mathbb{Z}_2^{2^{3m-2}5+2^{2m-2}9+2^m}] \\
&\quad \cong \mathbb{Z}_2^{2^{3m-3}15+2^{2m-2}13+2^{m-1}3}
\end{aligned}$$

and hence, $K_1(C^*(\Pi^3 H)) \cong \mathbb{Z}_2^{2^{3m-1}5+2^{2m-2}13+2^{m-1}3}$.

By induction, for $G = \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$ with $l \geq 2$, we may assume that

$$K_0(C^*(G)) \cong \mathbb{Z}^{s(l)} \oplus \mathbb{Z}_2^{t(l)}, \quad K_1(C^*(G)) \cong \mathbb{Z}_2^{v(l)},$$

for some $s(l), t(l), v(l) \in \mathbb{N}$. Then

$$\mathrm{Tor}_1^{\mathbb{Z}}(K_*(C^*(G)), K_*(C^*(H))) = \mathbb{Z}_2^{2^{m-1}t(l)} \oplus \mathbb{Z}_2^{2^{m-1}v(l)},$$

where the first summand corresponds to the pair $(K_0(C^*(G)), K_1(C^*(H)))$ and the second summand corresponds to the pair $(K_1(C^*(G)), K_1(C^*(H)))$. Therefore, we have

$$\begin{aligned} &K_0(C^*(G \times H))/\mathbb{Z}_2^{2^{m-1}t(l)} \\ &\quad \cong [(\mathbb{Z}^{s(l)} \oplus \mathbb{Z}_2^{t(l)}) \otimes \mathbb{Z}^{2^m+1}] \oplus [\mathbb{Z}_2^{v(l)} \otimes \mathbb{Z}_2^{2^{m-1}}] \\ &\quad \cong \mathbb{Z}^{(2^m+1)s(l)} \oplus \mathbb{Z}_2^{(2^m+1)t(l)+2^{m-1}v(l)}, \end{aligned}$$

and also

$$\begin{aligned} &K_1(C^*(G \times H))/\mathbb{Z}_2^{2^{m-1}v(l)} \\ &\quad \cong [\mathbb{Z}_2^{v(l)} \otimes \mathbb{Z}^{2^m+1}] \oplus [(\mathbb{Z}^{s(l)} \oplus \mathbb{Z}_2^{t(l)}) \otimes \mathbb{Z}_2^{2^{m-1}}] \\ &\quad \cong \mathbb{Z}_2^{(2^m+1)v(l)+2^{m-1}(s(l)+t(l))}. \end{aligned}$$

Hence we get

$$\begin{aligned} K_0(C^*(\Pi^{l+1}(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}))) &\cong \mathbb{Z}^{(2^m+1)s(l)} \oplus \mathbb{Z}_2^{(2^{m-1}3+1)t(l)+2^{m-1}v(l)}, \\ K_1(C^*(\Pi^{l+1}(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}))) &\cong \mathbb{Z}_2^{(2^{m-1}3+1)v(l)+2^{m-1}(s(l)+t(l))}. \end{aligned}$$

Moreover, it follows from the inductive equation $s(l+1) = (2^m+1)s(l)$ for $l \geq 1$ with $s(1) = 2^m + 1$ that $s(l) = (2^m+1)^l$ for $l \geq 1$. The indexes $t(l+1)$ and $v(l+1)$ are viewed as the vector $X(l+1)$ in the following equation with matrix multiplication:

$$\begin{aligned} X(l+1) &\equiv \begin{pmatrix} t(l+1) \\ v(l+1) \end{pmatrix} \\ &= \begin{pmatrix} 2^{m-1}3+1 & 2^{m-1} \\ 2^{m-1} & 2^{m-1}3+1 \end{pmatrix} \begin{pmatrix} t(l) \\ v(l) \end{pmatrix} + \begin{pmatrix} 0 \\ 2^{m-1}(2^m+1)^l \end{pmatrix} \\ &\equiv MX(l) + Y(l) \end{aligned}$$

$(l \geq 1)$. Inductively, it follows that

$$\begin{aligned} X(l) &= MX(l-1) + Y(l-1) \\ &= M(MX(l-2) + Y(l-2)) + Y(l-1) \\ &= M^2X(l-2) + MY(l-2) + Y(l-1) \\ &= \cdots\cdots\cdots \\ &= M^{l-1}X(1) + M^{l-2}Y(1) + \cdots + MY(l-2) + Y(l-1). \end{aligned}$$

Since the matrices $D = \left(\begin{smallmatrix} 2^{m-1}3+1 & 0 \\ 0 & 2^{m-1}3+1 \end{smallmatrix}\right)$, $F = \left(\begin{smallmatrix} 0 & 2^{m-1} \\ 2^{m-1} & 0 \end{smallmatrix}\right)$ commute, we compute the matrix product M^l by binary expansion:

$$\begin{aligned} M^l &= (D+F)^l \\ &= D^l + \binom{l}{1} D^{l-1}F + \cdots + \binom{l}{k} D^{l-k}F^k + \cdots + \binom{l}{l} F^l. \end{aligned}$$

Since we have

$$D^k = \begin{pmatrix} (2^{m-1}3+1)^k & 0 \\ 0 & (2^{m-1}3+1)^k \end{pmatrix}$$

and

$$F^k = \begin{cases} \begin{pmatrix} 2^{km-k} & 0 \\ 0 & 2^{km-k} \end{pmatrix} & \text{if } k \text{ is even,} \\ \begin{pmatrix} 0 & 2^{km-k} \\ 2^{km-k} & 0 \end{pmatrix} & \text{if } k \text{ is odd,} \end{cases}$$

if l is even, then the components $(M^l)_{ij}$ of M^l $(i = 1, 2, j = 1, 2)$ are given by

$$
\begin{aligned}
(M^l)_{11} &= (M^l)_{22} \\
&= (2^{m-1}3+1)^l + \binom{l}{2}(2^{m-1}3+1)^{l-2}2^{2m-2} + \cdots + \binom{l}{l}2^{lm-l} \\
&= \frac{1}{2}[(2^{m-1}3+1+2^{m-1})^l + (2^{m-1}3+1-2^{m-1})^l] \\
&= \frac{1}{2}[(2^{m+1}+1)^l + (2^m+1)^l],
\end{aligned}
$$

$$
\begin{aligned}
(M^l)_{12} &= (M^l)_{21} \\
&= \binom{l}{1}(2^{m-1}3+1)^{l-1}2^{m-1} + \binom{l}{3}(2^{m-1}3+1)^{l-3}2^{3m-3} \\
&\quad + \cdots + \binom{l}{l-1}(2^{m-1}3+1)2^{(l-1)m-l+1} \\
&= \frac{1}{2}[(2^{m-1}3+1+2^{m-1})^l - (2^{m-1}3+1-2^{m-1})^l] \\
&= \frac{1}{2}[(2^{m+1}+1)^l - (2^m+1)^l].
\end{aligned}
$$

Similarly, if l is odd, then the components $(M^l)_{ij}$ of M^l $(i = 1, 2, j = 1, 2)$ are given by

$$
\begin{aligned}
(M^l)_{11} &= (M^l)_{22} \\
&= (2^{m-1}3+1)^l + \binom{l}{2}(2^{m-1}3+1)^{l-2}2^{2m-2} \\
&\quad + \cdots + \binom{l}{l-1}(2^{m-1}3+1)2^{(l-1)m-l+1} \\
&= \frac{1}{2}[(2^{m-1}3+1+2^{m-1})^l + (2^{m-1}3+1-2^{m-1})^l] \\
&= \frac{1}{2}[(2^{m+1}+1)^l + (2^m+1)^l],
\end{aligned}
$$

$$\begin{aligned}(M^l)_{12} &= (M^l)_{21} \\ &= \binom{l}{1}(2^{m-1}3+1)^{l-1}2^{m-1} + \binom{l}{3}(2^{m-1}3+1)^{l-3}2^{3m-3} \\ &\quad + \cdots + \binom{l}{l}2^{lm-l} \\ &= \frac{1}{2}[(2^{m-1}3+1+2^{m-1})^l - (2^{m-1}3+1-2^{m-1})^l] \\ &= \frac{1}{2}[(2^{m+1}+1)^l - (2^m+1)^l].\end{aligned}$$

Note that as suggested by the referee, one may use linear algebra theory to compute the product M^l to be diagonalized as Jordan normal form by an invertible matrix. Indeed, the eigenvalues λ of the 2×2 matrix M are given by $2^m + 1$ and $2^{m+1} + 1$ by computing the determinant $\left|\begin{smallmatrix} 2^{m-1}3+1-\lambda & 2^{m-1} \\ 2^{m-1} & 2^{m-1}3+1-\lambda \end{smallmatrix}\right| = 0$, and the corresponding eigenvectors are given by $\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ respectively. It follows that

$$P^{-1}MP \equiv \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} M \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2^m+1 & 0 \\ 0 & 2^{m+1}+1 \end{pmatrix},$$

i.e., M is diagonalizable, and hence, we have

$$\begin{aligned}M^l &= P\begin{pmatrix} (2^m+1)^l & 0 \\ 0 & (2^{m+1}+1)^l \end{pmatrix} P^{-1} \\ &= \frac{1}{2}\begin{pmatrix} (2^{m+1}+1)^l + (2^m+1)^l & (2^{m+1}+1)^l - (2^m+1)^l \\ (2^{m+1}+1)^l - (2^m+1)^l & (2^{m+1}+1)^l + (2^m+1)^l \end{pmatrix}.\end{aligned}$$

Therefore, we obtain that for $l \geq 2$,

$$\begin{aligned}t(l) &= 2^{m-2}[(2^{m+1}+1)^{l-1} - (2^m+1)^{l-1}] \\ &\quad + 2^{m-2}[(2^{m+1}+1)^{l-2} - (2^m+1)^{l-2}](2^m+1) \\ &\quad + \cdots + 2^{m-2}[(2^{m+1}+1) - (2^m+1)](2^m+1)^{l-2} \\ &= 2^{m-2}[(2^{m+1}+1)^{l-1} - (2^m+1)^{l-1}] \\ &\quad + 2^{m-2}\sum_{k=1}^{l-2}[(2^{m+1}+1)^{l-1-k} - (2^m+1)^{l-1-k}](2^m+1)^k\end{aligned}$$

$$= 2^{m-2}[(2^{m+1}+1)^{l-1} - (2^m+1)^{l-1}]$$
$$+ 2^{m-2}\sum_{k=1}^{l-2}(2^{m+1}+1)^{l-1-k}(2^m+1)^k - 2^{m-2}(l-2)(2^m+1)^{l-1}$$

and

$$\begin{aligned} v(l) &= 2^{m-2}[(2^{m+1}+1)^{l-1} + (2^m+1)^{l-1}] \\ &\quad + 2^{m-2}[(2^{m+1}+1)^{l-2} + (2^m+1)^{l-2}](2^m+1) \\ &\quad + \cdots + 2^{m-2}[(2^{m+1}+1) + (2^m+1)](2^m+1)^{l-2} + 2^{m-1}(2^m+1)^{l-1} \\ &= 2^{m-2}[(2^{m+1}+1)^{l-1} + (2^m+1)^{l-1}] + 2^{m-1}(2^m+1)^{l-1} \\ &\quad + 2^{m-2}\sum_{k=1}^{l-2}[(2^{m+1}+1)^{l-1-k} + (2^m+1)^{l-1-k}](2^m+1)^k \\ &= 2^{m-2}[(2^{m+1}+1)^{l-1} + (2^m+1)^{l-1}] + 2^{m-1}(2^m+1)^{l-1} \\ &\quad + 2^{m-2}\sum_{k=1}^{l-2}(2^{m+1}+1)^{l-1-k}(2^m+1)^k + 2^{m-2}(l-2)(2^m+1)^{l-1}. \end{aligned}$$

Furthermore, we now put

$$S_l = \sum_{k=1}^{l-2}(2^{m+1}+1)^{l-1-k}(2^m+1)^k.$$

Then we have

$$S_l - \frac{2^m+1}{2^{m+1}+1}\, S_l = (2^{m+1}+1)^{l-2}(2^m+1) - (2^m+1)^{l-1}.$$

Hence,

$$\begin{aligned} S_l &= \frac{2^{m+1}+1}{2^m}[(2^{m+1}+1)^{l-2}(2^m+1) - (2^m+1)^{l-1}] \\ &= (2^{m+1}+1)^{l-1}\frac{2^m+1}{2^m} - \frac{2^{m+1}+1}{2^m}(2^m+1)^{l-1}. \end{aligned}$$

Therefore, we finally get

$$\begin{aligned} t(l) &= 2^{m-2}[(2^{m+1}+1)^{l-1} - (2^m+1)^{l-1}] \\ &\quad + (2^{m+1}+1)^{l-1}\left(\frac{2^m+1}{2^2}\right) - \frac{2^{m+1}+1}{2^2}(2^m+1)^{l-1} \\ &\quad - 2^{m-2}(l-2)(2^m+1)^{l-1} \\ &= (2^{m+1}+1)^{l-1}\frac{2^{m+1}+1}{2^2} - (2^m+1)^{l-1}\frac{2^m(l+1)+1}{2^2} \\ &= 2^{-2}[(2^{m+1}+1)^l - (2^m+1)^{l-1}(2^m(l+1)+1)], \\ v(l) &= 2^{m-2}[(2^{m+1}+1)^{l-1} + (2^m+1)^{l-1}] + 2^{m-1}(2^m+1)^{l-1} \\ &\quad + (2^{m+1}+1)^{l-1}\left(\frac{2^m+1}{2^2}\right) - \frac{2^{m+1}+1}{2^2}(2^m+1)^{l-1} \\ &\quad + 2^{m-2}(l-2)(2^m+1)^{l-1} \\ &= (2^{m+1}+1)^{l-1}\frac{2^{m+1}+1}{2^2} + (2^m+1)^{l-1}\frac{2^m(l-1)-1}{2^2} \\ &= 2^{-2}[(2^{m+1}+1)^l + (2^m+1)^{l-1}(2^m(l-1)-1)]. \end{aligned}$$

□

Proposition 5.4 *The group C^*-algebra $C^*(G)$ of $G = \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$ a finite product of the proper generalized discrete elementary $ax+b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$ has a composition series $\{\mathfrak{I}_j\}_{j=1}^{(m+1)^l}$ of closed ideals such that $\mathfrak{I}_0 = \{0\}$ and $\mathfrak{I}_{(m+1)^l} = C^*(G)$ and*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \mathfrak{L}_{1_j} \otimes \cdots \otimes \mathfrak{L}_{s_j} \otimes \cdots \otimes \mathfrak{L}_{l_j}$$

combinatoricly, for some $1 \le s_j \le m+1$ and $s_{j-1} \le s_j$ with $(1_j, \ldots, l_j)$ totally ordered properly as in Proposition 4.4, and

$$\begin{aligned} \mathfrak{L}_k &= \oplus^{\binom{m}{k-1}2^{m-1}}[C_0(\mathbb{R}^{m-k+1}) \otimes M_2(\mathbb{C})] \quad \text{for } 1 \le k \le m, \\ \mathfrak{L}_{m+1} &= \mathbb{C}^{2^{m+1}}. \end{aligned}$$

Proof. This is obtained from that $C^*(G) \cong \otimes^l C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$ and the composition series $\{\mathfrak{I}'_k\}_{k=0}^{m+1}$ of $C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$ obtained in Proposition 3.2. Indeed, the closed ideals $\mathfrak{I}_j$ are defined inductively as in the following:

$$\mathfrak{I}_1 = \mathfrak{L}_1 \otimes \mathfrak{L}_1 \otimes \cdots \otimes \mathfrak{L}_1 = \mathfrak{I}'_1 \otimes \mathfrak{I}'_1 \otimes \cdots \otimes \mathfrak{I}'_1,$$
$$\mathfrak{I}_2 = \mathfrak{I}'_2 \otimes \mathfrak{I}'_1 \otimes \cdots \otimes \mathfrak{I}'_1,$$

$\mathfrak{I}_3$ is generated by $\mathfrak{I}_2$ and $\mathfrak{I}'_1 \otimes \mathfrak{I}'_2 \otimes \mathfrak{I}'_1 \otimes \cdots \otimes \mathfrak{I}'_1$;

$$\mathfrak{I}_4 = \mathfrak{I}'_2 \otimes \mathfrak{I}'_2 \otimes \mathfrak{I}'_1 \cdots \otimes \mathfrak{I}'_1,$$

$\mathfrak{I}_5$ is generated by $\mathfrak{I}_4$ and $\mathfrak{I}'_1 \otimes \mathfrak{I}'_1 \otimes \mathfrak{I}'_2 \otimes \mathfrak{I}'_1 \cdots \otimes \mathfrak{I}'_1$; and similarly, ..., and finally, $\mathfrak{I}_{(m+1)^l-1}$ is generated by

$$C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2) \otimes C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2) \otimes \cdots \otimes C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2) \otimes \mathfrak{I}'_m,$$
$$\mathfrak{I}'_m \otimes C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2) \otimes \cdots \otimes C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2),$$
$$C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2) \otimes \mathfrak{I}'_m \otimes C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2) \otimes \cdots \otimes C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2),$$
$$\cdots\cdots\cdots\cdots \quad \text{and}$$
$$C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2) \otimes \cdots \otimes C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2) \otimes \mathfrak{I}'_m \otimes C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2). \qquad \square$$

Proposition 5.5 *The group C^*-algebra $C^*(G)$ of $G = \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$ a finite product of the proper generalized discrete elementary $ax + b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$ has the following topological stable rank estimate*:

$$\lceil (\lfloor m/2 \rfloor)/2 \rceil + 1 \leq \mathrm{sr}(C^*(G)) \leq \lceil (\lfloor (m+1)/2 \rfloor)/2 \rceil + 1.$$

and the following connected stable rank estimate:

$$2 \leq \mathrm{csr}(C^*(G)) \leq \lceil (\lfloor (m+1)/2 \rfloor)/2 \rceil + 1.$$

Proof. Applying the stable rank formulae as used in Propositions 2.3 and 3.4 to the composition series of $C^*(G)$ obtained in Proposition 5.4, we estimate it by reducing to the following maximum:

$$\begin{aligned}
\mathrm{sr}(C^*(G)) &\geq \max_{1\leq k\leq l} \mathrm{sr}(\otimes^k (C_0(\mathbb{R}^m) \otimes M_2(\mathbb{C}))) \\
&= \max_{1\leq k\leq l} \mathrm{sr}(C_0(\mathbb{R}^{km}) \otimes M_{2^k}(\mathbb{C}))) \\
&= \max_{1\leq k\leq l} \lceil (\mathrm{sr}(C_0(\mathbb{R}^{km})) - 1)2^{-k} \rceil + 1
\end{aligned}$$

$$= \max_{1 \leq k \leq l} \lceil (\lfloor km/2 \rfloor) 2^{-k} \rceil + 1$$

and note that if $k = 2l$ even, then

$$\frac{lm}{2^{2l}} - \frac{(l+1)m}{2^{2(l+1)}} = \frac{3}{4} \cdot \frac{lm}{2^{2l}} - \frac{1}{4} \cdot \frac{m}{2^{2l}} > 0$$

and if $k = 2l + 1$ odd and m is even, then

$$\begin{aligned} &\frac{(2l+1)(m/2)}{2^{2l+1}} - \frac{(2(l+1)+1)(m/2)}{2^{2(l+1)+1}} \\ &= \frac{3}{4} \cdot \frac{(2l+1)(m/2)}{2^{2l+1}} - \frac{1}{4} \cdot \frac{m}{2^{2l+1}} > 0 \end{aligned}$$

and if $k = 2l + 1$ odd and m is odd, then

$$\begin{aligned} &\frac{((2l+1)m-1)/2}{2^{2l+1}} - \frac{((2(l+1)+1)m-1)/2}{2^{2(l+1)+1}} \\ &= \frac{3}{8} \cdot \frac{(2l+1)m-1}{2^{2l+1}} - \frac{1}{8} \cdot \frac{2m}{2^{2l+1}} > 0 \end{aligned}$$

and therefore, the maximum is equal to

$$\lceil (\lfloor m/2 \rfloor) 2^{-1} \rceil + 1$$

in the case where $k = 1$.

Also, we estimate it by reducing to the following maximum:

$$\begin{aligned} \mathrm{sr}(C^*(G)) &\leq \max_{1 \leq k \leq l} \{\mathrm{sr}(\otimes^k (C_0(\mathbb{R}^m) \otimes M_2(\mathbb{C}))), \mathrm{csr}(\otimes^k (C_0(\mathbb{R}^m) \otimes M_2(\mathbb{C})))\} \\ &\leq \max_{1 \leq k \leq l} \{\lceil (\lfloor m/2 \rfloor) 2^{-1} \rceil + 1, \mathrm{csr}(C_0(\mathbb{R}^{km}) \otimes M_{2^k}(\mathbb{C}))\} \\ &\leq \max_{1 \leq k \leq l} \{\lceil (\lfloor m/2 \rfloor) 2^{-1} \rceil + 1, \lceil (\mathrm{csr}(C_0(\mathbb{R}^{km})) - 1)/2^k \rceil + 1\} \\ &\leq \max_{1 \leq k \leq l} \{\lceil (\lfloor m/2 \rfloor) 2^{-1} \rceil + 1, \lceil (\lfloor (km+1)/2 \rfloor)/2^k \rceil + 1\} \end{aligned}$$

and the similar computation as above shows that the maximum is equal to

$$\lceil (\lfloor (m+1)/2 \rfloor)/2 \rceil + 1.$$

Moreover, we obtain

$$\begin{aligned}\mathrm{csr}(C^*(G)) &\le \max_{1\le k\le l} \mathrm{csr}(\otimes^k (C_0(\mathbb{R}^m)\otimes M_2(\mathbb{C}))) \\ &\le \lceil(\lfloor (m+1)/2\rfloor)/2\rceil + 1.\end{aligned}$$

Since we have $K_1(C^*(G)) \neq 0$ by Theorem 5.3, we obtain $\mathrm{csr}(C^*(G)) \ge 2$.

□

Remark If $G = \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$ is a finite product of the (non-proper) generalized discrete elementary $ax+b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$, then we can obtain the similar composition series of $C^*(G) \cong \otimes^l C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$ as given in the case of $\Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$, by using the finite composition series of $C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$ obtained in Proposition 3.2 and viewing $C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$ as the mapping torus over $C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$. Consequently, we can deduce that

$$\lceil(\lfloor (m+1)/2\rfloor)/2\rceil + 1 \le \mathrm{sr}(C^*(G)) \le \lceil(\lfloor (m+2)/2\rfloor)/2\rceil + 1.$$

and

$$2 \le \mathrm{csr}(C^*(G)) \le \lceil(\lfloor (m+2)/2\rfloor)/2\rceil + 1.$$

by the same way as given in the proof above. These estimates give corrections to the case where $l = 1$ given in [11].

In both the proper and non-proper cases, it is worth noting that those stable ranks do not depend on the multiple l, i.e., a sort of stability of the stable ranks in taking tensor products, in those cases.

6. Their inductive limits

Corollary 6.1 *Let $G = \varinjlim \Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z})$ be the inductive limit (i.e., the direct sum) of finite products of the discrete elementary $ax+b$ group $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$ under the canonical inclusions. Then*

$$\begin{aligned}K_0(C^*(G)) &\cong K_1(C^*(G)) \\ &\cong \varinjlim(\mathbb{Z}^{2^{m-1}} \oplus \mathbb{Z}_2^{2^{2m-2}-2^{m-2}})\end{aligned}$$

Proof. Use the result on the K-theory groups for the generalized discrete elementary Mautner groups $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}^m \cong \Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z})$ in Theorem 4.2. Note that $C^*(G)$ is an inductive limit of the group C^*-algebras $C^*(\Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z}))$ and

$$\begin{aligned} K_j(C^*(G)) &= K_j(\varinjlim C^*(\Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z}))) \\ &\cong \varinjlim K_j(C^*(\Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z}))) \end{aligned}$$

for $j = 0, 1$, by continuity of K-theory (see [13]). □

Corollary 6.2 *Let $G = \varinjlim \Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2)$ be the inductive limit of finite products of the proper discrete elementary $ax + b$ group $\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$ under the canonical inclusions. Then*

$$\begin{aligned} K_0(C^*(G)) &\cong \varinjlim(\mathbb{Z}^{3^m} \oplus \mathbb{Z}_2^{2^{-2}[5^m-(2m+3)3^{m-1}]}), \\ K_1(C^*(G)) &\cong \varinjlim \mathbb{Z}_2^{2^{-2}[5^m+(2m-3)3^{m-1}]}. \end{aligned}$$

Proof. Use the result on the K-theory groups for the proper generalized discrete Mautner groups $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2^m \cong \Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2)$ in Theorem 4.3. The rest of the proof is the same as that of Corollary 6.1. □

Corollary 6.3 *Let $G = \varinjlim \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z})$ be the inductive limit of finite products of the generalized discrete elementary $ax + b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$ under the canonical inclusions. Then*

$$\begin{aligned} K_0(C^*(G)) &\cong K_1(C^*(G)) \\ &\cong \varinjlim(\mathbb{Z}^{2^{lm-1}} \oplus \mathbb{Z}_2^{2^{(m+1)l-2}-2^{lm-2}}). \end{aligned}$$

Proof. Use the result on the K-theory groups for finite products of the generalized discrete elementary $ax+b$ groups $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}$ in Theorem 5.2. The rest of the proof is the same as that of Corollary 6.1. □

Corollary 6.4 *Let $G = \varinjlim \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$ be the inductive limit of finite products of the proper generalized discrete elementary $ax+b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$ under the canonical inclusions. Then*

$$\begin{aligned} K_0(C^*(G)) &\cong \varinjlim(\mathbb{Z}^{(2^m+1)^l} \oplus \mathbb{Z}_2^{2^{-2}[(2^{m+1}+1)^l-(2^m+1)^{l-1}(2^m(l+1)+1)]}), \\ K_1(C^*(G)) &\cong \varinjlim \mathbb{Z}_2^{2^{-2}[(2^{m+1}+1)^l+(2^m+1)^{l-1}(2^m(l-1)-1)]} \end{aligned}$$

Proof. Use the result on the K-theory groups for finite products of the proper generalized discrete elementary $ax+b$ groups $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$ in Theorem 5.3. The rest of the proof is the same as that of Corollary 6.1. □

Remark It is shown in [12] that the K-theory groups K_0 and K_1 of the group C^*-algebras of finitely generated, discrete nilpotent groups without torsion (and even their inductive limits) are isomorphic. But now we can know the difference between the nilpotent case considered in [12] and the non-nilpotent, solvable case as revealed concretely above. This is also the main purpose to exhibit those concrete examples.

Corollary 6.5 *Let $G = \varinjlim \Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2)$ be the inductive limit of finite products of the proper discrete elementary $ax+b$ group $\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$ under the canonical inclusions. Then*

$$\mathrm{sr}(C^*(G)) \leq 2, \quad \textit{and} \quad \mathrm{csr}(C^*(G)) = 2.$$

Proof. By [7] and [5] we have

$$\mathrm{sr}(C^*(G)) \leq \liminf \mathrm{sr}(C^*(\Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2))) = 2,$$
$$\mathrm{csr}(C^*(G)) \leq \liminf \mathrm{sr}(C^*(\Pi^m(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2))) = 2$$

by our result Proposition 4.5 on the stable ranks for $C^*(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2^m)$. Since $K_1(C^*(G)) \neq 0$ by Corollary 6.2, we have $\mathrm{csr}(C^*(G)) \geq 2$. □

Remark It is very likely that $\mathrm{sr}(C^*(G)) = 2$, but this would be another task to check this. What we need to show it would be a sort of reverse inequality for the stable ranks of inductive limits if any. Or another K-theory obstruction (possibly Fredholm index, already known, or not it) to have stable rank more than one.

Corollary 6.6 *Let $G = \varinjlim \Pi^l(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2)$ be the inductive limit of finite products of the proper generalized discrete elementary $ax+b$ group $\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2$ under the canonical inclusions. Then*

$$\mathrm{sr}(C^*(G)) \leq \lceil (\lfloor (m+1)/2 \rfloor)/2 \rceil + 1.$$

and

$$2 \leq \mathrm{csr}(C^*(G)) \leq \lceil (\lfloor (m+1)/2 \rfloor)/2 \rceil + 1.$$

Proof. The proof is the same as above by using Proposition 5.5 and Corollary 6.4. □

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