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A remark on the Navier-Stokes flow with bounded initial data having a special structure

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Abstract. The Navier-Stokes equations with bounded initial data admit unique local-in-time smooth mild solutions. It is shown that the solution can be extended globally-in-time, if the initial velocity has a special structure. Thanks to the structure, the annihilation of the pressure occurs, and then the mild solution is a solution to the viscous Burgers equations. By the maximum principle, it is derived an a priori bound for velocity, uniformly in time and space.

Key words: Navier-Stokes equations, maximum principle, renormalization structure.

1. Introduction

We consider the Navier-Stokes equations in $\mathbb{R}^n$ with $n \in \mathbb{N}$ and $n \geq 2$, which describe the motion of incompressible viscous fluids:

$$\begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0, & x \in \mathbb{R}^n,\ t > 0, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^n. \end{cases} \qquad \text{(NS)}$$

Here $u = (u^1, \dots, u^n) = (u^1(x,t), \dots, u^n(x,t))$ and $p = p(x,t)$ denote the velocity and pressure of fluids, respectively. This Cauchy problem is called (NS) throughout this paper. Notations of derivatives are as follows: $u_t := \partial_t u := \partial u/\partial t$, $\partial_j := \partial/\partial x_j$ for $j = 1, \dots, n$, $\Delta := \sum_{j=1}^n \partial_j^2$ and $\nabla := (\partial_1, \dots, \partial_n)$. For vectors $a = (a^1, \dots, a^n)$ and $b = (b^1, \dots, b^n)$, we denote by $a \cdot b = (a, b) := \sum_{j=1}^n a^j b^j$. The problem is to determine a pair of a solution (u, p) to (NS), uniquely from the initial velocity $u_0 := (u_0^1(x), \dots, u_0^n(x))$. We deal with bounded, solenoidal and non-small initial data.

It is well known by e.g. [6] that (NS) admits a local-in-time mild solution when $u_0 \in L^\infty_\sigma(\mathbb{R}^n)$ for $n \geq 2$. The definition of function spaces is given in Section 3, as well as mild solutions. The mild solution is constructed by a

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successive approximation in $C(0,T_*;L^\infty_\sigma)$. The mild solution u is unique, as long as u exists. Moreover, the mild solution is smooth, which was shown by e.g. [8]. Once we construct a smooth mild solution, a pair (u,p) satisfies (NS) in the classical sense, provided the pressure

$$p=\sum_{i,j=1}^{n} R_iR_ju^iu^j \tag{1.1}$$

is chosen. This strategy was developed by Fujita and Kato [2].

Furthermore, in [6] the existence time of the mild solution is estimated from below as $T_*\geq C\|u_0\|_\infty^{-2}$ with some positive constant C depending only on n. When $n=2$, one can extend the mild solution globally-in-time by [7] as follows. One can apply the maximum principle (see Lemma 3.1 below) for the vorticity ω to the 2-D vorticity equation to derive a priori bounds as $\|\omega(t)\|_\infty\leq\|\omega_0\|_\infty$ for $t>0$. This leads us to a priori bounds for velocity:

$$\|u(t)\|_\infty\leq C\|u_0\|_\infty\exp\{C\|\omega_0\|_\infty t\} \quad \text{for} \quad t>0, \tag{1.2}$$

provided $\omega_0\in L^\infty$. For the details, see e.g. [12].

When $n\geq 3$, the existence of unique global-in-time smooth solutions to (NS) is a famous open problem. In general, it is not known whether a priori bounds like (1.2) can be derived. In this paper, we treat the initial velocity u_0 having the following special structure:

$$u_0(x)=\big(a,u_0^2(x_1),u_0^3(x_1,x_2),\dots,u_0^n(x_1,\dots,x_{n-1})\big) \tag{1.3}$$

with some constant a and bounded functions u_0^k of variables $x_1,\dots,x_{k-1}$. They satisfy the compatibility condition, i.e., $\nabla\cdot u_0=0$ holds for all $x\in\mathbb{R}^n$. These initial data appear in [11] with $n=3$ and $a=0$ for proving the ill-posedness theorem of (NS) in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$.

The purpose of this paper is to establish the existence theory of a unique global-in-time smooth mild solution to (NS) with initial velocity of the form (1.3). We see that the mild solution u has the same structure to u_0, i.e.,

$$u(x,t)=\big(a,u^2(x_1,t),u^3(x_1,x_2,t),\dots,u^n(x_1,\dots,x_{n-1},t)\big). \tag{1.4}$$

Thanks to (1.4), the annihilation of the pressure given by (1.1) occurs. So, our mild solution satisfies the vector valued viscous Burgers equations. We apply the maximum principle to derive the uniform in time and space a priori bound:

$$\|u(t)\|_\infty \le \|u_0\|_\infty \tag{1.5}$$

for $t > 0$. Obviously, (1.5) is a better estimate than (1.2).

2. Main Results

In this section we state the main results.

Theorem 2.1 *Let $n \ge 2$. Let $u_0 \in L^\infty(\mathbb{R}^n)$ be of the form* (1.3). *Then there exists a unique global-in-time smooth mild solution u in $C(0, \infty; L^\infty_\sigma)$ satisfying* (1.5) *for $t > 0$.*

We restrict ourself that the solution treated here is a mild solution, only. In general, the uniqueness of classical solutions does not hold in the framework of L^∞. In fact, for $g \in C^1([0, T))^n$ and for $c \in \mathbb{R}$, a pair $u(t) = g(t)$ and $p(t) = -g'(t) \cdot x + c$ satisfies (NS) for $t \in (0, T)$ with $u_0 = g(0)$. Nevertheless, the mild solution is uniquely determined by $u_0 = g(0)$ as $u(t) = u_0$. The uniqueness of such solutions in L^∞-framework was studied by [5], [10].

It is not known whether the similar theorem holds for the boundary value problem. Even local-in-time solvability is not known except for the half space [13]. Recently, the analyticity of the Stokes semigroup in L^∞ type spaces is established by [1]. A similar solution which is not a mild solution for the boundary value problem in the half space is often called a Poiseuille flow; see e.g. [4].

3. Maximum Principle

This section is devoted to the definition of function spaces, the notion of a mild solution and the maximum principle.

We first define function spaces. Let $L^\infty(\mathbb{R}^n)$ be the space of all bounded functions on $\mathbb{R}^n$ with a norm $\|f\|_\infty := \text{ess.sup}_{x \in \mathbb{R}^n} |f(x)|$. Let L^∞_σ be the solenoidal subspace of L^∞; we sometimes omit the notation $(\mathbb{R}^n)$, if no confusion occurs, likely. Also, we do not distinguish vector valued and scalar functions as well as function spaces. Let BMO be the space of all bounded mean oscillation functions. Let $C(0,T;X)$ be the space of all continuous functions on $(0,T)$ with value in X. Since L^∞ and BMO are subspaces of $\mathcal{S}'$, all calculation can be justified in the tempered distribution sense.

Let u be called a mild solution if it satisfies the integral equation:

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}(u(s),\nabla)u(s)ds, \tag{INT}$$

where $e^{t\Delta} := G_t*$ stands for the solution operator of the heat equation, $G_t(x) := (1/(4\pi t)^{n/2}\exp\{-|x|^2/4t\}$ is the Gauss kernel. The Helmholtz projection onto solenoidal subspace denotes $\mathbb{P}$, which is the matrix operator whose ij-component is given by $\delta_{ij}+R_iR_j$. Here δ_{ij} is Kronecker's delta, and $R_i := \partial_i(-\Delta)^{-1/2}$ denotes the Riesz transform. Note that R_i is a bounded operator from L^∞ to BMO, and from BMO to BMO. Thus, (1.1) makes BMO sense, if u is bounded. A mild solution is constructed as the limit of successive approximation in $C(0,T_*;L^\infty_\sigma)$ when $u_0 \in L^\infty_\sigma$: let us define $\{u_j\}_{j=1}^\infty$ by

$$u_1(t) := e^{t\Delta}u_0 \quad \text{and} \quad u_{j+1}(t) := u_1(t) - \mathcal{B}(u_j) \quad \text{for } j \in \mathbb{N},$$
$$\mathcal{B}(u_j) := \mathcal{B}(u_j)(t) := \int_0^t e^{(t-s)\Delta}\mathbb{P}(u_j(s),\nabla)u_j(s)ds.$$

We now recall the maximum principle for solutions to the equations:

$$\begin{cases} v_t - \Delta v + (w,\nabla)v = 0, & x \in \mathbb{R}^n,\ t > 0, \\ v|_{t=0} = v_0, & x \in \mathbb{R}^n. \end{cases} \tag{P}$$

Here w is some function. Note that (P) is the Cauchy problem of the vector valued viscous Burgers equations, provided $w := v \in \mathbb{R}^n$. Also, (P) is equivalent to the 2-D vorticity equation, provided $v := \omega := \partial_1 u^2 - \partial_2 u^1 \in \mathbb{R}$ and $w := u \in \mathbb{R}^2$. However, the 3-D vorticity equations are not of this type.

Lemma 3.1 *Let $v_0 \in L^\infty(\mathbb{R}^n)$, and let $v, w \in C(0,\infty;L^\infty)$. Assume that v is a smooth solution to* (P). *Then $\|v(t)\|_\infty \le \|v_0\|_\infty$ for $t > 0$.*

The maximum principle of this type was originally developed by Oleinik in [9]. The parabolicity of (P) is not needed, i.e., the terms Δv is removable. Since this lemma is precisely proved by [3, Theorem 2.3.8], we skip its proof.

4. Proof

We shall give a proof of Theorem 2.1. The following property is essential.

Proposition 4.1 *Let $n \geq 2$ and $T \in (0,\infty]$. Assume that $u_0 \in L^\infty(\mathbb{R}^n)$ is of the form* (1.3)*. Let $u \in C(0,T;L^\infty_\sigma)$ be a mild solution. Then p given by* (1.1) *vanishes.*

Proof of Proposition 4.1. In what follows, we argue an induction with respect to n. Let us start at $n = 2$. Let $\Delta_k := \sum_{i=1}^k \partial_i^2$ for $k = 1,\dots,n-1$.

$\boldsymbol{n = 2.}$ Since $e^{t\Delta} = e^{t\partial_1^2}e^{t\partial_2^2}$ and $e^{t\partial_j^2}a = a$, we see that the first approximation

$$u_1(x,t) = e^{t\Delta}u_0(x) = \big(a, e^{t\Delta_1}u_0^2(x_1)\big) = \big(a, u_1^2(x_1,t)\big).$$

If $a = 0$, then u_1 is a mild solution. Indeed, $\mathcal{B}(u_1) = 0$, since $(u_1,\nabla)u_1 = 0$.

If $a \neq 0$, then we obtain that $U_1 := (u_1,\nabla)u_1 = (0, a\partial_1 u_1^2(x_1,t))$. We thus see that

$$\mathcal{B}(u_1) = \int_0^t e^{(t-s)\Delta}U_1(s)ds = \left(0, \int_0^t e^{(t-s)\Delta}U_1^2(s)ds\right) =: \big(0, w_2^2(x_1,t)\big).$$

Here $\mathbb{P}U_1 = U_1$, since $\nabla\cdot U_1 = 0$. So, $u_2 = (a, u_2^2(x_1,t))$ with $u_2^2 = u_1^2 - w_2^2$.

We next compute u_3. It has the same structure, i.e., $U_2 := (u_2,\nabla)u_2 = (0, U_2^2(x_1,t))$, and then $\nabla\cdot U_2 = 0$. So, we see

$$\mathcal{B}(u_2) = \left(0, \int_0^t e^{(t-s)\Delta}U_2^2(s)ds\right) =: \big(0, w_3^2(x_1,t)\big).$$

This yields that $u_3 = (a, u_3^2(x_1,t))$ with $u_3^2 = u_1^2 - w_3^2$. For $j \geq 2$, u_j has the same structure (1.4). Therefore, (1.4) holds.

Note that R_i maps the constant $(u^1)^2 = a^2$ to zero, and u^2 is not a function of x_2, which heuristically imply

$$\begin{aligned} p &= \sum_{i,j=1}^2 R_iR_ju^iu^j = R_1R_1a^2 + 2aR_1R_2u^2 + R_2^2(u^2)^2 \\ &= 2a(-\Delta)^{-1}\partial_1\partial_2u^2 + (-\Delta)^{-1}\partial_2^2(u^2)^2 = 0. \end{aligned} \tag{4.1}$$

These equalities hold in BMO sense; $\nabla p = 0$ holds in L^∞ sense.

$\boldsymbol{n \geq 3}$. We now check (1.4) for all n. Let $n \geq 3$, and assume that (1.4) holds for $n-1$. We see that

$$\begin{aligned} u_1(x,t) &= \big(a, e^{t\Delta_1}u_0^2(x_1), \ldots, e^{t\Delta_{n-1}}u_0^n(x_1, \ldots, x_{n-1})\big) \\ &= \big(a, u_1^2(x_1,t), \ldots, u_1^n(x_1, \ldots, x_{n-1}, t)\big). \end{aligned}$$

Note that $u_j^1 = a$, and that u_j^k is a function of $x_1, \ldots, x_{k-1}$ and t for all $j \in \mathbb{N}$ and $k = 2, \ldots, n-1$, since u_j^k is independent of u_ℓ^n for all $\ell \in \mathbb{N}$ by the construction. Hence, it is enough to calculate the n-th component, only. We have that $U_1^n := (u_1, \nabla)u_1^n = \sum_{k=1}^{n-1} u_1^k \partial_k u_1^n$ is a function of $x_1, \ldots, x_{n-1}$ and t, and then $\nabla \cdot U_1 = 0$. Via $\mathcal{B}(u_1)$, we see that u_2^n is a function of $x_1, \ldots, x_{n-1}$ and t. Analogously, for $j \geq 3$, we see that

$$u_j^n = u_1^n(t) - \int_0^t e^{(t-s)\Delta} \sum_{k=1}^{n-1} u_{j-1}^k(s)\partial_k u_{j-1}^n(s)ds$$

is a function of $x_1, \ldots, x_{n-1}$ and t. The point is that $\mathbb{P}$ always disappears in the bilinear terms. Therefore, the mild solution u is of the form (1.4).

Finally, we arrive at the fact that $R_i R_j u^i u^j$ vanishes for any i and j, provided u is of (1.4), since $u^i u^j$ is not a function of $x_{i \vee j}$, where $i \vee j := \max\{i, j\}$. Hence, similarly to (4.1), we easily seek that $p = 0$ in BMO sense. This completes the proof of Proposition 4.1. □

We are in a position to give the complete proof of our main results.

Proof of Theorem 2.1. Let $u_0 \in L^\infty(\mathbb{R}^n)$ be of (1.3). As is shown by [6], one can construct a unique local-in-time smooth mild solution u up to time T_*, and $T_* \geq C\|u_0\|_\infty^{-2}$ with some constant C. By Proposition 4.1, p vanishes, then u satisfies (P) with $v = w := u$. By Lemma 3.1, we have (1.5) for $t \in (0, T_*]$. Taking $\varepsilon \in (0, T_*)$, and let $T_1 := T_* - \varepsilon$ be fixed, we see that the mild solution u can be constructed from $u(T_1)$ regarded as the initial velocity. Since $\|u(T_1)\|_\infty \leq \|u_0\|_\infty$, the mild solution is constructed at least up to $T_1 + T_* = 2T_* - \varepsilon$. Note that $u(T_1)$ is also of the form (1.3), then Propostion 4.1 and Lemma 3.1 are applicable again. Thus, (1.5) holds for $t \in (0, T_1 + T_*]$. We may repeat this procedure to get the unique global-in-time smooth mild solution u satisfying (1.5) for all $t > 0$. □

Remark 4.2 (i) One can also prove Theorem 2.1 as follows. Firstly, we construct a unique global-in-time smooth solution u in $C(0,\infty;L^\infty)$ to (P) with $w := v := u$ and $v_0 := u_0$ of (1.3), since one may solve the linear transport equation (with the viscous term) in order, component-wisely. Secondly, Lemma 3.1 produces (1.5) for $t > 0$. Thirdly, p of (1.1) vanishes by Proposition 4.1. Thanks to the uniqueness theorem of mild solutions by Kato [10], u is the only one mild solution determined by u_0. This method also works on the Euler equations if the initial velocity $u_0 \in B^1_{\infty,1}$.
(ii) The structure (1.3) is not a necessary condition to gain (1.5). Indeed, $n = 2$, if $u_0(x) = (\sin x_2, \sin x_1)$, then $u = u_1 = (e^{-t}\sin x_2, e^{-t}\sin x_1)$ is a mild solution, and the pressure $p = e^{-2t}(\cos x_1)\cos x_2$.

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