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Cohomological equations and invariant distributions on a compact Lie group

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Abstract. This paper deals with two analytic questions on a connected compact Lie group G. i) Let $a \in G$ and denote by γ the diffeomorphism of G given by $\gamma(x) = ax$ (left translation by a). We give necessary and sufficient conditions for the existence of solutions of the cohomological equation $f - f \circ \gamma = g$ on the Fréchet space $C^\infty(G)$ of complex C^∞ functions on G. ii) When G is the torus $\mathbb{T}^n$, we compute explicitly the distributions on $\mathbb{T}^n$ invariant by an affine automorphism γ, that is, $\gamma(x) = A(x + a)$ with $A \in \mathrm{GL}(n, \mathbb{Z})$ and $a \in \mathbb{T}^n$. iii) We apply these results to describe the infinitesimal deformations of some Lie foliations.

Key words: Lie group, cohomological equation, distribution, foliation, deformation.

0. Preliminaries

Let M be a manifold and γ a diffeomorphism of M. Usually, the couple (M, γ) is called a *discrete dynamical system.* Natural question: *What are the geometric objects invariant under the action of* γ? Formulated as such, this question is far to be trivial. However one can answer it in special situations for a given manifold if we specify the diffeomorphism γ and the nature of the geometrical objects. It has been customary, in the theory of dynamical systems, to seek an *invariant measure.* But this problem is very hard in general. Instead of this, it is more easier to seek an *invariant distribution.* This leads systematically to solving certain equations (called *cohomological equations*) on the Fréchet space $C^\infty(G)$ of complex C^∞ functions on G. Regardless of this, these equations constitute a theme currently booming.

The purpose of this paper is to answer these questions for some diffeomorphisms of a (connected) Lie group G: First, G is arbitrary compact and γ is a translation and then, G is the torus $\mathbb{T}^n$ and γ is an affine automorphism.

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Let G be a connected compact Lie group of dimension n. We denote by $C^\infty(G)$ the space of complex C^∞-functions on G equipped with the C^∞-topology. This topology can be defined as follows. Let $\{U_1,\dots,U_k\}$ be an open cover of G such that, for each $i\in\{1,\dots,k\}$, there exists a C^∞-diffeomorphism $\phi_i:\mathbb{R}^n\longrightarrow U_i$. Let $\{\rho_1,\dots,\rho_k\}$ be a C^∞-partition of 1 such that the support (which is compact) of each ρ_i is contained in U_i. Then, if $f\in C^\infty(G)$, one may write $f=\sum_{i=1}^k \rho_i f$.

For any $\mathbf{s}=(s_1,\dots,s_n)\in\mathbb{N}^n$, $|s|=s_1+\cdots+s_n$ is the *length* of $\mathbf{s}$; we denote by $D^{\mathbf{s}}$ the differential operator:

$$\frac{\partial^{|\mathbf{s}|}}{\partial x_1^{s_1}\dots\partial x_n^{s_n}}.$$

Let $i\in\{1,\dots,k\}$; the function $(\rho_i f)\circ\phi_i:\mathbb{R}^n\longrightarrow\mathbb{C}$ is of class C^∞ and with compact support. For any $r\in\mathbb{N}$, we set:

$$\|f\|_r=\sum_{i=1}^k\left(\sum_{|\mathbf{s}|\le r}\sup_{u\in\mathbb{R}^n}|D^{\mathbf{s}}((\rho_i f)\circ\phi_i)(u)|\right).$$

It is easy to see that $\|\ \|_r$ is a norm on $C^\infty(G)$ and that the family $\{\|\ \|_r\}_{r\ge 0}$ defines the C^∞-topology on $C^\infty(G)$ which makes it a Fréchet space.

Let $a\in G$ and γ be the analytic diffeomorphism $x\in G\longmapsto ax\in G$ (left translation by a). Then γ acts on functions by composition: $f\in C^\infty(G)\longmapsto f\circ\gamma\in C^\infty(G)$. We say that $f\in C^\infty(G)$ is *γ-invariant* if it satisfies $f=f\circ\gamma$; the quantity $f-f\circ\gamma$ is a measure of the 'invariance defect' of f; it is called a *divergence* of f for γ. The divergence functions generate a vector subspace $\mathcal{C}$ of $C^\infty(G)$. Its determination led to the problem:

$$\begin{array}{l}\text{Given } g\in C^\infty(G)\text{, does there exists } f\in C^\infty(G)\\ \text{such that } f-f\circ\gamma=g?\end{array}\qquad(1)$$

This consists exactly to the determination of the cokernel $C^\infty(G)/\mathcal{C}$ of the continuous operator:

$$\delta: f\in C^\infty(G)\longmapsto (f-f\circ\gamma)\in C^\infty(G).$$

This cokernel is the first *cohomology* vector space $H^1(\mathbb{Z},C^\infty(G))$ of the *discrete group* $\mathbb{Z}$ with coefficients in the $\mathbb{Z}$-module $C^\infty(G)$. The quotient

$C^\infty(G)/\overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}$, is the first *reduced cohomology* vector space of $\mathbb{Z}$ with coefficients in $C^\infty(G)$; we denote it $\overline{H}^1(\mathbb{Z}, C^\infty(G))$.

An element T of the topological dual of $C^\infty(G)$ is a *distribution* on G. (The evaluation of T on $f \in C^\infty(G)$ will be denoted $\langle T, f\rangle$.) We say that T is γ-*invariant* if, for any function $f \in C^\infty(G)$, we have $\langle T, f \circ \gamma\rangle = \langle T, f\rangle$; so it is a continuous linear functional $C^\infty(G) \longrightarrow \mathbb{C}$ which vanishes on the subspace $\mathcal{C}$. Then, the space $\mathcal{D}_\gamma(G)$ of γ-invariant distributions on G can be identified to the topological dual of the quotient $C^\infty(G)/\mathcal{C}$ (and then the dual of $C^\infty(G)/\overline{\mathcal{C}}$).

According to the different situations that arise, we will give some answers to the following questions:

- Under what conditions the cohomological equation (1) admits solutions?
- Determine the subspace $\mathcal{C}$ or, failing that, its closure $\overline{\mathcal{C}}$.
- Calculate the space $\mathcal{D}_\gamma(G)$ of distributions on G invariant under the action of γ.

We begin with the torus $\mathbb{T}^n$ and a translation γ. It will be a decisive step to solve the problem in the two situations: i) γ is a translation on an arbitrary compact G; ii) G again is the torus $\mathbb{T}^n$ and γ is an affine automorphism, that is, an element of the group $\mathbb{T}^n \rtimes \mathrm{GL}(n, \mathbb{Z})$ (of affine automorphisms of the Lie group $\mathbb{T}^n$).

1. *G* is a torus

Let $n \geq 1$ be an integer. The real vector space $\mathbb{R}^n$ will be equipped with its usual scalar product $\langle\, ,\, \rangle$ and the associated norm $|\cdot|$. The torus $\mathbb{T}^n$ is obtained as the quotient of $\mathbb{R}^n$ by its standard lattice $\mathbb{Z}^n$. A function on $\mathbb{T}^n$ is just a function $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ satisfying the relation $f(x + \mathbf{m}) = f(x)$ for any $x \in \mathbb{R}^n$ and any $\mathbf{m} \in \mathbb{Z}^n$. For $\mathbf{m} \in \mathbb{Z}^n$, we denote by $\Theta_{\mathbf{m}}$ the function $\Theta_{\mathbf{m}}(x) = e^{2i\pi\langle \mathbf{m}, x\rangle}$. If f is integrable, it admits a Fourier series expansion:

$$f(x) = \sum_{\mathbf{m}\in\mathbb{Z}^n} f_{\mathbf{m}}\Theta_{\mathbf{m}}(x)$$

where the complex numbers $f_{\mathbf{m}}$ are its Fourier coefficients given by the integral formula:

$$f_{\mathbf{m}} = \int_{\mathbb{T}^n} f(x) e^{-2i\pi\langle \mathbf{m}, x\rangle} dx.$$

If f is square integrable, the coefficients $f_{\mathbf{m}}$ satisfy the condition $\sum_{\mathbf{m}\in\mathbb{Z}^n} |f_{\mathbf{m}}|^2 < +\infty$.

In the same way, any distribution T on $\mathbb{T}^n$ (viewed as a $\mathbb{Z}^n$-periodic distribution on $\mathbb{R}^n$) can be written:

$$T = \sum_{\mathbf{m}\in\mathbb{Z}^n} T_{\mathbf{m}} \Theta_{\mathbf{m}}$$

where the family of complex numbers $T_{\mathbf{m}}$ (indexed by $\mathbf{m} \in \mathbb{Z}^n$) is of *polynomial growth*, that is, there exists $r \in \mathbb{N}$ and a constant $C > 0$ such that $|T_{\mathbf{m}}| \leq C|\mathbf{m}|^r$ for any $\mathbf{m} \in \mathbb{Z}^n$.

For any $r \in \mathbb{N}$, we denote by $W^{1,r}$ the space of functions f on $\mathbb{T}^n$ whose Fourier coefficients $(f_{\mathbf{m}})_{\mathbf{m}\in\mathbb{Z}^n}$ satisfy the condition $\sum_{\mathbf{m}\in\mathbb{Z}^n} |\mathbf{m}|^r |f_{\mathbf{m}}| < +\infty$. Similarly, $W^{2,r}$ will be the space of functions f whose Fourier coefficients $(f_{\mathbf{m}})_{\mathbf{m}\in\mathbb{Z}^n}$ satisfy the convergence condition $\sum_{\mathbf{m}\in\mathbb{Z}^n} |\mathbf{m}|^{2r} |f_{\mathbf{m}}|^2 < +\infty$. These are Banach spaces respectively for the norms:

$$\|f\|_{1,r} = |f_{\mathbf{0}}| + \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} |\mathbf{m}|^r |f_{\mathbf{m}}| \quad \text{for } f \in W^{1,r}$$

and:

$$\|f\|_{2,r} = \sqrt{|f_{\mathbf{0}}|^2 + \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} |\mathbf{m}|^{2r} |f_{\mathbf{m}}|^2} \quad \text{for } f \in W^{2,r}.$$

$W^{2,r}$ is the r^{th} Sobolev space of the torus $\mathbb{T}^n$; it has a Hilbert structure given by the Hermitian product:

$$\langle f, g\rangle_r = f_{\mathbf{0}}\overline{g}_{\mathbf{0}} + \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} |\mathbf{m}|^{2r} f_{\mathbf{m}} \overline{g}_{\mathbf{m}}.$$

We have natural inclusions:

$$C^\infty(\mathbb{T}^n) \subset \cdots \subset W^{1,r+1} \subset W^{1,r} \subset \cdots \subset W^{1,0}$$

and:

$$C^\infty(\mathbb{T}^n) \subset \cdots \subset W^{2,r+1} \subset W^{2,r} \subset \cdots \subset W^{2,0} = L^2(\mathbb{T}^n).$$

The following proposition is easy to establish.

Proposition 1.1 *Let $T = \sum_{\mathbf{m}\in\mathbb{Z}^n} T_{\mathbf{m}}\Theta_{\mathbf{m}}$ be a series (where $T_{\mathbf{m}}$ are complex numbers). Then the following assertions* i), ii) *and* iii) *are equivalent:*

i) *T is a regular distribution, that is, T is a C^∞-function.*
ii) *For any $r \in \mathbb{N}$, the series $\sum_{\mathbf{m}\in\mathbb{Z}^n} |\mathbf{m}|^{2r}|T_{\mathbf{m}}|^2$ converges.*
iii) *For any $r \in \mathbb{N}$, the series $\sum_{\mathbf{m}\in\mathbb{Z}^n} |\mathbf{m}|^{r}|T_{\mathbf{m}}|$convergenes.*

The injections $j_{1,r} : W^{1,r+1} \hookrightarrow W^{1,r}$ and $j_{2,r} : W^{2,r+1} \hookrightarrow W^{2,r}$ are compact operators.

The first three points of the proposition say:

$$\bigcap_{r\in\mathbb{N}} W^{1,r} = \bigcap_{r\in\mathbb{N}} W^{2,r} = C^\infty(\mathbb{T}^n).$$

Any vector $a \in \mathbb{R}^n$ defines a linear functional on $\mathbb{R}^n : x \in \mathbb{R}^n \longmapsto \langle a, x\rangle \in \mathbb{R}$ and then on the lattice $\mathbb{Z}^n$.

Definition 1.2 ([Sc]) Let $a = (a_1, \ldots, a_n)$ be a vector of $\mathbb{R}^n$ such that the subgroup generated by its projection on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is dense in $\mathbb{T}^n$. (This implies in particular that the numbers $1, a_1, \ldots, a_n$ are linearly independent over $\mathbb{Q}$.)

i) We say that a is ***Diophantine*** if there exist real numbers $C > 0$ and $\tau > 0$ such that $|1 - e^{2i\pi\langle\mathbf{m},a\rangle}| \geq C/|\mathbf{m}|^\tau$ for any nonzero $\mathbf{m} \in \mathbb{Z}^n$.
ii) We say that a is ***Liouville vector*** if there exists $C > 0$ such that, for any $\tau > 0$, there exists $\mathbf{m}_\tau \in \mathbb{Z}^n$ satisfying $|1 - e^{2i\pi\langle\mathbf{m},a\rangle}| \leq C/|\mathbf{m}_\tau|^\tau$.

For instance, any vector a of $\mathbb{R}^n$ as in Definition 1.2 and for which the components $a_1, \ldots, a_n$ are algebraic numbers is Diophantine.

Indeed, by multiplying the components by a common denominator, one may suppose that the a_i are algebraic integers. Let $\mathbb{Q}[a_1, \ldots, a_n]$ be the number field generated by $a_1, \ldots, a_n$. Denote by d the degree of the Galois extension $\mathbb{Q}[a_1, \ldots, a_n]$ and G its Galois group. Let σ_i $(i = 1, \ldots, d)$ be the different embeddings of $\mathbb{Q}[a_1, \ldots, a_n]$ in $\overline{\mathbb{Q}}$. For any $\mathbf{m} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$, the product $\prod_j \sigma_j(\langle a, \mathbf{m}\rangle)$ is a non zero algebraic integer invariant by G, then it is a non zero integer. This implies:

$$\left|\prod_j \sigma_j(\langle a, \mathbf{m}\rangle)\right| \geq 1.$$

Then, if $\sigma_1 = \mathrm{Id}$:

$$|\langle a, \mathbf{m}\rangle| \geq \frac{1}{\left|\prod_{j\geq 2} \sigma_j(\langle a, \mathbf{m}\rangle)\right|} \geq \frac{C}{|\mathbf{m}|^{d-1}}$$

where C is a real positive constant.

Remark 1.3 The Diophantine condition is invariant under the action of $\mathrm{GL}(n, \mathbb{Z})$, that is, if a is Diophantine, for any $\xi \in \mathrm{GL}(n, \mathbb{Z})$, $\xi(a)$ is Diophantine.

The proof is immediate. Let ξ^* be the transpose of ξ and let C and τ be the constants given by i) in Definition 1.2. For any $\mathbf{m} \in \mathbb{Z}^n$ different from $\mathbf{0}$, we have:

$$\begin{aligned}\langle \mathbf{m}, \xi(a)\rangle &= \langle \xi^*(\mathbf{m}), a\rangle \\ &\geq \frac{C}{|\xi^*(\mathbf{m})|^\tau} \\ &\geq \frac{C}{\|\xi^*\|^\tau} \cdot \frac{1}{|\mathbf{m}|^\tau}.\end{aligned}$$

This remark will justify what does 'Diophantine condition' mean for an element a in an arbitrary compact Lie group considered in Theorem 2.5.

We define a continuous linear functional $\mathcal{L} : C^\infty(\mathbb{T}^n) \longrightarrow \mathbb{C}$ by $\mathcal{L}(g) = \int_{\mathbb{T}^n} g(x)dx = g_{\mathbf{0}}$ for any function $g = \sum_{\mathbf{m}\in\mathbb{Z}^n} g_{\mathbf{m}}\Theta_{\mathbf{m}}$. One can interpret $\mathcal{L}$ as an operator on $C^\infty(\mathbb{T}^n)$ which associates to each g the function $\mathcal{L}(g)\mathbf{1}$ where $\mathbf{1}$ is the constant function equal to 1; it is a compact operator because its rank is finite (equal to 1). Its kernel $\mathcal{N}$ is closed and such that $C^\infty(\mathbb{T}^n) = \mathcal{N} \oplus \mathbb{C}\cdot\mathbf{1}$. Denote by P the first projection $C^\infty(\mathbb{T}^n) = \mathcal{N} \oplus \mathbb{C}\cdot\mathbf{1} \longrightarrow \mathcal{N}$. It satisfies $P \oplus \mathcal{L} = I$ (where I is the identity operator on $C^\infty(\mathbb{T}^n)$). We have the:

Theorem 1.4 *Let γ be the diffeomorphism of $\mathbb{T}^n$ associated to a transla-*

tion by the vector $a = (a_1, \dots, a_n)$ where $a_1, \dots, a_n$ are linearly independent over $\mathbb{Q}$ and the subgroup generated by a is dense in $\mathbb{T}^n$.

i) *Suppose a Diophantine. Then there exists a bounded operator $C^\infty(\mathbb{T}^n) \xrightarrow{G} C^\infty(\mathbb{T}^n)$ such that $\delta G = I - \mathcal{L}$. Consequently, the equation $f - f \circ \gamma = g$ has a solution $f \in C^\infty(\mathbb{T}^n)$ if and only if, $\mathcal{L}(g) = 0$. Moreover, the vector space $H^1(\mathbb{Z}, C^\infty(\mathbb{T}^n))$ has dimension 1 and is generated by the constant function **1**.*
ii) *Suppose a is Liouville vector. Then, there exists an infinite family of linearly independent functions g satisfying $\mathcal{L}(g) = 0$ and such that the equation $f - f \circ \gamma = g$ has no solution. In this case, $H^1(\mathbb{Z}, C^\infty(\mathbb{T}^n))$ is an infinite dimensional and non Hausdorff topological vector space. But its associted Hausdorff space $\overline{H}^1(\mathbb{Z}, C^\infty(\mathbb{T}^n))$ is one dimensional and also generated by the constant function equal to **1**.*
In the two cases the space $\mathcal{D}_\gamma(\mathbb{T}^n)$ of γ-invariant distributions has dimension 1 and is generated by the Haar measure $dx = dx_1 \otimes \cdots \otimes dx_n$.

Proof. If we integrate both sides of the equation (1), the left gives 0. Therefore a necessary condition for the existence of a solution is $\int_{\mathbb{T}^n} g(x)dx = 0$. Suppose that it is filled. The Fourier expansions of the functions f and g:

$$f(x) = \sum_{\mathbf{m} \in Z^n} f_{\mathbf{m}} e^{2i\pi \langle x, \mathbf{m} \rangle} \quad \text{and} \quad g(x) = \sum_{\mathbf{m} \in Z^n} g_{\mathbf{m}} e^{2i\pi \langle x, \mathbf{m} \rangle}$$

reduce the equation to the system:

$$\big(1 - e^{2i\pi \langle \mathbf{m}, a \rangle}\big) f_{\mathbf{m}} = g_{\mathbf{m}} \quad \text{with} \quad \mathbf{m} \in \mathbb{Z}^n. \tag{2}$$

The necessary condition $\int_{\mathbb{T}^n} g(x)dx = 0$ means in fact $g_{\mathbf{0}} = 0$. We set:

$$f_{\mathbf{m}} = \begin{cases} 0 & \text{if } \mathbf{m} = \mathbf{0} \\ \dfrac{g_{\mathbf{m}}}{1 - e^{2i\pi \langle \mathbf{m}, a \rangle}} & \text{if } \mathbf{m} \neq \mathbf{0}. \end{cases} \tag{3}$$

The function f is then formally given by its Fourier coefficients $(f_{\mathbf{m}})_{\mathbf{m} \in \mathbb{Z}^n}$. Let us study its regularity by using Proposition 1.1. Let $r \in \mathbb{N}$.

i) <u>a is Diophantine</u>
We have:

$$|\mathbf{m}|^{2r}|f_{\mathbf{m}}|^2 = |\mathbf{m}|^{2r}\left|\frac{g_{\mathbf{m}}}{1-e^{2i\pi\langle \mathbf{m},a\rangle}}\right|^2 \le \frac{1}{C^2}|g_{\mathbf{m}}|^2|\mathbf{m}|^{2(r+\tau)}.$$

Since g is C^∞, the series $\sum_{\mathbf{m}\in\mathbb{Z}^n}|g_{\mathbf{m}}|^2|\mathbf{m}|^{2(r+\tau)}$ converges, then $\sum_{\mathbf{m}\in\mathbb{Z}^n}|\mathbf{m}|^{2r}|f_{\mathbf{m}}|^2 < +\infty$ which shows that f is C^∞.

The image of the operator $\delta : C^\infty(\mathbb{T}^n) \longrightarrow C^\infty(\mathbb{T}^n)$ is exactly the subspace $\mathcal{N}$; in fact the restriction of δ to $\mathcal{N}$ is an (algebraic and continuous) isomorphism on $\mathcal{N}$; let G_0 denote its inverse: to g in $\mathcal{N}$ we associate the unique f in $\mathcal{N}$ which is the solution of the equation $\delta f = g$. We then set $G = G_0P$; we verify easily that $\delta G = I - \mathcal{L}$.

Let us show that G is bounded. It is sufficient to prove that G_0 is. The inequality (3) shows that, for any positive s, the operator:

$$G_0 : g \in \mathcal{N} \subset W^{2,s+r} \longmapsto G_0(g) = f \in C^\infty(\mathbb{T}^n) \subset W^{2,s}$$

satisfies the inequality:

$$\|G_0(g)\|_{2,s} \le \beta\|g\|_{2,s+r}$$

where $r = 1+$ (integral part of τ) and β is a positive real constant. This proves that G_0 is bounded.

The fact that the vector space $H^1(\mathbb{Z}, C^\infty(\mathbb{T}^n))$ has dimension 1 and that $\mathcal{D}_\gamma(\mathbb{T}^n)$ is one dimensional generated by the n-form $dx = dx_1 \otimes \cdots \otimes dx_n$ is immediate.

ii) a is Liouville

In this case, there exists $C > 0$ such that, for any $\tau \in \mathbb{N}^*$, there exists $\mathbf{m}_\tau \in \mathbb{Z}^n$ satisfying:

$$\left|1 - e^{2i\pi\langle \mathbf{m}_\tau, a\rangle}\right| \le \frac{C}{|\mathbf{m}_\tau|^\tau}.$$

Let $(\tau_k)_k$ be an increasing sequence in $\mathbb{N}^*$; the corresponding $\mathbf{m}_{\tau_k}$ will be denoted $\mathbf{m}_k$. We define a function g by its Fourier coefficients:

$$g_{\mathbf{m}} = \begin{cases} |\mathbf{m}_k|^{-\tau_k/2} & \text{if } \mathbf{m} = \mathbf{m}_k \\ 0 & \text{if not.} \end{cases} \tag{4}$$

The function g is of class C^∞ and satisfies $\int_{\mathbb{T}^n} g(x)dx = g_{\mathbf{0}} = 0$. But:

$$
\begin{aligned}
|f_{\mathbf{m}_k}|^2 &= \left|\frac{g_{\mathbf{m}_k}}{1-e^{2i\pi\langle \mathbf{m},a\rangle}}\right|^2 \\
&= \frac{|\mathbf{m}_k|^{-\tau_k}}{|1-e^{2i\pi\langle \mathbf{m},\alpha\rangle}|^2} \\
&\geq \frac{1}{C^2}|\mathbf{m}_k|^{\tau_k}.
\end{aligned}
$$

Then the coefficients $f_{\mathbf{m}}$ are not of polynomial growth and even cannot define a distribution f which is solution of equation (1)! By this way, one can construct an infinite family of linearly independent functions $(g^\ell)_{\ell\in\mathbb{N}^*}$ of class C^∞ for which the equation (1) has no solution. Therefore the cokernel of the operator $\delta : C^\infty(\mathbb{T}^n) \longrightarrow C^\infty(\mathbb{T}^n)$ is infinite dimensional.

If g is a trigonometric polynomial whose constant term $g_{\mathbf{0}}$ is zero, the cohomological equation always has a solution: the problem of convergence does not arise. Because the closure of the subspace algebraically generated by these polynomials has codimension 1 (it is the orthogonal subspace of the constant function $\mathbf{1}$), the image of the operator:

$$\delta : C^\infty(\mathbb{T}^n) \longrightarrow C^\infty(\mathbb{T}^n)$$

is not closed, then $H^1(\mathbb{Z}, C^\infty(\mathbb{T}^n))$ is not Hausdorff but its associated Hausdorff space $\overline{H}^1(\mathbb{Z}, C^\infty(\mathbb{T}^n))$ has dimension 1. This shows that the vector space $\mathcal{D}_\gamma(\mathbb{T}^n)$ has dimension 1 generated by the n-form $dx = dx_1 \otimes \cdots \otimes dx_n$. □

The case where the subgroup generated by a is not dense in $\mathbb{T}^n$ was not covered here but it will be contained in Theorem 2.5.

2. The general case

Let G be a compact connected Lie group. Let $a \in G$; so a generates an Abelian subgroup Γ whose closure K is a compact Abelian subgroup of G. Therefore:

(1) - The group K is finite if Γ is already closed in G. (In this case $K = \Gamma$.)
(2) - The group K is an extension of a torus $\mathbb{T}^n$ by a finite group Λ if the subgroup Γ is not closed in G. The torus $\mathbb{T}^n$ is the connected component of the identity element.

We will consider only the case where Λ is the trivial group, that is, K is a connected subgroup of G. A slight modification of our method may probably permit to study the general situation, that is, Λ is nontrivial.

2.1. The group Γ is finite

The principal fibration $\Gamma \hookrightarrow G \xrightarrow{\pi} B = G/K$ is then a covering with group Γ over the compact manifold B. We have a map $\pi_* : C^\infty(G) \longrightarrow C^\infty(B)$ which is linear, continuous and surjective defined by:

$$\pi_*(f) = \sum_{\sigma \in \Gamma} f \circ \sigma. \tag{5}$$

The kernel π_* contains the subspace $\mathcal{C}$ whose elements are functions of the form $f - f \circ \gamma$. In fact, it was proved in [EMM] that the sequence:

$$0 \longrightarrow \mathcal{C} \hookrightarrow C^\infty(G) \xrightarrow{\pi_*} C^\infty(B) \longrightarrow 0 \tag{6}$$

is exact. This clearly shows that the equation $f - f \circ \gamma = g$ has a solution if and only if $\sum_{\sigma \in \Gamma} g \circ \sigma = 0$.

2.2. The group Γ is infinite

In this situation Γ is not closed and is strictly contained in K. As we have said, K is a torus $\mathbb{T}^n$. Its left action on G defines a principal bundle:

$$\mathbb{T}^n \hookrightarrow G \xrightarrow{\pi} B = G/\mathbb{T}^n.$$

For example, if $G = \mathrm{SO}(3)$, γ is a 3×3-matrix and then it defines a rotation with respect to an axis Δ in $\mathbb{R}^3$. Any element of Γ is also a rotation with axis Δ. Hence the group K is the cyclic group $\mathbb{Z}/p\mathbb{Z}$ or the special orthogonal group $\mathrm{SO}(2)$. In the latter case, the manifold $B = G/K$ is the sphere $\mathbb{S}^2$.

Let $U_1, \ldots, U_p$ be a cover of B by open sets all diffeomorphic to $\mathbb{R}^d$ (where d is the dimension of B) and such that, if U is one of them, there exists a diffeomorphism $\Psi : \pi^{-1}(U) \longrightarrow U \times \mathbb{T}^n$ which makes the following diagram commutative:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Psi} & U \times \mathbb{T}^n \\ \pi \downarrow & & \downarrow p_1 \\ U & \xrightarrow{\mathrm{id}} & U. \end{array}$$

(p_1 is the projection on the first factor.) The diffeomorphism Ψ can be constructed as follows. Let $\sigma : U \longrightarrow \pi^{-1}(U)$ be a C^∞ cross section of π; we define $\Psi^{-1} : U \times \mathbb{T}^n \longrightarrow \pi^{-1}(U)$ by $\Psi^{-1}(u, x) = x \cdot \sigma(u)$ where the sign dot is the group multiplication on $\mathbb{T}^n$.

Moreover there exists a continuous function $u \in U \overset{a}{\longmapsto} a(u) = (a_1(u), \ldots, a_n(u)) \in \mathbb{T}^n$ with the $a_1(u), \ldots, a_n(u)$ linearly independent over $\mathbb{Q}$ and such that the action of Γ on the open set $\pi^{-1}(U)$ is equivalent (via Ψ) to the action:

$$(u, x) \in U \times \mathbb{T}^n \longrightarrow (u, x + a(u)) \in U \times \mathbb{T}^n.$$

Remark 2.3 The element $a(u)$ is independent of u. (We denote it a.)

This is a consequence of the choice of the trivialization of π over U and the fact that the left action of γ on G commutes with its right action. Then, the right translations on G are automorphisms of the action of K generated by γ. In other words, the translation in each torus $F_u = \pi^{-1}(u) = \mathbb{T}^n$ is by the vector a. This will enable one to apply Theorem 1.4 in each fibre F_u but independently of $u \in B$.

2.4.

We will put a topology on the space of functions $f : G \longrightarrow \mathbb{C}$ adapted to this fibred structure and which permits the control of their regularity. Let U be one of the open sets $U_1, \ldots, U_p$ and Ψ the associated trivializing diffeomorphism. Let $f : G \longrightarrow \mathbb{C}$ be a measurable function. The restriction of f to $V = \pi^{-1}(U)$ (via the diffeomorphism Ψ) can be viewed as a function $f_U : U \times \mathbb{T}^n \longrightarrow \mathbb{C}$; then we can use the coordinates $(u, x) = (u_1, \ldots, u_d, x_1, \ldots, x_n)$. We suppose that f is square integrable and we consider it as a distribution. Let us fix some notations. If $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ is a multi-index, we set:

(i) $\mathbf{k}^{\mathbf{s}} = k_1^{s_1} \ldots k_d^{s_d}$ for any $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{N}^d$,
(ii) $|\mathbf{k}| = k_1 + \cdots + k_d$ (this is the *length* of $\mathbf{k}$),
(iii) $\partial^{|\mathbf{k}|}/\partial u^{\mathbf{k}} = \partial^{|\mathbf{k}|}/\partial u_1^{k_1} \ldots \partial u_d^{k_d}$.

For fixed $u \in B$ and multi-indices $\mathbf{r} \in \mathbb{N}^d$ and $\mathbf{s} \in \mathbb{N}^n$, the distribution $\partial^{|\mathbf{r}|+|\mathbf{s}|} f/\partial u^{\mathbf{r}} \partial x^{\mathbf{s}}$ on $F_u = \pi^{-1}(u) = \mathbb{T}^n$ admits a Fourier expansion:

$$\frac{\partial^{|\mathbf{r}|+|\mathbf{s}|} f}{\partial u^{\mathbf{r}} \partial x^{\mathbf{s}}}(u,x) = \sum_{\mathbf{m}\in\mathbb{Z}^{\mathbf{n}}} \frac{\partial^{|\mathbf{r}|} f_{\mathbf{m}}}{\partial u^{\mathbf{r}}}(u)(2i\pi)^{|\mathbf{s}|}\mathbf{m}^{\mathbf{s}}\Theta_{\mathbf{m}}$$

Let $r, s \in \mathbb{N}$ and set:

$$\|f\|^U_{r,s} = \Bigg(\sum_{|\mathbf{r}|\le r} \int_U \left| \frac{\partial^{|\mathbf{r}|} f_{\mathbf{0}}}{\partial u^{\mathbf{r}}}(u) \right|^2 du + \sum_{|\mathbf{r}|\le r} \sum_{|\mathbf{s}|\le s} \int_U \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} |\mathbf{m}|^{2|\mathbf{s}|} \left| \frac{\partial^{|\mathbf{r}|} f_{\mathbf{m}}}{\partial u^{\mathbf{r}}}(u) \right|^2 du \Bigg)^{1/2}.$$

(Exceptionally here $|\mathbf{m}|$ denotes the Euclidean norm of the vector $\mathbf{m} \in \mathbb{Z}^n$.) Let $L^2(G)$ be the space of square integrable functions $f : G \longrightarrow \mathbb{C}$. The set of $f \in L^2(G)$ satisfying $\|f\|^U_{r,s} < +\infty$ for any open set U of B and trivializing the fibration π is a vector space $W^{r,s}(G)$ on which $\| \ \|^U_{r,s}$ is a norm. This norm is in fact associated to the Hermitian product:

$$\langle f, g\rangle^U_{r,s} = \sum_{|\mathbf{r}|\le r} \int_U \frac{\partial^{|\mathbf{r}|} f_{\mathbf{0}}}{\partial u^{\mathbf{r}}} \overline{\frac{\partial^{|\mathbf{r}|} g_{\mathbf{0}}}{\partial u^{\mathbf{r}}}} du + \sum_{|\mathbf{r}|\le r} \sum_{|\mathbf{s}|\le s} \int_U \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} |\mathbf{m}|^{2|\mathbf{s}|} \frac{\partial^{|\mathbf{r}|} f_{\mathbf{m}}}{\partial u^{\mathbf{r}}} \overline{\frac{\partial^{|\mathbf{r}|} g_{\mathbf{m}}}{\partial u^{\mathbf{r}}}} du.$$

Let us take again the cover of B by the open sets $U_1, \ldots, U_p$; denote $\| \ \|^{U_i}_{r,s}$ by $\| \ \|^i_{r,s}$. A function $f \in L^2(G)$ is of class C^∞ if and only if $\|f\|^i_{r,s} < +\infty$ for any $i = 1, \ldots, p$ and all $r, s \in \mathbb{N}$.

For any function $f \in L^2(G)$, let $I(f)$ be the function on B defined by:

$$I(f)(u) = \int_{\mathbb{T}^n} f(u, x_1, \ldots, x_n) dx_1 \otimes \cdots \otimes dx_n. \qquad (7)$$

It is easy to see (via Fubini's theorem) that the function $I(f) \in L^2(B)$ and that the map $I : L^2(G) \longrightarrow L^2(B)$ is linear, continuous and surjective. Moreover, it sends the space $C^\infty(G)$ onto the space $C^\infty(B)$.

Theorem 2.5 *Let γ be the diffeomorphism of G associated to a translation. (The notations are the same as before.) Then:*

i) *If a is Diophantine, the equation $f - f \circ \gamma = g$ has a solution $f \in C^\infty(G)$*

if and only if, $I(g) = 0$. In this case, the vector space $H^1(\mathbb{Z}, C^\infty(G))$ is canonically isomorphic to the space $C^\infty(B)$ of C^∞-functions on B.

ii) *Suppose a is Liouville. For each $u \in B$, there exists an infinite family of linearly independent functions $(g^\ell)_{\ell\in\mathbb{N}^*}$ in $C^\infty(G)$ such that $I(g^\ell) = 0$ and for which the cohomological equation $f - f \circ \gamma = g^\ell$ has no solution on the fibre F_u and then it has no solution on the group G.*

In case ii), we can interpret improperly $H^1(\mathbb{Z}, C^\infty(G))$ as a space of 'C^∞-functions on B with values in $H^1(\mathbb{Z}, C^\infty(\mathbb{T}^n))$' even $H^1(\mathbb{Z}, C^\infty(\mathbb{T}^n))$ is non Hausdorff!

Proof. i) The proof of this point consists on two steps. First on each open set $V_i = \pi^{-1}(U_i)$ diffeomorphic to a product $U_i \times \mathbb{T}^n$, and then globally on the group G.

• We identify V_i to $U_i \times \mathbb{T}^n$. Using Fourier coefficients, the equation $f - f \circ \gamma = g$ gives rise to the system:

$$\left(1 - e^{2i\pi\langle \mathbf{m}, a\rangle}\right) f_{\mathbf{m}} = g_{\mathbf{m}} \quad \text{for} \quad \mathbf{m} \in \mathbb{Z}^n.$$

Then, a necessary condition for g to be of the form $f - f \circ \gamma$ is $g_{\mathbf{0}} = 0$, which is just the condition $I(g) = 0$. Suppose it filled. Then we have a formal solution:

$$f_{\mathbf{m}}(u) = \begin{cases} 0 & \text{if } \mathbf{m} = \mathbf{0} \\ \dfrac{g_{\mathbf{m}}(u)}{1 - e^{2i\pi\langle \mathbf{m}, a\rangle}} & \text{if } \mathbf{m} \neq \mathbf{0}. \end{cases}$$

It is easy to prove that these Fourier coefficients define a C^∞-function: It is sufficient to verify that, for $r, s \in \mathbb{N}$ and $i = 1, \ldots, p$, we have $\|f\|^i_{r,s} < +\infty$.

• Let $\{\overline{\rho}_i\}$ be a partition of unity on B subordinated to the open cover $\{U_i\}$. Then $\{\rho_i\}$ where $\rho_i = \overline{\rho}_i \circ \pi$ is a partition of unity on G subordinated to the cover $\{V_i\}$ with ρ_i constant on the fibres of π. For each $i = 1, \ldots, p$, we denote by g_i the restriction of g to the open set V_i; then g_i is a function of class C^∞ on V_i satisfying the condition $I(g_i) = 0$. As was shown above, there exists a C^∞-function f_i on V_i which satisfies $f_i - f_i \circ \gamma = g_i$. Then it is immediate to see that $f = \sum_{i=1}^p \rho_i f_i$ is a C^∞-function on G and is a solution of the equation $f - f \circ \gamma = g$.

ii) Let $u \in B$ and $\ell \in \mathbb{N}^*$. We take $g^\ell_u = g$ the C^∞-function on the torus $\mathbb{T}^n = F_u$ constructed in point ii) in the proof of Theorem 1.4. Let $\overline{\psi} : B \longrightarrow$

$\mathbb{R}_+$ be a function with support in a neighborhood W of u contained in a trivializing open set of π and equal to 1 in a neighborhood $W' \subset W$ of u. Let $\psi = \overline{\psi} \circ \pi$. Then ψ is a C^∞-function on G equal to 1 on a neighborhood of the fibre $F_u = \mathbb{T}^n$; moreover, it is constant on the fibres of π. We set $g^\ell = \psi g_u^\ell$; this is a C^∞-function on G such that $I(g^\ell) = 0$ but which is not a solution of the cohomological equation $f - f \circ \gamma = g^\ell$. By varying ℓ in $\mathbb{N}^*$, we obtain the desired sequence $(g^\ell)_\ell$. □

Remark 2.6 We have restricted our attention to compact Lie groups to simplify the presentation. But the proofs can be adapted to any Lie group G and almost without any change if, in addition, G is exponential.

2.7. What about continuous functions?

We have seen that the regularity of the solutions of the cohomological equation (1) depends on the arithmetic nature of the vector a. What happens, instead of working on $C^\infty(G)$, we consider the functions that are only continuous? The situation is different. For instance, the following result was proved in [MS]:

Let $C_0^{bv}(\mathbb{S}^1)$ be the Banach space of continuous functions with bounded variation and which integrates to 0 on the circle $G = \mathbb{S}^1$. Let γ be an irrational rotation of $\mathbb{S}^1$. Then, there exists a residual set $\mathcal{R} \subset C_0^{bv}(\mathbb{S}^1)$ such that, for any $g \in \mathcal{R}$, there is no continuous function f on $\mathbb{S}^1$ satisfying the cohomological equation $f - f \circ \gamma = g$.

3. Affine automorphism of a torus $\mathbb{T}^n$

In this section the group G will be the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and γ the affine automorphism $\gamma(x) = A(x + a)$ where A is a matrix in $\mathrm{GL}(n, \mathbb{Z})$, that is, A is with integer coefficients and its determinant is equal to 1 or -1; $a = (a_1, \ldots, a_n)$ is an element of $\mathbb{T}^n$ (viewed as a vector of $\mathbb{R}^n$).

3.1. Notations

We will fix few notations we shall use in this subsection. This will allow us to work smoother and easier to understand the different steps of the calculation.

The diffeomorphism γ is an affine automorphism of $\mathbb{T}^n$ whose direction is given by the matrix A. Let A' be the transpose of A and B the inverse of A'. For any $k \in \mathbb{Z}$, B^k will be the $|k|^{\text{th}}$ power of B if $k \geq 0$ and of B^{-1}

if $k < 0$. We define the sequence of matrices (S_k) indexed by $\mathbb{Z}$ as follows:

$$\begin{cases} S_0 = 0 \\ S_{k+1} = S_k B + I. \end{cases} \tag{8}$$

Such sequence is easy to construct by induction. We set:

$$a^{\perp} = \{\mathbf{m} \in \mathbb{Z}^n : \langle \mathbf{m}, a \rangle = 0\} \quad \text{and} \quad F_B = \{\mathbf{m} \in \mathbb{Z}^n : B(\mathbf{m}) = \mathbf{m}\}.$$

Then $a^{\perp}$ and F_B are subgroups of $\mathbb{Z}^n$. From now on we will suppose that the following hypothesis is satisfied:

The matrix B has no periodic vector $\mathbf{m} \in \mathbb{Z}^n$ of period $q \in \mathbb{N} \setminus \{0, 1\}$, that is, no vector $\mathbf{m} \in \mathbb{Z}^n$ satisfies $B^q(\mathbf{m}) = \mathbf{m}$.

3.2.

An immediate computation shows that the action of γ on a function $f : \mathbb{T}^n \longrightarrow \mathbb{C}$ is given as follows:

$$f \circ \gamma = \sum_{\mathbf{m} \in \mathbb{Z}^n} e^{2i\pi \langle A'(\mathbf{m}), a \rangle} f_{\mathbf{m}} \Theta_{A'(\mathbf{m})} = \sum_{\mathbf{m} \in \mathbb{Z}^n} e^{2i\pi \langle \mathbf{m}, a \rangle} f_{B(\mathbf{m})} \Theta_{\mathbf{m}}.$$

Then the cohomological equation (1) gives the system:

$$f_{\mathbf{m}} - e^{2i\pi \langle \mathbf{m}, a \rangle} f_{B(\mathbf{m})} = g_{\mathbf{m}} \quad \text{pour} \quad \mathbf{m} \in \mathbb{Z}^n. \tag{9}$$

So it is easy to see that the kernel L of the operator δ is generated by $\Theta_{\mathbf{m}}$ with $\mathbf{m}$ varying in the subgroup $a^{\perp} \cap F_B$ that is:

$$L = \{ f \in C^{\infty}(\mathbb{T}^n) : f_{\mathbf{m}} = 0 \text{ for } \mathbf{m} \notin a^{\perp} \cap F_B \}. \tag{10}$$

Clearly we have $L \subset \operatorname{Ker}(\delta)$. Let us prove the inclusion $\operatorname{Ker}(\delta) \subset L$. Recall that:

$$\delta(f)_{\mathbf{m}} = f_{\mathbf{m}} - e^{2i\pi \langle \mathbf{m}, a \rangle} f_{B(\mathbf{m})}.$$

Thus if $f \in \operatorname{Ker}(\delta)$ and if $\mathbf{m} \in F_B \setminus a^{\perp}$, then it is easy to deduce that $f_{\mathbf{m}} = 0$. Consider the case where $\mathbf{m} \notin F_B$. Then if $\delta(f) = 0$, we have $|f_{\mathbf{m}}| = |f_{B(\mathbf{m})}|$. Thus nonvanishing of $f_{\mathbf{m}}$ contradicts the rapid decrease of the coefficients $f_{\mathbf{m}}$ by Proposition 1.1. Finally $\operatorname{Ker}(\delta) = L$.

A necessary condition for the equation (9) to admit a solution is then $g_{\mathbf{m}} = 0$ for $\mathbf{m}$ in $a^{\perp} \cap F_B$. Let V be the subspace of $C^{\infty}(\mathbb{T}^n)$ defined by:

$$V = \big\{ f \in C^{\infty}(\mathbb{T}^n) : f_{\mathbf{m}} = 0 \text{ for } \mathbf{m} \in a^{\perp} \cap F_B \big\}.$$

The operator $\delta : C^{\infty}(\mathbb{T}^n) \longrightarrow C^{\infty}(\mathbb{T}^n)$ preserves each factor of the decomposition:

$$C^{\infty}(\mathbb{T}^n) = V \oplus L. \tag{11}$$

Indeed, the subspace L, being the kernel of δ, is of course left invariant by δ. On the other hand, if $f \in V$, then $f_{\mathbf{m}} = 0$ for any $\mathbf{m} \in a^{\perp} \cap F_B$. Since $B(\mathbf{m}) = \mathbf{m}$, this implies $\delta(f)_{\mathbf{m}} = f_{\mathbf{m}} - e^{2i\pi\langle \mathbf{m}, a\rangle} f_{B(\mathbf{m})} = f_{\mathbf{m}} - f_{\mathbf{m}} = 0$, showing that $\delta(f) \in V$.

We can then restrict ourselves to the study of the operator δ on the subspace V. For $g \in V$ given, the system (9) has a priori two formal solutions:

$$f^{+}_{\mathbf{m}} = \begin{cases} 0 & \text{if } \mathbf{m} \in a^{\perp} \cap F_B \\ \dfrac{g_{\mathbf{m}}}{1 - e^{2i\pi\langle \mathbf{m}, a\rangle}} & \text{if } \mathbf{m} \in (a^{\perp})^c \cap F_B \\ f_{\mathbf{m}} = \displaystyle\sum_{k=0}^{\infty} e^{2i\pi\langle S_k(\mathbf{m}), a\rangle} g_{B^k(\mathbf{m})} & \text{if not.} \end{cases} \tag{12}$$

or:

$$f^{-}_{\mathbf{m}} = \begin{cases} 0 & \text{if } \mathbf{m} \in a^{\perp} \cap F_B \\ \dfrac{g_{\mathbf{m}}}{1 - e^{2i\pi\langle \mathbf{m}, a\rangle}} & \text{if } \mathbf{m} \in \left(a^{\perp}\right)^c \cap F_B \\ f_{\mathbf{m}} = -\displaystyle\sum_{k=-1}^{-\infty} e^{2i\pi\langle S_k(\mathbf{m}), a\rangle} g_{B^k(\mathbf{m})} & \text{if not.} \end{cases} \tag{12$'$}$$

The injectivity of δ forces these two solutions to coincide, which then imposes the condition:

$$\sum_{k \in \mathbb{Z}} e^{2i\pi\langle S_k(\mathbf{m}), a\rangle} g_{B^k(\mathbf{m})} = 0.$$

We will see that this condition does nothing but impose to γ-invariant distributions to be zero on the function g.

3.3.

We have already noted that the vanishing of the γ-invariant distributions on a function $g \in \mathbb{T}^n$ is a necessary condition for the cohomological equation (1) to admit a solution for the given g. So it seems natural to determine such distributions. This is what we shall do in this subsection.

Let Σ_0 be a subset of $\mathbb{Z}^n$ containing one and only one element of each orbit under the action of B on $\mathbb{Z}^n \setminus F_B$. We set $\Sigma = (a^\perp \cap F_B) \cup (\Sigma_0 \setminus F_B)$. For any $\mathbf{m} \in \Sigma$, we denote by $T_\mathbf{m}$ the linear functional $C^\infty(\mathbb{T}^n) \longrightarrow \mathbb{C}$ defined by:

$$\langle T_\mathbf{m}, g\rangle = \begin{cases} \displaystyle\int_{\mathbb{T}^n} \overline{\Theta}_\mathbf{m}(x) g(x) dx = g_\mathbf{m} & \text{for } \mathbf{m} \in a^\perp \cap F_B \\ \displaystyle\sum_{k\in\mathbb{Z}} e^{2i\pi\langle S_k(\mathbf{m}),a\rangle} g_{B^k(\mathbf{m})} & \text{for } \mathbf{m} \in \Sigma_0 \setminus F_B \end{cases} \tag{13}$$

where $\overline{\Theta}_\mathbf{m}(x) = e^{-2i\pi\langle \mathbf{m},x\rangle}$. An immediate computation shows that $T_\mathbf{m}$ is continuous and verifies $\langle T_\mathbf{m}, f\circ\gamma\rangle = \langle T_\mathbf{m}, f\rangle$; so it is a γ-invariant distribution on $\mathbb{T}^n$.

Before proceeding we give a concrete example on which we can see the whole Σ and the nature of invariant distributions. We take $n = 2$. Let $a = (a_1, 0)$ where a_1 is a non rational real number in $]0,1[$ and $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1\end{smallmatrix}\right)$. Then:

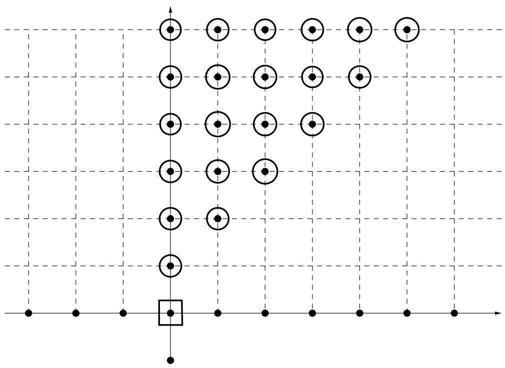
$$A' = \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}, \quad \text{and} \quad B^k = \begin{pmatrix} 1 & -k \\ 0 & 1\end{pmatrix} \quad \text{for } k \in \mathbb{Z}.$$

It is easy to see that:

$$a^\perp = \{(0, m_2) \in \mathbb{Z}^2 : m_2 \in \mathbb{Z}\} \quad \text{and} \quad F_B = \{(m_1, 0) \in \mathbb{Z}^2 : m_1 \in \mathbb{Z}\}.$$

The matrix B acts on the lattice $\mathbb{Z}^2$. The equivalence of two elements (m_1, m_2) and (m_1', m_2') is given as follows: $(m_1, 0) \sim (m_1', 0) \iff m_1 = m_1'$; for m_2 and m_2', both of them different from 0:

$$(m_1, m_2) \sim (m_1', m_2') \iff m_2 = m_2' \text{ and } m_1 - m_1' \in m_2\mathbb{Z}.$$



Points of the set Σ which indexes the invariant distributions: $a^\perp \cap F_B$ is reduced to $(0,0)$ and represents the only invariant measure which is the Lebesgue measure $dx_1 \otimes dx_2$; points in appearance circled represent the other invariant distributions (which are only of order 1).

Denote by $N_{\mathbf{m}}$ the kernel of $T_{\mathbf{m}}$ and by $\mathcal{N}$ the intersection of all $N_{\mathbf{m}}$; $\mathcal{N}$ is a closed subspace of $C^\infty(\mathbb{T}^n)$. Finally, we have all the ingredients needed to show that, for a trigonometric polynomial g in the space $\mathcal{N}$, the numbers $f_{\mathbf{m}}$ given by the expressions (12) or (12′) are the Fourier coefficients of a function f of class C^∞ solution of the equation $\delta f = g$. We have then proved the:

Theorem 3.4 *The closure of the image of the operator $\delta : C^\infty(\mathbb{T}^n) \longrightarrow C^\infty(\mathbb{T}^n)$ is equal to the subspace $\mathcal{N}$. This shows that the first reduced cohomology topological vector space $\overline{H}^1(\mathbb{Z}, C^\infty(G))$ is isomorphic to $C^\infty(\mathbb{T}^n)/\mathcal{N}$ and that the space $\mathcal{D}_\gamma$ of γ-invariant distributions is generated by the $T_{\mathbf{m}}$.*

4. Application to the deformations of some foliations

The vector space $H^1(\mathbb{Z}, C^\infty(M))$ associated to an action of the discrete group $\mathbb{Z}$ on a compact manifold M via a diffeomorphism γ plays a fundamental role in the theory of deformations: it is related to the *infinitesimal deformations* of γ in the diffeomorphism group $\mathrm{Diff}(M)$ of M and also to that of the foliation obtained by suspension of γ. We shall apply our computations to describe such infinitesimal deformations in the case of a translation on a non Abelian compact Lie G.

4.1. Some definitions

• Recall that a *foliation* $\mathcal{F}$ of dimension m on a manifold N is given by a subbundle τ of rank m of the tangent bundle TN which is *completely integrable*, that is, for two arbitrary sections $X, Y \in C^\infty(\tau)$ of τ (vector fields tangent to τ), the bracket $[X, Y]$ is also a section of τ. A connected submanifold tangent to τ is called a *leaf* of $\mathcal{F}$.

• Let N be a compact manifold of dimension $m + n$. For any $x \in N$, we denote $G_x(N, n)$ the Grassmannian of planes of codimension n of T_xN. We obtain a locally trivial fibre bundle $\mathcal{G}(N, n) \longrightarrow N$ whose typical fibre is the Grassmannian $G(m+n, n)$ of the vector space $\mathbb{R}^{m+n}$. A C^∞-*field* (or just a *field*) of m-planes on N is nothing but a section of $\mathcal{G}(M, n) \longrightarrow M$. Let τ be a field of m-planes and $(\tau_1, \ldots, \tau_m)$ a basis of local sections of τ. If $X = \sum_{i=1}^m a_i\tau_i$ and $Y = \sum_{j=1}^m b_j\tau_j$ are two local sections of τ, we have:

$$[X, Y] = \sum_{i,j=1}^{m} a_ib_j[\tau_i, \tau_j] + \sum_{i,j=1}^{m} \big\{a_i(\tau_i \cdot b_j)\tau_j - b_j(\tau_j \cdot a_i)\tau_i\big\}. \tag{14}$$

In the quotient $\mathcal{V} = TN/\tau$, the value $[X, Y]$ at a point $x \in N$ depends only on the values of X and Y at this point and not on the values of their derivatives. This gives rise to a 2-form $Q_\tau : \tau \times \tau \longrightarrow \mathcal{V}$ whose value at X_x and Y_x is the class in the quotient $\mathcal{V}_x = T_xN/\tau_x$ of the vector $[X, Y]_x$. By Frobenius theorem, the field of m-planes τ is integrable if and only if the 2-form Q_τ is identically zero. In this case, τ defines a foliation $\mathcal{F}$ of codimension n on N.

The space $C^\infty(\mathcal{G}(N, n))$ of C^∞-sections of $\mathcal{G}(N, n)$ equipped with the C^∞-topology is a Fréchet manifold [Ham2]. The subset $\mathfrak{F}(M, n)$ of foliations of codimension n on N ("zero set of Q") is closed; we equip it with the induced topology.

• A *deformation* of $\mathcal{F}$ parameterized by an open neighborhood T of 0 in the Euclidean space $\mathbb{R}^d$ is a continuous map $t \in T \longrightarrow \mathcal{F}_t \in \mathfrak{F}(N, n)$ such that $\mathcal{F}_0 = \mathcal{F}$. The study of deformations of foliations is a very hard subject in general. We shall focus our attention just on *infinitesimal deformations* which are described exactly by the elements of the group $H^1_{\mathcal{F}}(N, \mathcal{V})$ of *foliated cohomology* with values in the normal bundle $\mathcal{V} = TN/T\mathcal{F}$ which we shall introduce below (for more details see [Ha1] or [EN]).

• For any $r \in \mathbb{N}$, we denote $\Lambda^r(T^*\mathcal{F})$ the bundle of exterior algebras of degree r over $T\mathcal{F}$ (tangent bundle to $\mathcal{F}$). Its sections are the *foliated forms* of degree r; they form a vector space $\Omega^r_{\mathcal{F}}(N)$. We have an operator along the leaves $d_{\mathcal{F}} : \Omega^r_{\mathcal{F}}(N) \longrightarrow \Omega^{r+1}_{\mathcal{F}}(N)$ defined (as in the classical case) by the formula:

$$\begin{aligned} d_{\mathcal{F}}\alpha(X_1,\ldots,X_{r+1}) &= \sum_{i=1}^{r+1}(-1)^i X_i \cdot \alpha(X_1,\ldots,\widehat{X}_i,\ldots,X_{r+1}) \\ &\quad + \sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_1,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_{r+1}) \end{aligned}$$

where $\widehat{X}_i$ means that the argument X_i is omitted. We easily verify that $d^2_{\mathcal{F}} = 0$. So we obtain a differential complex (called the *de Rham foliated complex* of $\mathcal{F}$):

$$0 \longrightarrow \Omega^0_{\mathcal{F}}(N) \xrightarrow{d_{\mathcal{F}}} \Omega^1_{\mathcal{F}}(N) \xrightarrow{d_{\mathcal{F}}} \cdots \xrightarrow{d_{\mathcal{F}}} \Omega^{m-1}_{\mathcal{F}}(N) \xrightarrow{d_{\mathcal{F}}} \Omega^m_{\mathcal{F}}(N) \longrightarrow 0. \quad (15)$$

Let $Z^r_{\mathcal{F}}(N)$ be the kernel of $d_{\mathcal{F}} : \Omega^r_{\mathcal{F}}(N) \longrightarrow \Omega^{r+1}_{\mathcal{F}}(N)$ and $B^r_{\mathcal{F}}(N)$ the image of $d_{\mathcal{F}} : \Omega^{r-1}_{\mathcal{F}}(N) \longrightarrow \Omega^r_{\mathcal{F}}(N)$. The quotient $H^r_{\mathcal{F}}(N) = Z^r_{\mathcal{F}}(N)/B^r_{\mathcal{F}}(N)$ is the r^{th} vector space of *foliated cohomology* of $(N,\mathcal{F})$.

If X is a non singular vector field on N, it induces a foliation (or a flow) $\mathcal{F}$. One can define more simply its foliated cohomology. Let τ be the tangent bundle to $\mathcal{F}$ and ν a complement of τ in TN. Let χ be the differential 1-form such $\chi(X) = 1$ and $\chi_{|\nu} = 0$. Then it is easy to see that, for any $r \in \mathbb{N}$, we have:

$$\Omega^r_{\mathcal{F}}(N) = \begin{cases} C^\infty(N) & \text{if } r = 0 \\ C^\infty(N) \otimes \chi & \text{if } r = 1 \\ 0 & \text{if } r \geq 2 \end{cases}$$

and that the foliated complex is reduced to:

$$0 \longrightarrow \Omega^0_{\mathcal{F}}(N) \xrightarrow{d_X} \Omega^1_{\mathcal{F}}(N) \longrightarrow 0 \quad (16)$$

where d_X is the operator defined by $d_X f = (X \cdot f) \otimes \chi$. Its cokernel $\Omega^1_{\mathcal{F}}(N)/\mathrm{Im} d_X$ is exactly the first vector space $H^1_{\mathcal{F}}(N)$ of foliated cohomology of the foliation $\mathcal{F}$. It depends only on the foliation, not on the vector field X: one can easily verify, by exhibiting an explicit isomorphism of foliated complexes, that we obtain the same foliated cohomology if we consider the vector field $Z = hX$ with h any nonvanishing function. One can prove also that it does not depend on the choice of the complement ν.

• By the definition of a foliation the normal bundle $\mathcal{V}$ is a *foliated bundle*, that is, there exists an open cover $\{U_i\}_i$ trivializing the foliation and such that the transition functions $g_{ij} : U_i \cap U_j \longrightarrow \mathrm{GL}(n, \mathbb{R})$ which define $\mathcal{V}$ are constant on the leaves. Then $d_{\mathcal{F}}$ extends to an operator $d_{\mathcal{F}} : \Omega^r_{\mathcal{F}}(N, \mathcal{V}) \longrightarrow \Omega^{r+1}_{\mathcal{F}}(N, \mathcal{V})$ on the foliated forms with values in $\mathcal{V}$. This permits to define the foliated cohomology $H^*_{\mathcal{F}}(N, \mathcal{V})$ of $(N, \mathcal{F})$ with values in $\mathcal{V}$. The vector space $H^1_{\mathcal{F}}(N, \mathcal{V})$ contains exactly the infinitesimal deformations of $\mathcal{F}$ (see [Ha1]); its study is fundamental in deformation theory.

4.2. The case of a suspension

Let M be a (connected) compact manifold and γ a diffeomorphism of M. We denote by (x, t) the coordinates of a point z of $\widetilde{N} = M \times \mathbb{R}$ and $\widetilde{X}$ the vector field $\frac{\partial}{\partial t}$; $\widetilde{X}$ is invariant by the diffeomorphism $(x, t) \in M \times \mathbb{R} \longmapsto (\gamma(x), t + 1) \in M \times \mathbb{R}$ and then induces a non singular vector field X on the quotient manifold $N = M \times \mathbb{R}/(x, t) \simeq (\gamma(x), t + 1)$. The second projection $\widetilde{\pi} : \widetilde{N} = M \times \mathbb{R} \longrightarrow \mathbb{R}$ is equivariant under the two actions of the group $\mathbb{Z}$: $\tau_k : t \in \mathbb{R} \longrightarrow t + k \in \mathbb{R}$ and $(\gamma^k, \tau_k) : (x, t) \in \widetilde{N} \longrightarrow (\gamma^k(x), t + k) \in \widetilde{N}$; this means that, for any $k \in \mathbb{Z}$, the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{(\gamma^k, \tau_k)} & \widetilde{N} \\ {\scriptstyle \widetilde{\pi}} \downarrow & & \downarrow {\scriptstyle \widetilde{\pi}} \\ \mathbb{R} & \xrightarrow{\tau_k} & \mathbb{R} \end{array}$$

So $\widetilde{\pi}$ induces a submersion $\pi : N \longrightarrow \mathbb{S}^1$; it is a flat fibration with monodromy γ. Let $\mathcal{F}$ be the flow (or foliation of dimension 1) defined by X; we say that $(N, \mathcal{F})$ is the *suspension* of (M, γ). The following result was established in [DE]:

Theorem 4.3 *The differential equation $X \cdot f = g$ has a solution in $C^\infty(N)$ if, and only if, the cohomological equation $K - K \circ \gamma = \Phi$ has a solution in $C^\infty(M)$ for the function $\Phi = \int_0^1 g(\cdot, t)dt$. Then the two topological vector spaces $H^1_{\mathcal{F}}(N)$ and $H^1(\mathbb{Z}, C^\infty(M))$ are canonically isomorphic.*

We now have all the necessary ingredients that allow us to apply the results of the previous sections to deformations of certain foliations. Let G be a compact connected Lie group of dimension n with Lie algebra $\mathcal{G}$ and γ the diffeomorphism of G associated to a translation.

Theorem 4.4 *Let $\mathcal{F}$ be the foliation obtained on $N = G \times \mathbb{R}/(x,t) \simeq (\gamma x, t+1)$ by suspension of γ. Then, if γ is Diophantine, the topological vector space $H^1_{\mathcal{F}}(N,\mathcal{V})$ is isomorphic to the Fréchet space $C^\infty(B)\otimes\mathcal{G}$ where B is the homogeneous space $B = G/K$.*

Proof. It simply consists to get into some assumptions statements of Sections 1, 2 and 3.

Let $(X_1,\dots,X_n)$ be a basis of $\mathcal{G}$ where the $X_1,\dots,X_n$ are left invariant vector fields on G. The elements ω and α respectively of $\Omega^0_{\mathcal{F}}(N,\mathcal{V})$ and $\Omega^1_{\mathcal{F}}(N,\mathcal{V})$ are of the form:

$$\omega = \sum_{i=1}^{n} f_i \otimes X_i \quad \text{and} \quad \alpha = \sum_{j=1}^{n} g_j \otimes \chi \otimes X_j$$

where the f_i and the g_j are C^∞-functions on N. The operator $d_{\mathcal{F}} : \Omega^0_{\mathcal{F}}(N,\mathcal{V}) \longrightarrow \Omega^1_{\mathcal{F}}(N,\mathcal{V})$ can be written $d_{\mathcal{F}}\omega = \sum_{i=1}^{n}(X\cdot f_i)\otimes\chi\otimes X_i$. Then, solving $d_{\mathcal{F}}\omega = \alpha$ is equivalent to solve the system:

$$X\cdot f_i = g_i \quad \text{for} \quad i = 1,\dots,n$$

from which we deduce that $H^1_{\mathcal{F}}(N,\mathcal{V}) = H^1_{\mathcal{F}}(N)\otimes\mathcal{G}$. The remaining part of the proof results from Theorem 3.4 and Theorem 4.4. □

If G is non Abelian, the homogeneous space B is a manifold of positive dimension. Then Theorem 4.5 shows that the vector space $H^1_{\mathcal{F}}(N,\mathcal{V})$ is infinite dimensional. This suggests that the deformations of the foliation $\mathcal{F}$ abound.

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