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# Biharmonic maps into symmetric spaces and integrable systems

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**Abstract.** In this paper, the formulation of the biharmonic map equation in terms of the Maurer-Cartan form for all smooth maps of a compact Riemannian manifold into a Riemannian symmetric space $(G/K,h)$ induced from the bi-invariant Riemannian metric $h$ on $G$ is obtained. Using this, all the biharmonic curves into symmetric spaces are determined, and all the biharmonic maps of an open domain of $\mathbb{R}^2$ with the standard Riemannian metric into $(G/K,h)$ are characterized exactly.

*Key words*: harmonic map, biharmonic map, symmetric space, integrable system, Maurer-Cartan form.

## 1. Introduction and statement of results

This paper is a continuation of our previous one [16]. In our previous paper, we discussed the description of biharmonic maps into compact Lie groups in terms of the Maurer-Cartan form, and gave their exact constructions. In this paper, we consider biharmonic maps into Riemannian symmetric spaces.

The theory of harmonic maps into Lie groups, symmetric spaces or homogeneous spaces has been extensively studied in connection with the integrable systems by many authors (for instance, [1], [3], [4], [10], [11], [12], [14], [15], [18], [19]). Let us recall the loop group formulation of harmonic maps into symmetric spaces, briefly. Let $\varphi$ be a smooth map of a Riemann surface $M$ into a Riemannian symmetric space $(G/K,h)$ with a lift $\psi$ : $M \to G$ so that $\pi \circ \psi = \varphi$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of the Lie group $G$. Then, the pull back $\alpha = \psi^{-1}d\psi$ of the Maurer-Cartan form on $G$ is decomposed as $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$, correspondingly. Let us decompose $\alpha_{\mathfrak{m}}$ into the sum of the holomorphic part and the anti-holomorphic one: $\alpha_{\mathfrak{m}} = \alpha_{\mathfrak{m}}{}' + \alpha_{\mathfrak{m}}{}''$. Then, one can obtain the extended solution $\widetilde{\psi}$ of $M$ into a loop group $\Lambda G$ satisfying $\widetilde{\psi}^{-1}d\widetilde{\psi} = \lambda\alpha_{\mathfrak{m}}{}' + \alpha_{\mathfrak{k}} + \lambda^{-1}\alpha_{\mathfrak{m}}{}''$ for all $\lambda \in U(1) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ (cf. [4]). Then,

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$\varphi : M \to (G/K, h)$ is harmonic if and only if there exists a holomorphic and horizontal map $\widetilde{\psi}$ of $M$ into the homogeneous $\Lambda G/K$ with $\widetilde{\psi}_1 = \psi$ (cf. [4, p. 648]). Then, one can obtain a Weierstrass-type representation of harmonic maps (cf. [4, pp. 648–662]).

On the other hand, the notion of harmonic map has been extended to the one of biharmonic map (cf. [5], [8]). In this paper, we will describe biharmonic maps into Riemannian symmetric spaces in terms of the pull back $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$ of the Maurer-Cartan form (cf. Theorem 3.6), give some explicit solutions of the biharmonic map equation in Riemannian symmetric spaces, and construct several biharmonic maps into Riemannian symmetric spaces (Sections 4 and 5).

## 2. Preliminaries

In this section, we prepare general materials and facts on harmonic maps, biharmonic maps into Riemannian symmetric spaces (cf. [9]).

### 2.1.

Let $(M, g)$ be an $m$-dimensional compact Riemannian manifold, and the target space $(N, h)$, an $n$-dimensional Riemannian symmetric space $(G/K, h)$. Nemely, let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of $G$, $K$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the Cartan decomposition of $\mathfrak{g}$, and $h$, the $G$-invariant Riemannian metric on $G/K$ corresponding to the $\mathrm{Ad}(K)$-invariant inner product $\langle\, ,\, \rangle$ on $\mathfrak{m}$. Let $k$ be a left invariant Riemannian metric on $G$ such as the natural projection $\pi : G \to G/K$ is a Riemannian submersion of $(G, k)$ onto $(G/K, h)$. For every $C^\infty$ map $\varphi$ of $M$ into $G/K$, let us take its (local) *lift* $\psi : M \to G$ of $\varphi$, i.e., $\varphi = \pi \circ \psi$, $\varphi(x) = \psi(x)\, K \in G/K$ $(x \in U \subset M)$, where $U$ is an open subset of $M$.

The *energy functional* on the space $C^\infty(M, G/K)$ of all $C^\infty$ maps of $M$ into $G/K$ is defined by

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g,$$

and for a $C^\infty$ one parameter deformation $\varphi_t \in C^\infty(M, G/K)$ $(-\epsilon < t < \epsilon)$ of $\varphi$ with $\varphi_0 = \varphi$, the *first variation formula* is given by

$$\frac{d}{dt}\bigg|_{t=0} E(\varphi_t) = -\int_M \langle \tau(\varphi), V \rangle v_g,$$

where $V$ is a variation vector field along $\varphi$ defined by $V = (d/dt)|_{t=0}\varphi_t$ which belongs to the space $\Gamma(\varphi^{-1}T(G/K))$ of sections of the induced bundle of the tangent bundle $T(G/K)$ by $\varphi$. The *tension field* $\tau(\varphi)$ is defined by

$$\tau(\varphi) = \sum_{i=1}^{m} B(\varphi)(e_i, e_i), \tag{2.1}$$

where

$$B(\varphi)(X, Y) = \nabla^h_{d\varphi(X)} d\varphi(Y) - d\varphi(\nabla_X Y)$$

for $X, Y \in \mathfrak{X}(M)$. Here, $\nabla$, and $\nabla^h$, are the Levi-Civita connections of $(M, g)$ and $(G/K, h)$, respectively. For a harmonic map $\varphi : (M, g) \to (G/K, h)$, the *second variation formula* of the energy functional $E(\varphi)$ is

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\varphi_t) = \int_M \langle J(V), V\rangle v_g$$

where

$$J(V) := \overline{\Delta} V - \mathcal{R}(V), \tag{2.2}$$

$$\overline{\Delta} V := \overline{\overline{\nabla}}^* \, \overline{\overline{\nabla}} V = -\sum_{i=1}^{m} \left\{ \overline{\overline{\nabla}}_{e_i} (\overline{\overline{\nabla}}_{e_i} V) - \overline{\overline{\nabla}}_{\nabla_{e_i} e_i} V \right\}, \tag{2.3}$$

$$\mathcal{R}(V) := \sum_{i=1}^{m} R^h(V, d\varphi(e_i)) d\varphi(e_i). \tag{2.4}$$

Here, $\overline{\overline{\nabla}}$ is the induced connection on the induced bundle $\varphi^{-1}T(G/K)$, and is $R^h$ is the curvature tensor of $(G/K, h)$ given by $R^h(U, V)W = [\nabla^h_U, \nabla^h_V]W - \nabla^h_{[U,V]} W$ $(U, V, W \in \mathfrak{X}(G/K))$.

The *bienergy functional* is defined by

$$E_2(\varphi) = \frac{1}{2} \int_M |(d + \delta)^2 \varphi|^2 v_g = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g, \tag{2.5}$$

and the *first variation formula* of the bienergy is given (cf. [8]) by

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = -\int_M \langle \tau_2(\varphi), V\rangle v_g \tag{2.6}$$

where the *bitension field* $\tau_2(\varphi)$ is defined by

$$\tau_2(\varphi) = J(\tau(\varphi)) = \overline{\Delta}\tau(\varphi) - \mathcal{R}(\tau(\varphi)), \tag{2.7}$$

and a $C^\infty$ map $\varphi : (M, g) \to (G/K, h)$ is said to be *biharmonic* if

$$\tau_2(\varphi) = 0. \tag{2.8}$$

**2.2.**

Let $k$ be a left invariant Riemannian metric on $G$ corresponding to the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ given by $\langle\cdot,\cdot\rangle = -B(\cdot,\cdot)$ if $(G/K, h)$ is of compact type, and by $\langle U+X, V+Y\rangle = -B(U,V)+B(X,Y)$ $(U, V \in \mathfrak{k},\ X, Y \in \mathfrak{m})$ if $(G/K, h)$ is of non-compact type. Here, $B(\cdot,\cdot)$ is the Killing form of $\mathfrak{g}$. Then, the projection $\pi$ of $G$ onto $G/K$ is a Riemannian submersion of $(G, k)$ onto $(G/K, h)$, and we have also the orthogonal decomposition of the tangent space $T_{\psi(x)}G$ $(x \in M)$ with respect to the inner product $k_{\psi(x)}(\cdot,\cdot)$ $(x \in M)$ in such a way that

$$T_{\psi(x)}G = V_{\psi(x)} \oplus H_{\psi(x)}, \tag{2.9}$$

where the *vertical space* at $\psi(x) \in G$ is given by

$$V_{\psi(x)} = \mathrm{Ker}(\pi_{*\,\psi(x)}) = \{X_{\psi(x)} \mid X \in \mathfrak{k}\}, \tag{2.10}$$

and the *horizontal space* at $\psi(x)$ is given by

$$H_{\psi(x)} = \{Y_{\psi(x)} \mid Y \in \mathfrak{m}\}, \tag{2.11}$$

corresponding to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then, for every $C^\infty$ section $W \in \Gamma(\psi^{-1}TG)$, we have the decomposition corresponding to (2.9),

$$W(x) = W^V(x) + W^H(x) \quad (x \in M), \tag{2.12}$$

where $W^V$, $W^H$, (denoted also by $\mathcal{V}W$, $\mathcal{H}W$, respectively) belong to $\Gamma(\psi^{-1}TG)$. We denote by $\Gamma(E)$, the space of all $C^\infty$ sections of a vector bundle $E$. For $Y \in \mathfrak{m}$, define $\widetilde{Y} \in \Gamma(\psi^{-1}TG)$ by $\widetilde{Y}(x) := Y_{\psi(x)}$ $(x \in M)$. Let $\{X_i\}_{i=1}^n$ be an orthonormal basis of $\mathfrak{m}$ with respect to the inner product

$\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ corresponding to the left invariant Riemannian metric $k$ on $G$. Then, $W^H$ can be written in terms of $\widetilde{X_i}$ as

$$W^H = \sum_{i=1}^{n} f_i \, \widetilde{X_i}$$

where $f_i \in C^\infty(M)$ $(i = 1, \ldots, n)$. Because, for every $x \in M$, $W^H(x) \in H_{\psi(x)}$, so that we have

$$W^H(x) = \sum_{i=1}^{n} f_i(x) X_{i\,\psi(x)} = \sum_{i=1}^{n} f_i(x) \widetilde{X_i}(x).$$

We say $W \in \Gamma(\psi^{-1}TG)$ and $V \in \Gamma(\varphi^{-1}T(G/K))$ are $\pi$-*related*, denoted by $V = \pi_* W$, if it holds that

$$V(x) = \pi_* W(x) \quad (x \in M),$$

where $\pi_* : T_{\psi(x)}G \to T_{\varphi(x)}(G/K) = T_{\pi(\psi(x))}(G/K)$ is the differentiation of the projection $\pi$ of $G$ onto $G/K$ at $\psi(x)$ for each $x \in M$.

Let be $\nabla$, $\nabla^k$, $\nabla^h$, the Levi-Civita connections of $(M, g)$, $(G, k)$, $(G/K, h)$, and $\overline{\nabla}$, $\overline{\overline{\nabla}}$, the induced connection of $\nabla^k$ on the induced bundle $\psi^{-1}TG$ by $\psi : M \to G$, and the one of $\nabla^h$ on the induced bundle $\varphi^{-1}T(G/K)$ by $\varphi : M \to G/K$, respectively.

**Lemma 2.1** *Assume that $W \in \Gamma(\psi^{-1}TG)$ and $V \in \Gamma(\varphi^{-1}T(G/K))$ are $\pi$-related, i.e., $V = \pi_* W$.*

(1) *Then, we have*

$$\overline{\overline{\nabla}}_X V = \pi_* \nabla^k_{(\psi_* X)^H} W^H, \tag{2.13}$$

*where $(\psi_* X)^H$ is the horizontal component of $\psi_* X$ for every $C^\infty$ vector field $X$ on $M$.*

(2) *If we express $W^H = \sum_{i=1}^n f_i \, \widetilde{X_i}$ and $(\psi_* X)^H = \sum_{j=1}^n g_j \, \widetilde{X_j}$ where $f_i$, $g_j \in C^\infty(M)$ $(i, j = 1, \ldots, n)$, then, it holds that*

$$\left(\nabla^k_{(\psi_* X)^H} W^H\right)_{\psi(x)} = \frac{1}{2} \sum_{i,j=1}^{n} f_i(x) g_j(x) [X_j, X_i]_{\psi(x)} + \sum_{i=1}^{n} X_x(f_i) \widetilde{X_i}(x)$$
$$\in V_{\psi(x)} \oplus H_{\psi(x)} \quad (x \in M), \tag{2.14}$$

*correspondingly.*

(3) *For every $x \in M$, we have*

$$\overline{\overline{\nabla}}_X V(x) = \sum_{i=1}^{n} X_x(\langle W, X_{i\,\psi(x)}\rangle)\pi_*(X_{i\,\psi(x)}). \tag{2.15}$$

*Here, it holds that $\pi_*(X_{\psi(x)}) = t_{\psi(x)\,*}\pi_*(X)$ $(X \in \mathfrak{m})$, where $t_a$ is the translation of $G/K$ by $a \in G$, i.e., $t_a(yK) := ayK$ $(y \in G)$.*

*Proof.* (1) Due to Lemmas 1 and 3 in [14, p. 460], we have

$$\begin{aligned}
\overline{\overline{\nabla}}_X V &= \nabla^h_{\varphi_* X} V \\
&= \nabla^h_{\pi_*(\psi_* X)} \pi_* W \\
&= \pi_*\big(\mathcal{H}\,\nabla^k_{(\psi_* X)^H} W^H\big) \\
&= \pi_* \nabla^k_{(\psi_* X)^H} W^H.
\end{aligned}$$

(2) Indeed, we have

$$\begin{aligned}
\big(\nabla^k_{(\psi_* X)^H} W^H\big)_{\psi(x)} &= \sum_{j=1}^{n} g_j(x)\big(\nabla^k_{X_j} W^H\big)_{\psi(x)} \\
&= \sum_{j=1}^{n} g_j(x)\bigg(\sum_{i=1}^{n} \nabla^k_{X_j}(f_i\,\widetilde{X_i})\bigg)_{\psi(x)} \\
&= \sum_{i,j=1}^{n} g_j(x)\big\{(X_j f_i)(x)\widetilde{X_i}(x) + f_i(x)(\nabla^k_{X_j} X_i)_{\psi(x)}\big\} \\
&= \sum_{i,j=1} g_j(x)\bigg\{(X_j f_i)(x)\widetilde{X_i}(x) + \frac{1}{2} f_i(x)[X_j, X_i]_{\psi(x)}\bigg\} \\
&= \frac{1}{2}\sum_{i,j=1}^{n} f_i(x) g_j(x)[X_j, X_i]_{\psi(x)} + \sum_{i=1}^{n} X_x(f_i)\widetilde{X_i}(x),
\end{aligned}$$

since it holds that

$$\begin{aligned}\left(\nabla^k_{X_j} X_i\right)_{\psi(x)} &= L_{\psi(x)*}\left(\nabla^k_{X_j} X_i\right)_e \\ &= L_{\psi(x)*}\left(\frac{1}{2}[X_j, X_i]_e\right) \\ &= \frac{1}{2}[X_j, X_i]_{\psi(x)}\end{aligned}$$

and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. For (3), notice that $W^H = \sum_{i=1}^n \langle W, X_{i\,\psi(\cdot)}\rangle \widetilde{X_i}$. Due to (1), (2), we have (3). □

**Lemma 2.2** *Under the same assumption of Lemma 2.1, we have,*

$$\overline{\overline{\nabla}}_X(\overline{\overline{\nabla}}_Y V) = \sum_{i=1}^n X_x(Y\langle W, X_{i\,\psi(\cdot)}\rangle)\pi_*(X_{i\,\psi(x)}) \in T_{\varphi(x)}(G/K), \tag{2.16}$$

*at each $x \in M$, for every $C^\infty$ vector fields $X$ and $Y$ on $M$.*

*Proof.* Let $Z := \overline{\overline{\nabla}}_Y V \in \Gamma(\varphi^{-1}T(G/K))$. Then, by Lemma 2.1 (1), we have

$$\overline{\overline{\nabla}}_X(\overline{\overline{\nabla}}_Y V) = \overline{\overline{\nabla}}_X Z = \pi_* \nabla^k_{(\psi_* X)^H} Z^H \tag{2.17}$$

where by Lemma 2.1 (3), we have for every $y \in M$,

$$Z^H(y) = \sum_{i=1}^n Y_y\langle W, X_{i\,\psi(\cdot)}\rangle X_{i\,\psi(y)},$$

$Z(y) = \pi_* Z^H(y) \in T_{\varphi(y)}(G/K)$ and $Z \in \Gamma(\varphi^{-1}T(G/K))$. Then, at each $x \in M$, the right hand side of (2.17) which belong to $T_{\varphi(x)}(G/K)$, coincides with the following:

$$\begin{aligned}&\sum_{j=1}^n X_x\langle Z^H, X_{j\,\psi(\cdot)}\rangle \pi_*(X_{j\,\psi(x)}) \\ &\quad= \sum_{j=1}^n X_x\left\langle \sum_{i=1}^n Y_\bullet\langle W, X_{i\,\psi}\rangle X_{i\,\psi(\cdot)}, X_{j\,\psi(\cdot)}\right\rangle \pi_*(X_{j\,\psi(x)}) \\ &\quad= \sum_{i,j=1}^n X_x(Y_\bullet\langle W, X_{i\,\psi}\rangle)\delta_{ij}\, \pi_*(X_{j\,\psi(x)})\end{aligned}$$

$$= \sum_{i=1}^{n} X_x(Y_\bullet\langle W, X_{i\,\psi}\rangle)\pi_*(X_{j\,\psi(x)}).$$

Thus, we have (2.16). □

**Proposition 2.3** *The rough Laplacian $\overline{\Delta}$ acting on $\Gamma(\varphi^{-1}T(G/K))$ can be calculated as follows: For $V \in \Gamma(\varphi^{-1}T(G/K))$ with $V = \pi_* W$ for $W \in \Gamma(\psi^{-1}TG)$,*

$$(\overline{\Delta}V)(x) = \sum_{i=1}^{n} \Delta_x\langle W, X_{i\,\psi(\cdot)}\rangle\pi_*(X_{i\,\psi(x)}) \in T_{\varphi(x)}(G/K), \tag{2.18}$$

*for each $x \in M$. Here, since $f : M \ni x \mapsto \langle W(x), X_{i\,\psi(x)}\rangle_{\psi(x)} \in \mathbb{R}$ is a (local) $C^\infty$ function on $M$, the Laplacian $\Delta_x = \delta\, d$ acting on $C^\infty(M)$ works well to this $f$.*

*Proof.* Indeed, if we recall the definition (2.3) of the rough Laplacian $\overline{\Delta}$, and due to Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
\overline{\Delta}V &= -\sum_{j=1}^{m}\left\{\overline{\overline{\nabla}}_{e_j}(\overline{\overline{\nabla}}_{e_j}V) - \overline{\overline{\nabla}}_{\nabla_{e_j}e_j}V\right\},\\
&= -\sum_{i=1}^{n}\sum_{j=1}^{n}({e_j}^2 - \nabla_{e_j}e_j)\langle W, X_{i\,\psi(\cdot)}\rangle\pi_*(X_{i\,\psi(x)})\\
&= \sum_{i=1}^{n}\Delta_x\langle W, X_{i\,\psi(\cdot)}\rangle\pi_*(X_{i\,\psi(x)}).
\end{aligned}$$

We have Proposition 2.3. □

## 3. Determination of the bitension field

Now, let $\theta$ be the Maurer-Cartan form on $G$, i.e., a $\mathfrak{g}$-valued left invariant 1-form on $G$ which is defined by $\theta_y(Z_y) = Z$ ($y \in G$, $Z \in \mathfrak{g}$). For every $C^\infty$ map $\varphi$ of $(M, g)$ into $(G/K, h)$ with a lift $\psi : M \to G$, let us consider a $\mathfrak{g}$-valued 1-form $\alpha$ on $M$ given by $\alpha = \psi^*\theta$ and the decomposition

$$\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}} \tag{3.1}$$

corresponding to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then, it is well known (see for example, [3]) that

**Lemma 3.1** *For every $C^\infty$ map $\varphi : (M, g) \to (G/K, h)$,*

$$t_{\psi(x)^{-1}*}\tau(\varphi) = -\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)], \quad (x \in M), \tag{3.2}$$

*where $\alpha = \varphi^*\theta$, and $\theta$ is the Maurer-Cartan form of $G$, $\delta(\alpha_{\mathfrak{m}})$ is the co-differentiation of $\mathfrak{m}$-valued 1-form $\alpha_{\mathfrak{m}}$ on $(M, g)$.*

*Thus, $\varphi : (M, g) \to (G/K, h)$ is harmonic if and only if*

$$-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] = 0. \tag{3.3}$$

Furthermore, we obtain

**Theorem 3.2** *We have*

$$\begin{aligned} t_{\psi(x)^{-1}*}\tau_2(\varphi) &= \Delta_g\Big( - \delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)]\Big) \\ &\quad + \sum_{s=1}^{m}\Big[\Big[ - \delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)], \alpha_{\mathfrak{m}}(e_s)\Big], \alpha_{\mathfrak{m}}(e_s)\Big], \end{aligned} \tag{3.4}$$

*where $\Delta_g$ is the (positive) Laplacian of $(M, g)$ acting on $C^\infty$ functions on $M$, and $\{e_i\}_{i=1}^m$ is a local orthonormal frame field on $(M, g)$.*

Therefore, we obtain immediately the following two corollaries.

**Corollary 3.3** *Let $(G/K, h)$ be a Riemannian symmetric space, and $\varphi : (M, g) \to (G/K, h)$, a $C^\infty$ mapping. Then, we have:*

(1) *the map $\varphi : (M, g) \to (G/K, h)$ is harmonic if and only if*

$$-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] = 0. \tag{3.5}$$

(2) *The map $\varphi : (M, g) \to (G/K, h)$ is biharmonic if and only if*

$$\Delta_g\Big( - \delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^m [\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)]\Big) + \sum_{s=1}^m \Big[\Big[ - \delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^m [\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)], \alpha_{\mathfrak{m}}(e_s)\Big], \alpha_{\mathfrak{m}}(e_s)\Big] = 0. \quad (3.6)$$

**Corollary 3.4** *Let $(G/K, h)$ be a Riemannian symmetric space, and $\varphi : (M, g) \to (G/K, h)$, a $C^\infty$ mapping with a* horizontal *lift $\psi : M \to G$, i.e., $\varphi = \pi \circ \psi$ and $\psi_x(T_xM) \subset H_{\psi(x)}$ which is equivalent to $\alpha_{\mathfrak{k}} \equiv 0$.*

*Then, we have:*

(1) *the map $\varphi : (M, g) \to (G/K, h)$ is harmonic if and only if*

$$\delta(\alpha_{\mathfrak{m}}) = 0, \quad (3.7)$$

(2) *and the map $\varphi : (M, g) \to (G/K, h)$ is biharmonic if and only if*

$$\delta\, d\, \delta(\alpha_{\mathfrak{m}}) + \sum_{s=1}^m [[\delta(\alpha_{\mathfrak{m}}), \alpha_{\mathfrak{m}}(e_s)], \alpha_{\mathfrak{m}}(e_s)] = 0. \quad (3.8)$$

*Proof of Theorem 3.2.* We need the following lemma:

**Lemma 3.5** *The tension field $\tau(\varphi)$ of a $C^\infty$ map $\varphi : (M, g) \to (G/K, h)$ can be expressed as*

$$\tau(\varphi) = \pi_* W = \pi_*(W^H),$$

*where $W \in \Gamma(\psi^{-1}TG)$, and $W^H$ is the horizontal component of $W$ in the decomposition $W(x) = W^V(x) + W^H(x) \in T_{\psi(x)}G = V_{\psi(x)} \oplus H_{\psi(x)}$ $(x \in M)$. If we define an $\mathfrak{m}$-valued function $\beta$ on $M$ by*

$$\beta := \sum_{i=1}^n \langle W, \widetilde{X_i}\rangle X_i = \sum_{i=1}^n \langle W^H, \widetilde{X_i}\rangle X_i, \quad (3.9)$$

*then, we have*

$$t_{\psi(x)\,*}{}^{-1}\tau(\varphi) = \pi_*\beta. \quad (3.10)$$

*If we define an $\mathfrak{m}$-valued functions $\beta_i$ $(i = 1, \dots, n)$ on $M$ by*

$$\beta_i := \sum_{j=1}^{n} \langle \psi_* e_i, X_{j\,\psi(\cdot)} \rangle X_j \in \mathfrak{m}. \tag{3.11}$$

*Then, it holds that*

$$t_{\psi(x)\,*}{}^{-1} \varphi_* e_i = \pi_* \beta_i \quad \text{and} \quad \beta_i = \alpha_{\mathfrak{m}}(e_i), \tag{3.12}$$

*where $\alpha_{\mathfrak{m}}$ is the $\mathfrak{m}$-component of $\alpha := \psi^*\theta$, and the Maurer-Cartan form on $G$.*

Indeed, (3.10) and the first part of (3.12) follow from the definition of $\beta$ and the fact that

$$\begin{aligned}
\alpha(e_i) &= (\psi^*\theta)(e_i) \\
&= \theta(\psi_* e_i) \\
&= \theta\Bigg( \sum_{j=1}^{n} \langle \psi_* e_i, X_{j\,\psi(x)} \rangle X_{j\,\psi(x)} + \sum_{j=n+1}^{\ell} \langle \psi_* e_i, X_{j\,\psi(x)} \rangle X_{j\,\psi(x)} \Bigg) \\
&= \sum_{j=1}^{n} \langle \psi_* e_i, X_{j\,\psi(x)} \rangle X_j + \sum_{j=n+1}^{\ell} \langle \psi_* e_i, X_{j\,\psi(x)} \rangle X_j \\
&\in \mathfrak{m} \oplus \mathfrak{k},
\end{aligned}$$

since $\alpha = \psi^*\theta$. Thus, we have $\beta_i = \alpha_{\mathfrak{m}}(e_i)$. □

(*Continued the proof of Theorem 3.2*). We have

$$t_{\psi(x)\,*}{}^{-1} \varphi_* e_i = \sum_{j=1}^{n} \langle \psi_* e_i, X_{j\,\psi(x)} \rangle \pi_*(X_j) \in T_o(G/K), \tag{3.13}$$

where $o = \{K\} \in G/K$ is the origin of $G/K$. Because,

$$t_{\psi(x)\,*}{}^{-1} \varphi_* e_i = t_{\psi(x)\,*}{}^{-1} \pi_*\, \psi_* e_i = \pi_*\, L_{\psi(x)*}{}^{-1}\, \psi_* e_i = \pi_* (L_{\psi(x)*}{}^{-1} \psi_*\, e_i)_{\mathfrak{m}}$$

which coincides with

$$\sum_{j=1}^{n} \langle L_{\psi(x)\,*}{}^{-1}\, \psi_* e_i, X_j \rangle \pi_*(X_j) = \sum_{j=1}^{n} \langle \psi_* e_i, X_{j\,\psi(x)} \rangle \pi_*(X_j),$$

which imply (3.13).

Thus, we have

$$\varphi_* e_i = \pi_* W_i \quad (i = 1, \dots, m), \tag{3.14}$$

where $W_i \in \Gamma(\psi^{-1}TG)$ and $\mathfrak{m}$-valued functions $\widetilde{W_i}$ on $M$ $(i = 1, \dots, m)$ are given by

$$W_i(x) := \sum_{j=1}^{n} \langle \psi_* e_i, X_{j\,\psi(x)} \rangle X_{j\,\psi(x)}, \tag{3.15}$$

$$\widetilde{W_i}(x) := \sum_{j=1}^{n} \langle \psi_* e_i, X_{j\,\psi(x)} \rangle X_j \in \mathfrak{m}, \tag{3.16}$$

for each $x \in M$.

On the other hand, we have

$$\tau(\varphi) = \pi_* W, \tag{3.17}$$

where $W \in \Gamma(\psi^{-1}TG)$ and an $\mathfrak{m}$-valued function $\widetilde{W}$ on $M$ are given by

$$W(x) := t_{\psi(x)\,*}\Big( -\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m} [\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] \Big), \tag{3.18}$$

$$\widetilde{W} := -\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m} [\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] \tag{3.19}$$

for each $x \in M$. And we also have

$$t_{\psi(x)\,*}{}^{-1}\, \overline{\Delta}\tau(\varphi)(x) = \Delta\Big( -\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m} [\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] \Big)(x) \quad (x \in M), \tag{3.20}$$

where $\Delta = \delta\, d$ is the positive Laplacian acting on the space of all $C^\infty$ $\mathfrak{m}$-valued functions on $M$.

We want to calculate $\mathcal{R}(\tau(\varphi)) = \sum_{i=1}^{m} R^h(\tau(\varphi), \varphi_* e_i)\varphi_* e_i$. Indeed, we have

$$
\begin{aligned}
{t_{\psi(x)}}_*^{-1}\mathcal{R}(\tau(\varphi)) &= -\sum_{i=1}^m [[\widetilde{W},\widetilde{W_i}],\widetilde{W_i}] \\
&= -\sum_{s=1}^m \Big[\Big[-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^m [\alpha_{\mathfrak{k}}(e_i),\alpha_{\mathfrak{m}}(e_i)],\alpha_{\mathfrak{m}}(e_s)\Big],\alpha_{\mathfrak{m}}(e_s)\Big].
\end{aligned}
\tag{3.21}
$$

Here, we used the formula of the curvature $R^h$ of the Riemannian symmetric space $(G/K,h)$ ([9, p. 202, p. 231, Theorem 3.2]):

$$(R^h(X,Y)Z)_o = -[[X,Y],Z]_o \quad (X,Y,Z\in\mathfrak{m}).$$

Thus, we obtain Theorem 3.2. □

Let us recall the *integrability condition* for a $C^\infty$ mapping $\varphi:(M,g)\to(G/K,h)$. The Maurer-Cartan form $\theta$ on $G$ satisfies

$$d\theta + \frac{1}{2}[\theta\wedge\theta] = 0, \tag{3.22}$$

so that the pull back $\alpha=\psi^*\theta$ of $\theta$ by the lift $\psi: M\to G$ of $\varphi: M\to G/K$ also satisfies that

$$d\alpha + \frac{1}{2}[\alpha\wedge\alpha] = 0, \tag{3.23}$$

which is equivalent to

$$
\begin{cases}
d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}}\wedge\alpha_{\mathfrak{k}}] + \frac{1}{2}[\alpha_{\mathfrak{m}}\wedge\alpha_{\mathfrak{m}}] = 0, \\
d\alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{k}}\wedge\alpha_{\mathfrak{m}}] = 0.
\end{cases}
\tag{3.24}
$$

Summarizing the above, we have

**Theorem 3.6** *Let $(M,g)$ be an $m$-dimensional compact Riemannian manifold, $(G/K,h)$, an $n$-dimensional Riemannian symmetric space, $\pi: G\to G/K$, the projection, and $\varphi:(M,g)\to(G/K,h)$, a $C^\infty$ mapping with a local lift $\psi: M\to G$, $\varphi=\pi\circ\psi$. Let $\alpha=\psi^*\theta$ be the pull back of the Maurer-Cartan form $\theta$, and $\alpha=\alpha_{\mathfrak{k}}+\alpha_{\mathfrak{m}}$, the decomposition of $\alpha$ corresponding to the Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$.*

(I) *The mapping* $\varphi : (M, g) \to (G/K, h)$ *is* harmonic *if and only if*

$$-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m} [\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] = 0, \tag{3.25}$$

*where* $\delta$ *is the co-differentiation, and* $\{e_i\}_{i=1}^{m}$ *is a local orthonormal frame field on* $(M, g)$.

*Furthermore,* $\varphi : (M, g) \to (G/K, h)$ *is* biharmonic *if and only if*

$$\Delta\Big( - \delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m} [\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)]\Big) + \sum_{s=1}^{m} \Big[\Big[ - \delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m} [\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)], \alpha_{\mathfrak{m}}(e_s)\Big], \alpha_{\mathfrak{m}}(e_s)\Big] = 0, \tag{3.26}$$

*where* $\Delta = \delta d$ *is the* (*positive*) *Laplacian of* $(M, g)$ *acting on the space of* $\mathfrak{g}$*-valued* $C^\infty$ *functions on* $(M, g)$.

(II) *Conversely, let* $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$ *be a* $\mathfrak{g}$*-valued* 1*-form on* $(M, g)$*. If* $\alpha$ *satisfies* (3.23) *or* (3.24)*, and satisfies* (3.25) (*resp.* (3.26))*, then, there exists a* $C^\infty$*-mapping* $\varphi$ *of* $M$ *into* $G$ *with a local lift* $\psi : M \to G$, $\varphi = \pi \circ \psi$ *and the initial value* $\varphi(p) = a \in G$ *at some* $p \in M$ *such that* $\alpha = \psi^*\theta$ *and* $\varphi$ *is a harmonic* (*resp. biharmonic*) *map of* $(M, g)$ *into* $(G/K, h)$.

## 4. Biharmonic curves into Riemannian symmetric spaces

### 4.1.

Let $\varphi : (\mathbb{R}, g_0) \to (G/K, h)$ be a $C^\infty$ curve, and $\psi : \mathbb{R} \to G$, a lift of $\varphi$, ($\varphi = \pi \circ \psi$). Then, $\alpha = \psi^*\theta = \psi^{-1}d\psi = F(t)dt$ is a $\mathfrak{g}$-valued 1-form on $\mathbb{R}$ and $F$ is a $\mathfrak{g}$-valued function on $\mathbb{R}$ satisfying $\psi(t)^{-1}(d\psi/dt) = F(t)$. Conversely, for a $\mathfrak{g}$-valued $C^\infty$ function $F(t)$ on $\mathbb{R}$, there exists a unique $C^\infty$-curve $\psi : \mathbb{R} \to G$ which satisfies

$$\begin{cases} \psi(t)^{-1} \dfrac{d\psi}{dt} = F(t), \\ \psi(0) = x \in G. \end{cases} \tag{4.1}$$

To give an explicit solution $\psi$ of (4.1) is very difficult, in general, since $G$ is not abelian. However, corresponding to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, we decompose $F(t) = F_{\mathfrak{k}}(t) + F_{\mathfrak{m}}(t)$, $\alpha_{\mathfrak{k}} = F_{\mathfrak{k}}(t)dt$, and $\alpha_{\mathfrak{m}} = F_{\mathfrak{m}}(t)dt$, so we have

$$\delta\alpha = -(\overline{\nabla}_{e_1})(\alpha(e_1)) = -\nabla^h_{e_1}(\alpha(e_1)) = -e_1(F(t)) = -F'(t),$$

and

$$\delta\alpha_{\mathfrak{m}} = -{F_{\mathfrak{m}}}'(t).$$

Thus the harmonic map equation (3.25) is

$${F_{\mathfrak{m}}}'(t) + [F_{\mathfrak{k}}(t), F_{\mathfrak{m}}(t)] = 0, \tag{4.2}$$

and the biharmonic map equation (3.26) is

$$\begin{aligned} &-\frac{d^2}{dt^2}\big({F_{\mathfrak{m}}}'(t) + [F_{\mathfrak{k}}(t), F_{\mathfrak{m}}(t)]\big) \\ &\quad + \big[[{F_{\mathfrak{m}}}'(t) + [F_{\mathfrak{k}}(t), F_{\mathfrak{m}}(t)], F_{\mathfrak{m}}], F_{\mathfrak{m}}\big] = 0. \end{aligned} \tag{4.3}$$

In these cases, the integrability condition (3.23) always holds, so that the existence of $\psi$ of (4.1) is always true.

Let us recall that a lift $\psi(t)$ is *horizontal* if $\psi_*(T_xM) \subset L_{*\,\psi(x)}(\mathfrak{m})$ if and only if $F_{\mathfrak{k}} \equiv 0$. In this case, (4.2) is equivalent to

$${F_{\mathfrak{m}}}'(t) = 0, \tag{4.4}$$

which implies that $F_{\mathfrak{m}}(t) = X \in \mathfrak{m}$ (constant). So that $F(t) = X \in \mathfrak{m}$. Then, we have

$$\psi(t) = x\exp(tX), \quad \varphi(t) = x\exp(tX)K \in G/K. \tag{4.5}$$

Furthermore, (4.3) is equivalent to

$$-{F_{\mathfrak{m}}}'''(t) + \big[[{F_{\mathfrak{m}}}'(t), F_{\mathfrak{m}}(t)], F_{\mathfrak{m}}(t)\big] = 0. \tag{4.6}$$

**Example 4.1** Assume that $(G/K, h)$ is *of the Euclidean type*. In this case, $\mathfrak{m}$ is an abelian ideal and $\mathfrak{k}$ acts on $\mathfrak{m}$ by $[T, X] = T \cdot X$ $(T \in \mathfrak{k}, X \in \mathfrak{m})$ regarding $\mathfrak{k}$ as a subalgebra of $\mathfrak{gl}(\mathfrak{m})$. Then, we have

(1) $\varphi : (\mathbb{R}, g_0) \to (G/K, h)$ is harmonic if and only if

$$F_{\mathfrak{m}}{}'(t) + F_{\mathfrak{k}}(t) \cdot F_{\mathfrak{m}}(t) = 0. \tag{4.7}$$

(2) $\varphi : (\mathbb{R}, g_0) \to (G/K, h)$ is biharmonic if and only if

$$\frac{d^2}{dt^2}\big(F_{\mathfrak{m}}{}'(t) + F_{\mathfrak{k}}(t) \cdot F_{\mathfrak{m}}(t)\big) = 0 \tag{4.8}$$

which is equivalent to

$$F_{\mathfrak{m}}{}'(t) + F_{\mathfrak{k}}(t) \cdot F_{\mathfrak{m}}(t) = At + B \tag{4.9}$$

for some $A$ and $B$ in $\mathfrak{m}$. Thus, if $\psi : (\mathbb{R}, g_0) \to G$ is horizontal, i.e., $F_{\mathfrak{k}} \equiv 0$, then, $F_{\mathfrak{m}}(t) = C$ (a constant vector in $\mathfrak{m}$) for the case (1), and $F_{\mathfrak{m}}(t) = At^2 + Bt + C$ for the case (2). If $[A, B] = [B, C] = [C, A] = 0$, then $\psi(t) = \exp(t^2\, A + t\, B + C)$ and $\varphi(t) = \psi(t) \cdot \{K\}$ is a biharmonic curve in a Riemannian symmetric space $(G/K, h)$ of the Euclidean type.

### 4.2. Biharmonic curves into rank one symmetric spaces

In this subsection, we study biharmonic curves in a compact symmetric spaces $(G/K, h)$.

(1) *Case of the unit sphere* $(S^n, h)$.

Let $G = SO(n+1)$ act on $\mathbb{R}^{n+1}$ linearly, and $K = SO(n)$ be the isotropy subgroup of $G$ at the origin $o = {}^{\mathrm{t}}(1, 0, \ldots, 0)$. Their Lie algebras $\mathfrak{g} = \mathfrak{so}(n+1)$, $\mathfrak{k} = \mathfrak{so}(n)$ and the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ are given by

$$\mathfrak{g} = \mathfrak{so}(n+1) = \{X \in \mathfrak{gl}(n+1) : X + {}^{\mathrm{t}}X = O\},$$

$$\mathfrak{k} = \mathfrak{so}(n) = \left\{ \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & X_1 & \\ 0 & & & \end{array}\right) : X_1 \in \mathfrak{gl}(n),\ X_1 + {}^{\mathrm{t}}X_1 = O \right\},$$

$$\mathfrak{m} = \left\{ \left(\begin{array}{c|c} 0 & -{}^{\mathrm{t}}u \\ \hline u & O \end{array}\right) : u = {}^{\mathrm{t}}(u_1, \ldots, u_n) \in \mathbb{R}^n \right\}.$$

For a $\mathfrak{m}$-valued $C^\infty$ function $F_{\mathfrak{m}}(t)$ given by

$$F_{\mathfrak{m}}(t) = \left(\begin{array}{c|ccc} 0 & -u_1(t) & \cdots & -u_n(t) \\ \hline u_1(t) & & & \\ \vdots & & O & \\ u_n(t) & & & \end{array}\right), \tag{4.10}$$

and $F_{\mathfrak{k}} \equiv 0$, the biharmonic map equation (4.7) is equivalent to

$$-{u_i}''' + \sum_{j=1}^{n} (u_i\, {u_j}' - {u_i}'\, u_j) u_j = 0 \quad (i = 1, \ldots, n) \tag{4.11}$$

which is also equivalent to

$$-u''' + \langle u', u \rangle u - \langle u, u \rangle u' = 0, \tag{4.12}$$

where the inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ is given by $\langle u, v \rangle = \sum_{i=1}^{n} u_i v_i$ for $u, v \in \mathbb{R}^n$.

<u>*Case of* $n = 2$</u>. Our problem is to find a $C^\infty$ plane curve which satisfies (4.12). To do it, we assume that $u(t)$ is reparametrized in such a way that $u(s)$ is a tangent curve of a plane curve $\mathbf{p}(s)$: $u(s) = \mathbf{p}'(s) = \mathbf{e}_1(s)$. For the other cases, we have no idea to solve (4.12). Recall the Frenet-Serret formula for a plane curve $\mathbf{p}(s)$:

$$\begin{cases} \mathbf{p}'(s) = \mathbf{e}_1(s), \\ \mathbf{e}_1'(s) = \kappa(s)\, \mathbf{e}_2(s), \\ \mathbf{e}_2'(s) = -\kappa(s)\, \mathbf{e}_1(s). \end{cases} \tag{4.13}$$

Now we have

$$u = \mathbf{e}_1, \tag{4.14}$$

$$u' = \mathbf{e}_1' = \kappa\, \mathbf{e}_2, \tag{4.15}$$

$$u'' = \kappa'\, \mathbf{e}_2 + \kappa\, \mathbf{e}_2' = -\kappa^2\, \mathbf{e}_1 + \kappa'\, \mathbf{e}_2, \tag{4.16}$$

$$u''' = -3\kappa\, \kappa'\, \mathbf{e}_1 + (\kappa'' - \kappa^3)\mathbf{e}_2. \tag{4.17}$$

Since $\langle u', u \rangle = 0$ and $\langle u, u \rangle = 1$, (4.12) is equivalent to

$$-3\kappa\,\kappa' = 0, \tag{4.18}$$

$$\kappa'' - \kappa^3 = -\kappa, \tag{4.19}$$

By (4.18), $\kappa = c$ (a constant), and by (4.19), $c = 0, 1, -1$. Thus, we have

( i ) In the case of $c = 0$,

$$\mathbf{p}(s) = s\,\mathbf{a} + \mathbf{b}, \quad u(s) = \mathbf{a}, \quad (\mathbf{a}, b \in \mathbf{R}^2), \tag{4.20}$$

( ii ) in the case of $c = 1$,

$$\mathbf{p}(s) = (\cos s, \sin s), \quad u(s) = (-\sin s, \cos s), \tag{4.21}$$

(iii) in the case of $c = -1$,

$$\mathbf{p}(s) = (\cos s, -\sin s), \quad u(s) = (-\sin s, -\cos s). \tag{4.22}$$

Now it is easy to find $\psi : \mathbb{R} \to G$ and $\varphi(t) = \psi(t)\{K\} \in G/K$ satisfying $\psi(t)^{-1}(d\psi/dt) = F(t) = F_{\mathfrak{m}}(t)$ for such $u(t)$ in (4.12).

Case (i): If $\mathbf{a} = {}^{t}(a, b) \in \mathbb{R}^2$, we have due to (4.1),

$$\varphi(t) = \psi(t)\{K\} = x \begin{pmatrix} \cos(t\sqrt{a^2+b^2}) \\ \frac{a}{\sqrt{a^2+b^2}} \sin(t\sqrt{a^2+b^2}) \\ \frac{b}{\sqrt{a^2+b^2}} \sin(t\sqrt{a^2+b^2}) \end{pmatrix}, \tag{4.23}$$

which is a great circle of the standard 2-sphere $(S^2, h)$.

Cases (ii) and (iii): In these cases, if we assume $F_{\mathfrak{k}} \equiv 0$, we have

$$F_{\mathfrak{m}}(t) = \left(\begin{array}{c|cc} 0 & \sin t & -\cos t \\ \hline -\sin t & 0 & 0 \\ \cos t & 0 & 0 \end{array}\right), \tag{4.24}$$

for Case (ii), and

$$F_{\mathfrak{m}}(t) = \left(\begin{array}{c|cc} 0 & \sin t & \cos t \\ \hline -\sin t & 0 & 0 \\ -\cos t & 0 & 0 \end{array}\right), \tag{4.25}$$

for Case (iii). In these cases, because of $[F_{\mathfrak{m}}(t), {F_{\mathfrak{m}}}'(t)] \neq 0$, it is difficult for us to give explicitly a unique solution of the initial value problem of

$$\psi(t)^{-1}\frac{d\psi(t)}{dt} = F(t) \quad \text{and } \psi(0) = a \in SO(3). \tag{4.26}$$

<u>*Case of* $n = 3$</u>. In this case, we have to solve for a $C^\infty$ curve $u : \mathbb{R} \to \mathbb{R}^3$, the equation (4.12) which is equivalent to

$$-u''' + u \times (u \times u') = 0. \tag{4.27}$$

To do it, we assume that $u(t)$ is parametrized in such a way that $u(s)$ is a tangent curve of a $C^\infty$ curve in $\mathbb{R}^3$, $\mathbf{p}(s) : u(s) = \mathbf{p}'(s) = \mathbf{e}_1(s)$. Recall the Frene-Serret formula for a curve $\mathbf{p}(s)$:

$$\begin{cases} \mathbf{p} = \mathbf{e}_1 \\ {\mathbf{e}_1}' = \kappa\mathbf{e}_2 \\ {\mathbf{e}_2}' = -\kappa\mathbf{e}_1 + \tau\mathbf{e}_3 \\ {\mathbf{e}_3}' = -\tau\mathbf{e}_2 \end{cases} \tag{4.28}$$

where $\kappa$ and $\tau$ are the curvature and torsion of $\mathbf{p}(s)$, respectively. By making use of (4.28), we have

$$\begin{cases} u' = \kappa\mathbf{e}_2 \\ u'' = -\kappa^2\mathbf{e}_1 + \kappa'\mathbf{e}_2 + \kappa\tau\mathbf{e}_3 \\ u''' = -3\kappa\kappa'\mathbf{e}_1 + (\kappa'' - \kappa^3 - \kappa\tau^2)\mathbf{e}_2 + (2\kappa'\tau + \kappa\tau')\mathbf{e}_3. \end{cases} \tag{4.29}$$

Thus, (4.29) is equivalent to

$$\begin{cases} -3\kappa\kappa' = 0 \\ \kappa'' - \kappa^3 - \kappa\tau^2 = -\kappa \\ 2\kappa'\tau + \kappa\tau' = 0. \end{cases} \tag{4.30}$$

By the first equation of (4.30), $\kappa = \kappa_0$ (a constant). In the case $\kappa_0 = 0$, $u(t) = \mathbf{a} \in \mathbb{R}^3$ (a constant vector). In the case $\kappa_0 \neq 0$, by the third equation of (4.30), $\tau = \tau_0$ (a constant). By the second equation of (4.30), ${\kappa_0}^2 + {\tau_0}^2 = 1$. Then, $\mathbf{p}(s) = {}^t(a\cos t, a\sin t, bt)$, with $s = \sqrt{a^2+b^2}\,t$. Here, $\kappa_0 = a/(a^2+b^2)$, and $\tau_0 = b/(a^2+b^2)$, and $1 = {\kappa_0}^2 + {\tau_0}^2 = 1/(a^2+b^2)$, i.e.,

$a^2 + b^2 = 1$. Therefore, we have

$$\begin{cases} \mathbf{p}(t) = {}^{\mathrm{t}}(a\cos t, a\sin t, bt), \\ u(t) = \mathbf{p}'(t) = {}^{\mathrm{t}}(-a\sin t, a\cos t, b), \end{cases} \tag{4.31}$$

where $a$ and $b$ are constants with $a^2 + b^2 = 1$. Thus, $F_{\mathfrak{m}}(t)$ with $F_{\mathfrak{k}} \equiv 0$, is given by

$$F_{\mathfrak{m}}(t) = \left(\begin{array}{c|ccc} 0 & a\sin t & -a\cos t & -b \\ \hline -a\sin t & & & \\ a\cos t & & O & \\ b & & & \end{array}\right). \tag{4.32}$$

<u>*Case of* $n \geq 2$.</u> In this case, the other-type solutions exist:

Let $u = (u_1, \ldots, u_n) = (0, \ldots, 0, \overbrace{v}^{i\text{ th}}, 0, \ldots, 0)$ $(i = 1, \ldots, n)$. Then, for such $u$, the equation (4.12) is reduced to $v''' = 0$. Thus, we have $v(t) = D_t := at^2 + bt + c$ for some constants $a$, $b$ and $c$. Thus, $F_{\mathfrak{m}}(t)$ is given by

$$F_{\mathfrak{m}}(t) = D_t \left(\begin{array}{c|ccccc} 0 & 0 & \ldots & -1 & \ldots & 0 \\ \hline 0 & & & & & \\ \vdots & & & & & \\ 1 & & & O & & \\ \vdots & & & & & \\ 0 & & & & & \end{array}\right). \tag{4.33}$$

Thus,

$$\psi(t) = x\exp\left(\int_0^t F(s)ds\right) = x\left(\begin{array}{c|ccccc} \cos d_t & 0 & \ldots & -\sin d_t & \ldots & 0 \\ \hline 0 & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \sin d_t & 0 & \ldots & \cos d_t & \ldots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 0 \end{array}\right),$$

where $d_t := (a/3)t^3 + (b/2)t^2 + ct$. So, we have a *biharmonic curve* into $(S^n, h)$:

$$\varphi(t) = \psi(t)\{K\} = x\,{}^{\mathrm{t}}(\cos d_t, 0, \dots, 0, \sin d_t, 0, \dots, 0), \tag{4.34}$$

for $x \in SO(n+1)$, where $d_t := (a/3)t^3 + (b/2)t^2 + ct$. Furthermore, $\varphi(t)$ is harmonic if and only if $a = b = 0$.

(2) *Case of the complex projective space* $(\mathbb{C}P^n, h)$.

Let $G = SU(n+1)$ act on the projective space linearly on $\mathbb{C}P^n = \{[z] : z \in \mathbb{C}^{n+1}\backslash\{\mathbf{0}\}\}$, and $K$, the isotropy subgroup of $G$ at $o = {}^{\mathrm{t}}[1, 0, \dots, 0]$. The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is given by

$$\mathfrak{g} = \{X \in \mathfrak{g}l(n+1, \mathbb{C}) : X + {}^{\mathrm{t}}\overline{X} = O,\ \mathrm{tr}X = 0\},$$

$$\mathfrak{k} = \left\{\begin{pmatrix} \sqrt{-1}\,a & 0 \\ 0 & X \end{pmatrix} : a \in \mathbb{R}, X \in \mathfrak{g}l(n, \mathbb{C}),\ {}^{\mathrm{t}}\overline{X} + X = O,\ \sqrt{-1}a + \mathrm{tr}X = 0\right\},$$

$$\mathfrak{m} = \left\{\begin{pmatrix} 0 & -{}^{\mathrm{t}}\overline{z} \\ z & O \end{pmatrix} : z \in \mathbb{C}^n\right\}.$$

For a $C^\infty$ $\mathfrak{m}$-valued function $F_{\mathfrak{m}}(t)$ given by

$$F_{\mathfrak{m}}(t) = \left(\begin{array}{c|ccc} 0 & -\overline{z_1(t)} & \cdots & -\overline{z_n(t)} \\ \hline z_1(t) & & & \\ \vdots & & O & \\ z_n(t) & & & \end{array}\right), \tag{4.35}$$

where $z_i(t) = u_i(t) + \sqrt{-1}v_i(t)u_i(t)$ and $v_i(t)$ are real valued $C^\infty$ functions $(i = 1, \dots, n)$, and $F_{\mathfrak{k}} \equiv 0$, the biharmonic map equation (4.6) is equivalent to

$$-{z_i}''' + \sum_{j=1}^{n} \left\{ (z_i\,\overline{z_j}' - {z_i}'\,\overline{z_j})\,z_j - z_i(\overline{z_j}{z_j}' - \overline{z_j}'\,z_j) \right\} = 0 \tag{4.36}$$

for all $i = 1, \dots, n$. Notice here that this (4.36) can be written as

$$-z''' + 2\langle z, z'\rangle z - \langle z', z\rangle z - \langle z, z\rangle z' = 0, \tag{4.37}$$

where $\langle z, w\rangle = \sum_{i=1}^n z_i\overline{w_i}$ for two $\mathbb{C}^n$-valued functions $z$ and $w$ in $t$. If we write $z = u + \sqrt{-1}v$, where $u$ and $v$ are $\mathbb{R}^n$-valued functions, then (4.37) is equivalent to

$$\begin{cases} -u''' + 4n\,(-v^2\,u' + u\,v\,v') = 0 \\ \quad -v''' + 4n\,(u\,v\,u' - u^2\,v') = 0. \end{cases} \tag{4.38}$$

One can find the following solutions of (4.38):

( i ) $u = D_t = a\,t^2 + b\,t + c$ and $v \equiv 0$,
( ii ) $u \equiv 0$ and $v = D_t = a\,t^2 + b\,t + c$, or
(iii) $u = v = D_t = a\,t^2 + b\,t + c$,

where $a$, $b$ and $c$ are constant vectors in $\mathbb{R}^n$. Corresponding to these, we can find $F_{\mathfrak{m}}(t)$ of (4.35) as follows:

$$F_{\mathfrak{m}}(t) = D_t \left(\begin{array}{c|ccc} 0 & -\overline{z_1(t)} & \cdots & -\overline{z_n(t)} \\ \hline z_1(t) & & & \\ \vdots & & O & \\ z_n(t) & & & \end{array}\right), \tag{4.39}$$

where $z_1(t), \dots, z_n(t)$ are

Case (i): $z_1(t) = \cdots = z_n(t) = 1$,
Case (ii): $z_1(t) = \cdots = z_n(t) = \sqrt{-1}$,
Case (iii): $z_1(t) = \cdots = z_n(t) = 1 + \sqrt{-1}$,

correspondingly. In each cases, we can find $\psi(t)$ by the same way as the case of $(S^n, h)$, and a *biharmonic curve* in $(\mathbb{C}P^n, h)$:

Case (i): $\varphi(t) = x\,{}^{\mathrm{t}}\Big[\cos(\sqrt{n}\,d_t), \frac{1}{\sqrt{n}}\sin(\sqrt{n}\,d_t), \dots, \frac{1}{\sqrt{n}}\sin(\sqrt{n}\,d_t)\Big]$,

Case (ii): $\varphi(t) = x\,{}^{\mathrm{t}}\Big[\cos(\sqrt{n}\,d_t), \frac{\sqrt{-1}}{\sqrt{n}}\sin(\sqrt{n}\,d_t), \dots, \frac{\sqrt{-1}}{\sqrt{n}}\sin(\sqrt{n}\,d_t)\Big]$,

Case (iii): $\varphi(t) = x\,{}^{\mathrm{t}}\Big[\cos(\sqrt{2n}\,d_t), \frac{1+\sqrt{-1}}{\sqrt{2n}}\sin(\sqrt{2n}\,d_t), \dots,$
$\frac{1+\sqrt{-1}}{\sqrt{2n}}\sin(\sqrt{2n}\,d_t)\Big]$,

where $d_t := (a/3)t^3 + (b/2)t^2 + c\,t$, $a$, $b$ and $c$ are constant real numbers, and $x \in SU(n+1)$. Each $\varphi : (\mathbb{R}, g_0) \to (\mathbb{C}P^n, h)$ is *harmonic* if and only if $a = b = 0$.

(3) *Case of the quaternion projective space* $(\mathbb{H}P^n, h)$.

Let $G = Sp(n+1) = \{x \in U(2n+2) \mid {}^{\mathrm{t}}x\,J_{n+1}x = J_{n+1}\}$, where $J_{n+1} = \left(\begin{smallmatrix} O & I_{n+1} \\ -I_{n+1} & O \end{smallmatrix}\right)$, and $I_{n+1}$ is the identity matrix of order $n+1$. $G$ acts on the quaternion projective space linearly on $\mathbb{H}P^n = \{[z] : z \in \mathbb{H}^{n+1}\backslash\{\mathbf{0}\}\}$, and $K = Sp(1) \times Sp(n)$ is the isotropy subgroup $K$ of $G$ at $o = {}^{\mathrm{t}}[1, 0, \ldots, 0]$. The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is given by

$$\mathfrak{g} = \mathfrak{sp}(n+1) = \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \,\middle|\, A, B \in M_{n+1}(\mathbb{C}),\ {}^{\mathrm{t}}\overline{A} + A = O, {}^{\mathrm{t}}B = B \right\},$$

$$\mathfrak{k} = \mathfrak{sp}(1) \times \mathfrak{sp}(n) = \left\{ \left(\begin{array}{cc|cc} x & 0 & y & 0 \\ 0 & X & 0 & Y \\ \hline -\overline{y} & 0 & \overline{x} & 0 \\ 0 & -\overline{Y} & 0 & \overline{X} \end{array}\right) \,\middle|\, x \in \sqrt{-1}\mathbb{R},\ y \in \mathbb{C},\right.$$

$$\left. X, Y \in M_n(\mathbb{C}), {}^{\mathrm{t}}\overline{X} + X = 0, {}^{\mathrm{t}}Y = Y \right\},$$

$$\mathfrak{m} = \left\{ \left(\begin{array}{cc|cc} 0 & Z & 0 & W \\ -{}^{\mathrm{t}}\overline{Z} & O & {}^{\mathrm{t}}W & O \\ \hline 0 & -\overline{W} & 0 & \overline{Z} \\ -{}^{\mathrm{t}}\overline{W} & O & -{}^{\mathrm{t}}Z & O \end{array}\right) \,\middle|\, Z, W \in M(1, n, \mathbb{C}) \right\}.$$

For a $C^\infty$ $\mathfrak{m}$-valued function $F_{\mathfrak{m}}(t)$ given by

$$F_{\mathfrak{m}}(t) = \begin{pmatrix} 0 & Z & 0 & W \\ -{}^{\mathrm{t}}\overline{Z} & O & {}^{\mathrm{t}}W & O \\ 0 & -\overline{W} & 0 & \overline{Z} \\ -{}^{\mathrm{t}}\overline{W} & O & -{}^{\mathrm{t}}Z & O \end{pmatrix}, \tag{4.40}$$

where $Z = Z(t) = (z_1(t), \ldots, z_n(t))$, $W = W(t) = (w_1(t), \ldots, w_n(t))$, and for $F_{\mathfrak{m}}$ in (4.40) with $F_{\mathfrak{k}} \equiv 0$, the biharmonic map equation (4.6) is equivalent to

$$\begin{cases} -Z''' - (|Z|^2 + |W|^2)Z \\ \qquad + (2\langle Z, Z'\rangle + 2\langle W, W'\rangle - \langle Z', Z\rangle - \langle W', W\rangle)Z \\ \qquad + (\langle Z', \overline{W}\rangle - \langle W', \overline{Z}\rangle)\overline{W} = 0, \\ -W''' - (|Z|^2 + |W|^2)W \\ \qquad + (2\langle Z, Z'\rangle + 2\langle W, W'\rangle - \langle Z', Z\rangle - \langle W', W\rangle)W \\ \qquad + 3(\langle Z', \overline{W}\rangle - \langle W', \overline{Z}\rangle)\overline{Z} = 0, \end{cases} \tag{4.41}$$

where $Z' = ({z_1}'(t), \dots, {z_n}'(t))$ and $\langle Z, W\rangle := \sum_{i=1}^n z_i(t)\,\overline{w_i(t)}$.

We find the following solutions of (4.41):

Case (i): $z_1(t) = \cdots = z_n(t) = D_t$ and $w_1(t) = \cdots = w_n(t) = 0$.
Case (ii): $z_1(t) = \cdots = z_n(t) = \sqrt{-1}D_t$ and $w_1(t) = \cdots = w_n(t) = 0$.
Case (iii): $z_1(t) = \cdots = z_n(t) = 0$ and $w_1(t) = \cdots = w_n(t) = D_t$.
Case (iv): $z_1(t) = \cdots = z_n(t) = 0$ and $w_1(t) = \cdots = w_n(t) = \sqrt{-1}D_t$.

The corresponding biharmonic curves into the quaternion projective spaces $\mathbb{H}P^n$ are given as follows:

Case (i): $$\varphi(t) = x\left[\cos(\sqrt{n}\,d_t), -\frac{1}{\sqrt{n}}\sin(\sqrt{n}\,d_t), \dots, -\frac{1}{\sqrt{n}}\sin(\sqrt{n}\,d_t)\right].$$

Case (ii): $$\varphi(t) = x\left[\cos(\sqrt{n}\,d_t), i\frac{1}{\sqrt{n}}\sin(\sqrt{n}\,d_t), \dots, i\frac{1}{\sqrt{n}}\sin(\sqrt{n}\,d_t)\right].$$

Case (iii): $$\varphi(t) = x\left[\cos(\sqrt{n}\,d_t), -j\frac{1}{\sqrt{n}}\sin(\sqrt{n}\,d_t), \dots, -j\frac{1}{\sqrt{n}}\sin(\sqrt{n}\,d_t)\right].$$

Case (iv): $$\varphi(t) = x\left[\cos(\sqrt{n}\,d_t), k\frac{1}{\sqrt{n}}\sin(\sqrt{n}\,d_t), \dots, k\frac{1}{\sqrt{n}}\sin(\sqrt{n}\,d_t)\right].$$

Here, $x \in Sp(n+1)$, $i$, $j$ and $k$ are the quaternions satisfying $i^2 = j^2 = k^2 = -1$ and $ij = k$, and $d_t = (a/3)t^3 + (b/2)t^2 + ct$, $a$, $b$ and $c$ are constant real numbers. In each case, $\varphi$ is harmonic if and only if $a = b = 0$.

## 5. Biharmonic maps from plane domains

### 5.1. Setting and deriving the equations

In this section, we will treat biharmonic maps of $(M, g)$ into a Riennian symmetric space $(G/K, h)$, with $\dim M = 2$. We assume that $(M, g) = (\Omega, g)$ is an open domain in the 2-dimensional Euclidean space $\mathbb{R}^2$ with $g = \mu^2\, g_0$, where $\mu$ is a positive $C^\infty$ function on $\Omega$, $g_0 = (dx)^2 + (dy)^2$ is the standard Euclidean metric and $(x, y)$ is the standard coordinate on $\mathbb{R}^2$.

Let $\varphi$ be a $C^\infty$ map from $\Omega$ into a symmetric space $N = G/K$ with a local lift $\psi : \Omega \to G$ satisfying $\varphi = \pi \circ \psi$, where $\pi : G \to G/K$ is the standard projection. The pull back of the Maurer-Cartan form $\theta$ on $G$ by $\psi$ is given by

$$\alpha = \psi^{-1} d\psi = \psi^{-1}\frac{\partial \psi}{\partial x} dx + \psi^{-1}\frac{\partial \psi}{\partial y} dy$$
$$= A_x\, dx + A_y\, dy,$$

where we decompose two $\mathfrak{g}$-valued functions $A_x := \psi^{-1}(\partial\psi/\partial x)$ and $A_y := \psi^{-1}(\partial\psi/\partial y)$ on $\Omega$ according to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ as follows:

$$A_x = A_{x,\mathfrak{k}} + A_{x,\mathfrak{m}}, \quad A_y = A_{y,\mathfrak{k}} + A_{y,\mathfrak{m}},$$

which yield the decomposition of $\alpha$: $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$, where

$$\alpha_{\mathfrak{k}} = A_{x,\mathfrak{k}}\, dx + A_{y,\mathfrak{k}}\, dy, \quad \alpha_{\mathfrak{m}} = A_{x,\mathfrak{m}}\, dx + A_{y,\mathfrak{m}}\, dy.$$

Then, we have by a direct computation,

$$\delta(\alpha_{\mathfrak{m}}) = -\mu^{-2}\left\{\frac{\partial A_{x,\mathfrak{m}}}{\partial x} + \frac{\partial A_{y,\mathfrak{m}}}{\partial y}\right\}. \tag{5.1}$$

Indeed, if we take, as an orthonormal frame field with respect to $g$, $e_1 = (1/\mu)(\partial/\partial x)$ and $e_2 = (1/\mu)(\partial/\partial y)$. Then, we have

$$\alpha_{\mathfrak{m}}(\nabla_{e_1} e_1) = -\mu^{-3}\frac{\partial \mu}{\partial y} A_{y,\,\mathfrak{m}}, \quad \alpha_{\mathfrak{m}}(\nabla_{e_2} e_2) = -\mu^{-3}\frac{\partial \mu}{\partial x} A_{x,\,\mathfrak{m}},$$

and

$$\delta(\alpha_{\mathfrak{m}}) = -\sum_{i=1}^{2}\left\{\nabla_{e_i}(\alpha_{\mathfrak{m}}(e_i)) - \alpha_{\mathfrak{m}}(\nabla_{e_i} e_i\right\},$$

we have (5.1). □

Next, we have to calculate the harmonic map equation (3.25), and the biharmonic map equation (3.26) in this case.

First, for the left hand side of (3.25), we have

$$-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)]$$

$$= \mu^{-2}\left\{\frac{\partial A_{x,\mathfrak{m}}}{\partial x} + \frac{\partial A_{y,\mathfrak{m}}}{\partial y}\right\} + [\mu^{-1}\,A_{x,\mathfrak{k}}, \mu^{-1}\,A_{x,\mathfrak{m}}] + [\mu^{-1}\,A_{y,\mathfrak{k}}, \mu^{-1}\,A_{y,\mathfrak{m}}]$$

$$= \mu^{-2}\left\{\frac{\partial A_{x,\mathfrak{m}}}{\partial x} + \frac{\partial A_{y,\mathfrak{m}}}{\partial y} + [A_{x,\mathfrak{k}}, A_{x,\mathfrak{m}}] + [A_{y,\mathfrak{k}}, A_{y,\mathfrak{m}}]\right\}. \tag{5.2}$$

For the left hand side of (3.26), since $\Delta_g = -\mu^{-2}\{\partial^2/\partial x^2 + \partial^2/\partial y^2\}$, we have

$$\begin{aligned}
&\Delta_g\Big( - \delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)]\Big) \\
&\quad + \sum_{s=1}^{m}\Big[\Big[ - \delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)], \alpha_{\mathfrak{m}}(e_s)\Big], \alpha_{\mathfrak{m}}(e_s)\Big] \\
&= -\mu^{-2}\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}\left(\mu^{-2}\left\{\frac{\partial A_{x,\mathfrak{m}}}{\partial x} + \frac{\partial A_{y,\mathfrak{m}}}{\partial y} + [A_{x,\mathfrak{k}}, A_{x,\mathfrak{m}}] + [A_{y,\mathfrak{k}}, A_{y,\mathfrak{m}}]\right\}\right) \\
&\quad + \mu^{-4}\left[\left[\frac{\partial A_{x,\mathfrak{m}}}{\partial x} + \frac{\partial A_{y,\mathfrak{m}}}{\partial y} + [A_{x,\mathfrak{k}}, A_{x,\mathfrak{m}}] + [A_{y,\mathfrak{k}}, A_{y,\mathfrak{m}}], A_{x,\mathfrak{m}}\right], A_{x,\mathfrak{m}}\right] \\
&\quad + \mu^{-4}\left[\left[\frac{\partial A_{x,\mathfrak{m}}}{\partial x} + \frac{\partial A_{y,\mathfrak{m}}}{\partial y} + [A_{x,\mathfrak{k}}, A_{x,\mathfrak{m}}] + [A_{y,\mathfrak{k}}, A_{y,\mathfrak{m}}], A_{y,\mathfrak{m}}\right], A_{y,\mathfrak{m}}\right].
\end{aligned} \tag{5.3}$$

Therefore, we have that $\varphi : (\Omega, g) \to (G/K, h)$ is biharmonic if and only if the right hand side of (5.3) vanishes.

Second, we have to examine the integrability condition (3.23) or (3.24). We have

$$\begin{aligned}
&d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + \frac{1}{2}[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}] \\
&\quad = \left\{ - \frac{\partial A_{x,\mathfrak{k}}}{\partial y} + \frac{\partial A_{y,\mathfrak{k}}}{\partial x} + [A_{x,\mathfrak{k}}, A_{y,\mathfrak{k}}] + [A_{x,\mathfrak{m}}, A_{y,\mathfrak{m}}]\right\} dx \wedge dy \\
&\quad = 0,
\end{aligned}$$

so that we have

$$-\frac{\partial A_{x,\mathfrak{k}}}{\partial y} + \frac{\partial A_{y,\mathfrak{k}}}{\partial x} + [A_{x,\mathfrak{k}}, A_{y,\mathfrak{k}}] + [A_{x,\mathfrak{m}}, A_{y,\mathfrak{m}}] = 0. \tag{5.4}$$

For the second equation of (3.24), $d\alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{m}}] = 0$, we have

$$-\frac{\partial A_{x,\mathfrak{m}}}{\partial y} + \frac{\partial A_{y,\mathfrak{m}}}{\partial x} + [A_{x,\mathfrak{k}}, A_{y,\mathfrak{m}}] + [A_{x,\mathfrak{m}}, A_{y,\mathfrak{k}}] = 0. \tag{5.5}$$

Summing up the above, we obtain

**Theorem 5.1** *Let $\Omega \subset \mathbb{R}^2$ an open domian, $g = \mu^2 g_0$, $\mu > 0$, a positive $C^\infty$ function on $\Omega$, and $g_0 = (dx)^2 + (dy)^2$ is standard Riemannian metric on $\mathbb{R}^2$. on which $(x, y)$ is the standard coordinate. Let $(G/K, h)$ a Riemannian symmetric space, with $\pi : G \to G/K$, the projection. For every $C^\infty$ map from $\Omega$ into $G/K$ with a local lift $\psi : \Omega \to G$ such that $\varphi = \pi \circ \psi$, let $\alpha = \psi^*\theta$, the pull back of the Maurer-Cartan form $\theta$ on $G$ by $\psi$ and decompose it in such a way that $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$ corresponding to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then,*

(1) *$\varphi : (\Omega, g) \to (G/K, h)$ is harmonic if and only if*

$$\frac{\partial A_{x,\mathfrak{m}}}{\partial x} + \frac{\partial A_{y,\mathfrak{m}}}{\partial y} + [A_{x,\mathfrak{k}}, A_{x,\mathfrak{m}}] + [A_{y,\mathfrak{k}}, A_{y,\mathfrak{m}}] = 0. \tag{5.6}$$

(2) *$\varphi : (\Omega, g) \to (G/K, h)$ is biharmonic if and only if* (5.3) *vanishes.*
(3) *For the integrability condition,* (5.4) *and* (5.5) *must hold.*
(4) *In particular, for a* horizontal *lift $\psi$, i.e., $\alpha_{\mathfrak{k}} \equiv 0$, we have*

$$-\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}\left(\mu^{-2}\left\{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right\}\right) + \left[\left[\mu^{-2}\left\{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right\}, P\right], P\right]$$
$$+ \left[\left[\mu^{-2}\left\{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right\}, Q\right], Q\right] = 0, \tag{5.7}$$

$$[P, Q] = 0, \tag{5.8}$$

$$-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0, \tag{5.9}$$

*where we put $P := \alpha_{x,\mathfrak{m}}$ and $Q := \alpha_{y,\mathfrak{m}}$. In the case $\mu = 1$, the following three equations must hold for the biharmonic map $\varphi$ :*

$$-P_{xxx} - P_{xyy} - Q_{xxy} - Q_{yyy}$$
$$+ [[P_x + Q_y, P], P] + [[P_x + Q_y, Q], Q] = 0, \tag{5.10}$$

$$[P, Q] = 0, \tag{5.11}$$

$$P_y - Q_x = 0, \tag{5.12}$$

*where we denote $P_x = \partial P/\partial x$, etc.*

**5.2. Solving the biharmonic map equations**

In this subsection, we want to give the solutions of the equations (5.10), (5.11) and (5.12).

To do it, let us consider the special case that $P_y \equiv 0$ and $Q_x \equiv 0$, i.e., $P(x,y) = P(x)$ and $Q(x,y) = Q(y)$. Then, (5.12) holds clearly. The left hand side of (5.10) coincides with

$$\begin{aligned}\{-P_{xxx} + [[P_x, P], P]\} + \{-Q_{yyy} + [[Q_y, Q], Q]\}\\ + [[Q_y, P], P] + [[P_x, Q], Q] = 0.\end{aligned} \tag{5.13}$$

Here, we have that $[[Q_y, P], P] = 0$ and $[[P_x, Q], Q] = 0$. Because, we have due to $Q_x = 0$

$$\frac{\partial}{\partial x}[[P, Q], Q] = [[P_x, Q], Q]. \tag{5.14}$$

But, due to (5.11) the left hand side of (5.14) must vanish. By the same way, we have $[[Q_y, P], P] = 0$.

Thus, (5.13) turns out that

$$-\{-P_{xxx} + [[P_x, P], P]\} = -Q_{yyy} + [[Q_y, Q], Q] \tag{5.15}$$

But, notice that the left hand side of (5.15) is an $\mathfrak{m}$-valued function only in $x$ and the right hand side of (5.15) is the one only in $y$, so we have

$$\begin{cases} -P_{xxx} + [[P_x, P], P] = c, \\ -Q_{yyy} + [[Q_y, Q], Q] = -c, \end{cases} \tag{5.16}$$

where $c \in \mathfrak{m}$ is a constant vector.

Notice here that both two equations of (5.16) are the same as (4.6) in the case $c = 0$. So, we can obtain the following two theorems by carrying out the similar calculations as in 4.2.

Thus, we have

**Theorem 5.2** *Let $(G/K, h)$ be a Riemannian symmetric space whose rank is bigger than or equal to two, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, the Cartan decomposition, $\mathfrak{a}$, a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{m}$. Let $X, Y \in \mathfrak{a}$ be two elements in $\mathfrak{a}$ which are linearly independent.*

(1) *Let us take two $\mathfrak{m}$-valued functions $P(x,y) = (a_1\,x^2 + b_1\,x + c_1)X$ and $Q(x,y) = (a_2\,y^2 + b_2\,y + c_2)Y$, where $a_i$, $b_i$ and $c_i$ $(i = 1, 2)$ are constant real numbers. Then, $P$ and $Q$ are solutions of* (5.10), (5.11) *and* (5.12). *For such $P$ and $Q$, there exists a unique $C^\infty$ map $\psi$ from $\Omega$ into $G$ such that $\varphi = \pi \circ \psi$ is a biharmonic mapping form $(\Omega, g_0)$ into $(G/K, h)$ with $\varphi(0,0) = x_0 \in G$ for a fixed point $x_0 \in G/K$. $\varphi : (\Omega, g) \to (G/K, h)$ is harmonic if and only if $a_i = b_i = 0$ $(i = 1, 2)$.*

(2) *Assume that $G$ is a matrix Lie group, i.e., a subgroup of $GL(N, \mathbb{C})$. Then, the above $C^\infty$ maps $\psi : \Omega \to G$ and $\varphi = \pi \circ \psi$ are given by*

$$\begin{cases} \psi(x,y) = x_0 \exp(d_x\,X + d_y\,Y) \in G, \\ \varphi(x,y) = x_0 \exp(d_x\,X + d_y\,Y) \cdot o \in G/K, \end{cases} \tag{5.17}$$

*where $o = \{K\} \in G/K$, $d_x = (a_1/3)x^3 + (b_1/2)x^2 + c_1\,x$ and $d_y = (a_2/3)y^3 + (b_2/2)y^2 + c_2\,y$, respectively.*

*Proof.* We only have to verify the statement (2). By the assumption that $\{X, Y\}$ is abelian, we have for the $\psi(x,y)$ of the form (5.17), as a matrix of degree $N$,

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= x_0 \exp(d_x\,X + d_y\,Y) \cdot \frac{\partial}{\partial x}(d_x\,X + d_y\,Y) \\ &= \psi \cdot (a_1\,x^2 + b_1\,x + c_1)X \\ &= \psi\,P, \end{aligned}$$

so we have $\psi^{-1}(\partial\psi/\partial x) = P$. By the same way, $\psi^{-1}(\partial\psi/\partial y) = Q$, so we have $\psi^{-1}d\psi = P\,dx + Q\,dy = \alpha$. The mapping $\psi$ is the desired $C^\infty$ mapping of $\Omega$ into $G$, and due to Theorem 3.6, we obtain a biharmonic mapping of $(\Omega, g_0)$ into $(G/K, h)$. □

**Remark** When $a_i = b_i = 0$ $(i = 1, 2)$, the mapping $\varphi : \mathbb{R}^2 \to (G/K, h)$ is a well known totally geodesic immersion into a Riemannian symmetric space $(G/K, h)$.

By a calculation similar to that in the subsection 4.2, we obtain

**Theorem 5.3** *For the cases of the standard unit sphere* $(S^n, h)$, *the complex projective space* $(\mathbb{C}P^n, h)$, *the quaternion one* $(\mathbb{H}P^n, h)$, *we obtain the following biharmonic mappings of* $(\mathbb{R}^2, g_0)$ *into them, respectively.*

(1) *Case of* $(S^n, h)$ :

$$\varphi_1(t) = x_0\, {}^{\mathrm{t}}\Big( \cos(\sqrt{n}(d_x + d_y)), \frac{1}{\sqrt{n}} \sin(\sqrt{n}(d_x + d_y)), \ldots, \frac{1}{\sqrt{n}} \sin(\sqrt{n}(d_x + d_y))\Big), \quad (5.18)$$

*is a biharmonic mapping of* $(\mathbb{R}^2, g_0)$ *into* $(S^n, h)$, *where* $x_0 \in G = SO(n+1)$.

(2) *Case of* $(\mathbb{C}P^n, h)$ :

$$\varphi_2(t) = x_0\, {}^{\mathrm{t}}\Big( \cos(\sqrt{n}(d_x + d_y)), \frac{\sqrt{-1}}{\sqrt{n}} \sin(\sqrt{n}(d_x + d_y)), \ldots, \frac{\sqrt{-1}}{\sqrt{n}} \sin(\sqrt{n}(d_x + d_y))\Big), \quad (5.19)$$

*is a biharmonic mapping of* $(\mathbb{R}^2, g_0)$ *into* $(\mathbb{C}P^n, h)$, *where* $x_0 \in G = SU(n+1)$.

(3) *Case of* $(\mathbb{H}P^n, h)$ :

$$\varphi_3(t) = x_0\, {}^{\mathrm{t}}\Big( \cos(\sqrt{n}(d_x + d_y)), \frac{k}{\sqrt{n}} \sin(\sqrt{n}(d_x + d_y)), \ldots, \frac{k}{\sqrt{n}} \sin(\sqrt{n}(d_x + d_y))\Big), \quad (5.20)$$

*is a biharmonic mapping of* $(\mathbb{R}^2, g_0)$ *into* $(\mathbb{H}P^n, h)$, *where* $x_0 \in G = Sp(n+1)$, *and* $i$, $j$ *and* $k$ *are the quaternions satisfying* $i^2 = j^2 = k^2 = -1$ *and* $ij = k$.

*Here, in all the cases,* $d_t = (a/3)t^3 + (b/2)t^2 + ct$ *for* $t = x$ *or* $t = y$.

*Furthermore,* $\varphi_i$ $(i = 1, 2, 3)$ *are harmonic maps of* $(\mathbb{R}^2, g_0)$ *into* $(S^n, h)$, $(\mathbb{C}P^n, h)$ *or* $(\mathbb{H}P^n, h)$ *if and only if* $a = b = 0$, *respectively.*

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