Hokkaido Mathematical Journal Vol. 43 (2014) p. 51–64

Quasi-invariance of measures of analytic type on locally compact abelian groups

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(Received March 19, 2012)

Abstract. Asmar, Montgomery-Smith and Saeki gave a generalization of a theorem of Bochner for a locally compact abelian group with certain direction. We show that a strong version of their result holds for a σ-compact, connected locally compact abelian group with certain direction. We also give several conditions for quasi-invariance of analytic measures and another proof of a theorem of deLeeuw and Glicksberg.

Key words: LCA group, measure, Fourier transform, quasi-invariant.

1. Introduction

Let G be a LCA group (locally compact abelian group) with the dual group $\hat{G}$. m_G stands for the Haar measure of G. Let $L^1(G)$ and $M(G)$ be the group algebra and the measure algebra, respectively. For μ in $M(G)$, $\hat{\mu}$ denotes the Fourier-Stieltjes transform of μ, i.e., $\hat{\mu}(\gamma) = \int_G (-x, \gamma)\, d\,\mu(x)$ for $\gamma \in \hat{G}$.

Let ψ be a nontrivial continuous homomorphism from $\hat{G}$ into $\mathbb{R}$ (the reals), and let $\phi : \mathbb{R} \to G$ be the dual homomorphism of ψ. We say that $\mu \in M(G)$ is of analytic type if $\hat{\mu}$ vanishes off $\psi^{-1}([0, \infty))$. Asmar, Montgomery-Smith and Saeki [2] obtained the following theorem.

Theorem A ([2, Theorem 4.5]) *Let $\mu \in M(G)$, and suppose that, for every $s \in \mathbb{R}$, $\psi^{-1}((-\infty, s]) \cap \mathrm{supp}(\hat{\mu})$ is compact. Then $\mu \ll m_G$.*

As for the above theorem, we consider the following:

(1.1) Are $\mu \neq 0$ and m_G mutually absolutely continuous under the condition in Theorem A?

As will be showed in the following example, (1.1) is not true, in general.

2000 Mathematics Subject Classification : 43A05, 43A10, 43A25.

Example 1.1 Let $G = \mathbb{T} \oplus F$, where $\mathbb{T}$ is the circle group and F is a nontrivial finite abelian group. Let $\psi : \hat{G} \cong \mathbb{Z} \oplus \hat{F} \to \mathbb{Z}$ $(\subset \mathbb{R})$ be the projection. Then ψ is a nontrivial continuous homomorphism. Put $\mu = m_{\mathbb{T}} \times \delta_0$. Since $\hat{\mu} = \chi_{\{0\}\times\hat{F}}$, $\psi^{-1}((-\infty, s]) \cap \mathrm{supp}(\hat{\mu})$ is compact for every $s \in \mathbb{R}$. However, μ and m_G are not mutually absolutely continuous.

On the other hand, the following holds.

Example 1.2 (cf. [19, Example 2.1]) Let f and g be functions on $\mathbb{Z}^+$ (the set of nonnegative integers) such that $g(n) \le f(n)$ for all $n \in \mathbb{Z}^+$, and put $F = \{(n, m) \in \mathbb{Z}^2 : n \in \mathbb{Z}^+,\ g(n) \le m \le f(n)\}$. Let μ be a nonzero measure in $M(\mathbb{T}^2)$ such that $\hat{\mu}$ vanishes off F. Then μ and $m_{\mathbb{T}^2}$ are mutually absolutely continuous.

We note that the group G in Example 1.1 is not connected. In this paper, we show that, if G is σ-compact and connected, (1.1) holds. We also give several conditions for quasi-invariance of measures of analytic type.

2. Notation and results

Let G be a LCA group with the dual group $\hat{G}$. For $x \in G, \delta_x$ denotes the point mass at x. For a closed subset E of $\hat{G}, M_E(G)$ stands for the space of measures in $M(G)$ whose Fourier-Stieltjes transform vanish off E. Let $C_o(G)$ be the Banach space of continuous functions on G which vanish at infinity. Then $M(G)$ is identified with the dual space of $C_o(G)$. Let $M^+(G)$ be the set of nonnegative measures in $M(G)$. For $\mu \in M(G)$ and $f \in L^1(|\mu|)$, we often use the notation $\mu(f)$ as $\int_G f(x) d\mu(x)$.

Let ψ be a nontrivial continuous homomorphism from $\hat{G}$ into $\mathbb{R}$. We may assume that there exists $\chi_o \in \hat{G}$ such that $\psi(\chi_o) = 1$ by considering a multiplication of ψ if necessary. Let $\phi : \mathbb{R} \to G$ be the dual homomorphism of ψ, i.e., $(\phi(t), \gamma) = e^{i\psi(\gamma)t}$ for $t \in \mathbb{R}$ and $\gamma \in \hat{G}$.

Let Λ be a discrete subgroup of $\hat{G}$ generated by χ_o, and let $K = \Lambda^{\perp}$, the annhilator of Λ. We define a continuous homomorphism $\alpha : \mathbb{R} \oplus K \to G$ by

$$\alpha(t, u) = \phi(t) + u. \tag{2.1}$$

Then $\alpha((-\pi, \pi] \times K) = G$ and α is a homeomorphism on the interior of $(-\pi, \pi] \times K$ (cf. [3, Lemma 6.1], [14, Lemma 2.3]). We note that $\ker(\alpha) =$

$\{(2\pi n, -\phi(2\pi n)) : n \in \mathbb{Z}\}$ and $\ker(\alpha)^\perp = \{(\psi(\gamma), \gamma|_K) : \gamma \in \hat{G}\} \cong \hat{G}$. For $\mu \in M(\mathbb{R} \oplus K)$, we have $\alpha(\mu)\hat{}(\gamma) = \hat{\mu}(\psi(\gamma), \gamma|_K)$ for $\gamma \in \hat{G}$. Moreover, we have the following (cf. [14, Proposition 2.2]):

$$\begin{aligned} &\alpha(L^1(\mathbb{R} \oplus K)) \subset L^1(G); \\ &\alpha(M_s(\mathbb{R} \oplus K)) \subset M_s(G), \end{aligned} \tag{2.2}$$

where $M_s(G)$ denotes the subspace of $M(G)$ consisting of singular measures. For $0 < \epsilon < 1/6$, we define a function $\Delta_\epsilon(t, \omega)$ on $\mathbb{R} \oplus \hat{K}$ by

$$\Delta_\epsilon(t, \omega) = \begin{cases} \max\left(1 - \dfrac{1}{\epsilon}|t|, 0\right) & (\omega = 0), \\ 0 & (\omega \neq 0). \end{cases}$$

For $\mu \in M(G)$, define a function Φ_μ^ϵ on $\mathbb{R} \oplus \hat{G}$ by

$$\Phi_\mu^\epsilon(t, \omega) = \sum_{\gamma \in \hat{G}} \hat{\mu}(\gamma) \Delta_\epsilon((t, \omega) - (\psi(\gamma), \gamma\,|_K)).$$

Then $\Phi_\mu^\epsilon \in M(\mathbb{R} \oplus K)\hat{}$, $\|(\Phi_\mu^\epsilon)\check{}\| = \|\mu\|$ and $\alpha((\Phi_\mu^\epsilon)\check{}) = \mu$ for $\mu \in M(G)$ (cf. [17, (3.4)–(3.7)]). We define an isometry $T_\psi^\epsilon : M(G) \to M(R \oplus K)$ by

$$T_\psi^\epsilon(\mu) = (\Phi_\mu^\epsilon)\check{}. \tag{2.3}$$

Let $k_\epsilon(t) = 1/\pi \cdot (1 - \cos(\epsilon t))/(\epsilon t)^2$. Then $\hat{k_\epsilon}(s) = \int_{-\infty}^{\infty} k_\epsilon(t) e^{-ist} dt = \max(1 - (1/\epsilon)|t|, 0)$. We define a function $\nabla_\epsilon(t, u)$ on $\mathbb{R} \oplus K$ by $\nabla_\epsilon(t, u) = k_\epsilon(t)$. The following theorem and proposition are due to [17].

Theorem B ([17, Theorem 3.1]) *For $\mu \in M^+(G)$, let $\tilde{\mu}$ be the periodic extension of μ to $\mathbb{R} \oplus K$, i.e., for a Borel set $E \subset \mathbb{R} \oplus K$,*

$$\tilde{\mu}(E) = \sum_{n \in \mathbb{Z}} \mu(\alpha(E \cap [2\pi n, 2\pi(n+1)) \times K)).$$

Then $T_\psi^\epsilon(\mu) = 2\pi \nabla_\epsilon \tilde{\mu}$.

Proposition A ([17, Proposition 3.1]) *Let $\mu \in M^+(G)$ and $f \in L^1(\mu)$. Then*

$$T_\psi^\epsilon(f\mu) = (f \circ \alpha)T_\psi^\epsilon(\mu).$$

Hence $f \circ \alpha \in L^1(T_\psi^\epsilon(\mu))$ *and* $T_\psi^\epsilon(f\mu) \ll T_\psi^\epsilon(\mu)$. *In particular,* $\xi \ll \mu$ $(\xi \in M(G))$ *implies* $T_\psi^\epsilon(\xi) \ll T_\psi^\epsilon(\mu)$.

For $\mu \in M(G)$, μ is said to be quasi-invariant if $|\mu| * \delta_x \ll |\mu|$ for every $x \in G$. μ is called quasi-invariant under ϕ if $|\mu| * \delta_{\phi(t)} \ll |\mu|$ for every $t \in \mathbb{R}$[1].

Remark 2.1 (cf. [16, Remark 4.1 and Proposition 4.1])

(i) Suppose there exists a nonzero measure $\mu \in M(G)$ that is quasi-invariant. Then, by regularity of μ, G must be σ-compact.

(ii) Let μ be a nonzero measure in $M(G)$. Then the following are equivarent.

(ii.a) μ is quasi-invariant.

(ii.b) $|\mu|$ and m_G are mutually absolutely continuos.

We state our first result.

Theorem 2.1 *Let* μ *be a nonzero measure in* $M(G)$ *which is of analytic type, and let* ν *be a nonzero measure in* $M^+(\mathbb{R})$. *Then* $(\nu \times \delta_0) * T_\psi^\epsilon(|\mu|)$ *and* $T_\psi^\epsilon(|\mu|)$ *are mutually absolutely continuous.*

Corollary 2.1 *Let* μ *and* ν *be as in Theorem* 2.1. *Then* $\phi(\nu) * |\mu|$ *and* $|\mu|$ *are mutually absolutely continuous.*

Proof. It follows from Theorem 2.1 that $(\nu \times \delta_0) * T_\psi^\epsilon(|\mu|)$ and $T_\psi^\epsilon(|\mu|)$ are mutually absolutely continuous. Since $\alpha((\nu \times \delta_0) * T_\psi^\epsilon(|\mu|)) = \phi(\nu) * |\mu|$ and $\alpha(T_\psi^\epsilon(|\mu|)) = |\mu|$, $\phi(\nu) * |\mu|$ and $|\mu|$ are also mutually absolutely continuous. □

Corollary 2.2 *Let* μ *be a nonzero measure in* $M(G)$, *and let* ν *be a nonzero measure in* $M^+(\mathbb{R})$. *Suppose that, for every* $s \in \mathbb{R}$, $\psi^{-1}((-\infty, s]) \cap \mathrm{supp}(\hat{\mu})$ *is compact. Then* $\phi(\nu) * |\mu|$ *and* $|\mu|$ *are mutually absolutely continuous.*

Proof. By assumption, $\psi(\psi^{-1}((-\infty, 0]) \cap \mathrm{supp}(\hat{\mu}))$ is a compact set in $\mathbb{R}$. Thus there exists $\gamma_o \in \hat{G}$, with $\psi(\gamma_o) < 0$, such that $\psi(\psi^{-1}((-\infty, 0]) \cap \mathrm{supp}(\hat{\mu})) \subset [\psi(\gamma_o), \infty)$. It is easy to see that $(-\gamma_o)\mu$ is of analytic type. Since $|(-\gamma_o)\mu| = |\mu|$, it follows from Corollary 2.1 that $\phi(\nu) * |\mu|$ and $|\mu|$ are

[1]In [17], we call simply "quasi-invariant under ϕ" by "quasi-invariant".

mutually absolutely continuous. □

Let ρ be the measure in $M^+(G)$ which is a continuous image of the measure $(1+x^2)^{-1}dx$ under ϕ. We obtain the following proposition as same as [3, Proposition 2.3].

Proposition 2.1 (cf. [3, Proposition 2.3]) *Let μ be a measure in $M(G)$. Then the following are equivalent.*

(i) *μ is quasi-invariant under ϕ.*
(ii) *$|\mu|$ and $\rho * |\mu|$ are mutually absolutely continuous.*

We have the following corollary, by Corollary 2.1 and Proposition 2.1.

Corollary 2.3 (cf. [3, Main Theorem]) *Let μ be a measure of analytic type in $M(G)$. Then μ is quasi-invariant under ϕ.*

Next we state our main theorem.

Theorem 2.2 *Let G be a σ-compact, connected LCA group. Let ψ be a nontrivial continuous homomorphism from $\hat{G}$ into $\mathbb{R}$. Let μ be a nonzero measure in $M(G)$, and suppose that, for every $s \in \mathbb{R}$, $\psi^{-1}((-\infty, s]) \cap \mathrm{supp}(\hat{\mu})$ is compact. Then μ and m_G are mutually absolutely continuous.*

Throughout this paper, for measures μ and ν, we write $\mu \sim \nu$ if they are mutually absolutely continuous.

3. Proofs of Theorem 2.1 and Theorem 2.2

In this section, we give proofs of Theorem 2.1 and Theorem 2.2. Let $\eta \in M^+(K)$. Let $\mu^{(i)} \in M^+(\mathbb{R} \oplus K)$, and let $\{\xi_u^{(i)}\}_{u \in K}$ be families of measures in $M^+(\mathbb{R})$ with the following properties ($i = 1, 2$):

(1) $u \to (\xi_u^{(i)} \times \delta_u)(f)$ is η-measurable for each bounded Borel function f on $\mathbb{R} \oplus K$,
(2) $\|\xi_u^{(i)}\| \le C$, and
(3) $\mu^{(i)}(f) = \int_K (\xi_u^{(i)} \times \delta_u)(f) d\eta(u)$ for each bounded Borel function f on $\mathbb{R} \oplus K$,

where C is a positive constant. Under this situation, we have the following proposition.

Proposition 3.1 *If $\xi_u^{(1)} \ll \xi_u^{(2)}$ $\eta - a.a.\, u \in K$, then $\mu^{(1)} \ll \mu^{(2)}$.*

Proof. Let A be a Borel set in $\mathbb{R} \oplus K$ such that $\mu^{(2)}(A) = 0$. Then

$$\begin{aligned} 0 &= \int_K (\xi_u^{(2)} \times \delta_u)(A) d\eta(u) \\ &= \int_K \xi_u^{(2)}(A_u) d\eta(u), \end{aligned}$$

where $A_u = \{x \in \mathbb{R} : (x, u) \in A\}$. Thus

$$\xi_u^{(2)}(A_u) = 0 \quad \eta - a.a.\, u \in K,$$

which implies

$$\xi_u^{(1)}(A_u) = 0 \quad \eta - a.a.\, u \in K.$$

Hence we have

$$\mu^{(1)}(A) = \int_K \xi_u^{(1)}(A_u) d\eta(u) = 0.$$

This shows that $\mu^{(1)} \ll \mu^{(2)}$, and the proof is complete. □

Now we prove Theorem 2.1. Put $\eta = \pi_K(T_\psi^\epsilon(|\mu|))$, where $\pi_K : \mathbb{R} \oplus K \to K$ is the projection. By the theory of disintegration of measures (cf. [15, Proposition 1.4]), there exists a family $\{\xi_u\}_{u \in K}$ of measures in $M^+(\mathbb{R})$ with the following properties:

(4) $u \to (\xi_u \times \delta_u)(f)$ is η-measurable for each bounded Borel function f on $\mathbb{R} \oplus K$,
(5) $\|\xi_u\| = 1$, and
(6) $T_\psi^\epsilon(|\mu|)(f) = \int_K (\xi_u \times \delta_u)(f) d\eta(u)$ for each bounded Borel function f on $\mathbb{R} \oplus K$.

Since $T_\psi^\epsilon(\mu)\hat{} = \Phi_\mu^\epsilon$, we note that

(7) $\mathrm{supp}(T_\psi^\epsilon(\mu)\hat{}) \subset [-\epsilon, \infty) \times \hat{K}$.

There exists a Borel measurable function h on G, with $|h| = 1$, such that $\mu = h|\mu|$. Hence Proposition A implies that

(8) $T_\psi^\epsilon(\mu) = (h \circ \alpha) T_\psi^\epsilon(|\mu|)$.

Since $|h \circ \alpha| = 1$, we have $\pi_K(|T_\psi^\epsilon(\mu)|) = \pi_K(T_\psi^\epsilon(|\mu|)) = \eta$. And, there exists a measure $\lambda_u \in M(\mathbb{R})$ such that

$$\lambda_u \times \delta_u = (h \circ \alpha)(\xi_u \times \delta_u).$$

Thus we have, by (4)–(6) and (8),

(9) $u \to (\lambda_u \times \delta_u)(f)$ is η-measurable for each bounded Borel function f on $\mathbb{R} \oplus K$,
(10) $\|\lambda_u\| = 1$, and
(11) $T_\psi^\epsilon(\mu)(f) = \int_K (\lambda_u \times \delta_u)(f) d\eta(u)$ for each bounded Borel function f on $\mathbb{R} \oplus K$.

(7) and [15, Lemma 2.1] imply

(12) $\mathrm{supp}(\hat{\lambda_u}) \subset [-\epsilon, \infty) \quad \eta - a.a.\, u \in K$,

which, together with the F. and M. Riesz theorem, yields

(13) $|\lambda_u| \sim m_{\mathbb{R}} \quad \eta - a.a.\, u \in K$.

Since $|\lambda_u| = \xi_u$, this shows that

$$\xi_u \sim m_{\mathbb{R}} \quad \eta - a.a.\, u \in K,$$

and we have

(14) $\xi_u \sim \nu * \xi_u \quad \eta - a.a.\, u \in K$.

On the other hand, we have, by (4)–(6),

(15) $u \to \{(\nu * \xi_u) \times \delta_u\}(f)$ is η-measurable for each bounded Borel function f on $\mathbb{R} \oplus K$,
(16) $\|\nu * \xi_u\| \le \|\nu\|$, and
(17) $(\nu \times \delta_0) * T_\psi^\epsilon(|\mu|)(f) = \int_K \{(\nu * \xi_u) \times \delta_u\}(f) d\eta(u)$ for each bounded Borel function f on $\mathbb{R} \oplus K$.

Thus Proposition 3.1, together with (4)–(6) and (14)–(17), yields that $(\nu \times \delta_0) * T_\psi^\epsilon(|\mu|) \sim T_\psi^\epsilon(|\mu|)$. This completes the proof of Theorem 2.1.

Before we prove Theorem 2.2, we state a definiion and lemmas.

Definition 3.1 Let G be a LCA group, and let E be a closed subset of $\hat{G}$. We say that E satisfies condition $(*)$ if the following holds:

(∗) For $\mu \in M_E(G)$, μ is quasi-invariant.

The following lemma is due to [16].

Lemma 3.1 ([16, Proposition 4.4]) *Let G_1 and G_2 be LCA groups. Let E_i be a closed subset of $\hat{G}_i$ satisfying condition (∗) ($i = 1, 2$). Then $E_1 \times E_2$ also satisfies condition (∗).*

Lemma 3.2 *Let G be a σ-compact, connected LCA group, and let E be a compact subset of $\hat{G}$. Let μ be a nonzero measure in $M_E(G)$. Then μ and m_G are mutually absolutely continuous.*

Proof. By the structure theorem of LCA groups (cf. [11, 2.4.1 Theorem]) and connectedness of G, we have $G \cong \mathbb{R}^n \oplus K$, where n is a nonnegative integer and K is a connected compact abelian group. Since E is a compact subset of $\hat{G} \cong \mathbb{R}^n \oplus \hat{K}$, there exist a compact set A in $\mathbb{R}^n$ and a compact set B in $\hat{K}$ such that $E \subset A \times B$. Evidently, A satisfies condition (∗), and B also satisfies condition (∗) (cf. [11, 8.4.1 Theorem]). Hence $A \times B$ satisfies condition (∗), by Lemma 3.1. Therefore μ and m_G are mutually absolutely continuous, and the proof is complete. □

Now we prove Theorem 2.2. By hypothesis, there exists a nonzero measure $\nu \in M(\mathbb{R})$ and $s > 0$ such that $\phi(\nu) * \mu \neq 0$ and $\mathrm{supp}(\hat{\nu}) \subset [-s, s]$. Then

$$\begin{aligned} \mathrm{supp}((\phi(\nu) * \mu)\hat{\ }) &= \mathrm{supp}(\hat{\nu}(\psi)\hat{\mu}) \\ &\subset \psi^{-1}((-\infty, s]) \cap \mathrm{supp}(\hat{\mu}), \end{aligned}$$

and $\psi^{-1}((-\infty, s]) \cap \mathrm{supp}(\hat{\mu})$ is compact, by assumption. It follows from Lemma 3.2 that $\phi(\nu) * \mu$ and m_G are mutually absolutely continuous. In particular,

(18) $m_G \ll \phi(|\nu|) * |\mu|$.

On the other hand, Theorem A implies

(19) $\phi(|\nu|) * |\mu| \ll m_G$.

(18), (19) and Corollary 2.2 imply that $|\mu|$ and m_G are mutually absolutely continuous, and the proof of Theorem 2.2 is complete.

4. Conditions for quasi-invariance of analytic measures

In this section, we give conditions for quasi-invariance of measures of analytic type.

Theorem 4.1 *Let G be a σ-compact LCA group, and let μ be a nonzero measure in $M(G)$ that is of analytic type. Then the following are equivalent*:

(i) *μ is quasi-invariant.*
(ii) *$\pi_K(T_\psi^\epsilon(|\mu|))$ is quasi-invariant.*

We need two lemmas to prove the above theorem.

Lemma 4.1 *Let G be a σ-compact LCA group, and let μ be a nonzero measure in $M^+(G)$ that is quasi-invariant. Then $T_\psi^\epsilon(\mu)$ is quasi-invariant.*

Proof. As we noted before, $\alpha((-\pi,\pi] \times K) = G$, $\ker(\alpha) = \{(2\pi n, -\phi(2\pi n)) : n \in \mathbb{Z}\}$ and α is a homeomorphism on the interior of $(-\pi,\pi]\times K$. It follows from the construction of $\tilde{\mu}$ that $\tilde{\mu}$ and $m_{\mathbb{R}\oplus K}$ are mutually absolutely continuos. Noting that $\{(t,u) \in \mathbb{R}\oplus K : \nabla_\epsilon(t,u) = 0\}$ is a $m_{\mathbb{R}\oplus K}$-null set, we have, by Theorem B,

$$T_\psi^\epsilon(\mu) \sim m_{\mathbb{R}\oplus K},$$

and the proof is complete. □

Lemma 4.2 *Let G be a σ-compact LCA group, and let μ be a nonzero measure in $M(G)$ that is of analytic type. Suppose that $\pi_K(T_\psi^\epsilon(|\mu|))$ is quasi-invariant. Then $T_\psi^\epsilon(|\mu|)$ is quasi-invariant.*

Proof. Put $\eta = \pi_K(T_\psi^\epsilon(|\mu|))$. By the theory of disintegration of measures, there exists a family $\{\xi_u\}_{u\in K}$ of measures in $M^+(\mathbb{R})$ with the following properties:

(1) $u \to (\xi_u \times \delta_u)(f)$ is η-measurable for each bounded Borel function f on $\mathbb{R}\oplus K$,
(2) $\|\xi_u\| = 1$, and
(3) $T_\psi^\epsilon(|\mu|)(f) = \int_K (\xi_u \times \delta_u)(f) d\eta(u)$ for each bounded Borel function f on $\mathbb{R}\oplus K$.

As seen in the proof of Theorem 2.1, we have

(4) $\xi_u \sim m_{\mathbb{R}}\ \eta - a.a.\, u \in K$.

Since G is σ-compact, there exists a measure $\omega \in M^+(K)$ such that $\omega \sim m_K$. Let ρ_o be the measure in $M^+(\mathbb{R})$ with $d\rho_o(t) = (1/(1+t^2))dt$. Then

(5) $\rho_o \times \omega \sim m_{\mathbb{R}\oplus K}$.

Claim. $\rho_o \times \omega \sim T_\psi^\epsilon(|\mu|)$.

In fact, let F be a Borel set in $\mathbb{R} \oplus K$ such that $T_\psi^\epsilon(|\mu|)(F) = 0$. Then (3) implies

$$0 = \int_K (\xi_u \times \delta_u)(F) d\eta(u).$$

Hence there exists a Borel set B in K such that

(6) $\eta(B) = 0$, and
(7) $\{u \in K : (\xi_u \times \delta_u)(F) > 0\} \subset B$.

Since η and m_K are mutually absolutely continuous, (6) implies

(8) $\omega(B) = 0$.

Hence we have

$$\begin{aligned}(\rho_o \times \omega)(F) &= \int_K (\rho_o \times \delta_u)(F) d\omega(u) \\ &= \int_B (\rho_o \times \delta_u)(F) d\omega(u) + \int_{K\setminus B} (\rho_o \times \delta_u)(F) d\omega(u) \\ &= 0 + \int_{K\setminus B} (\rho_o \times \delta_u)(F) d\omega(u).\end{aligned}$$

If $u \in K\setminus B$, (7) implies that $\xi_u(F_u) = 0$. Since η and ω are mutually absolutely continuous, it follows from (4) that

$$(\rho_o \times \delta_u)(F) = \rho_o(F_u) = 0 \quad \omega - a.a.\, u \in K\setminus B.$$

Hence

$$\begin{aligned}(\rho_o \times \omega)(F) &= 0 + \int_{K\setminus B} (\rho_o \times \delta_u)(F) d\omega(u) \\ &= 0.\end{aligned}$$

This shows that

(9) $\rho_o \times \omega \ll T_\psi^\epsilon(|\mu|)$.

By a similar argument, we have

(10) $T_\psi^\epsilon(|\mu|) \ll \rho_o \times \omega$.

Claim follows from (9) and (10). By (5) and Claim, $T_\psi^\epsilon(|\mu|)$ is quasi-invariant. This complete the proof. □

Now we prove Theorem 4.1. Suppose μ is quasi-invariant. Then $|\mu|$ is also quasi-invariant. Lemma 4.1 implies that $T_\psi^\epsilon(|\mu|)$ is quasi-invariant, and so $\pi_K(T_\psi^\epsilon(|\mu|))$ is. This shows that (i) implies (ii). Next suppose that $\pi_K(T_\psi^\epsilon(|\mu|))$ is quasi-invariant. It follows from Lemma 4.2 that $T_\psi^\epsilon(|\mu|)$ is quasi-invariant, which, together with Theorem B, yields that $|\tilde{\mu}|$ is quasi-invariant. Hence $|\mu|$ is quasi-invariant, by construction of $|\tilde{\mu}|$. Thus (ii) implies (i), and the proof is complete.

Before we close this section, we give another conditions for quasi-invariance of analytic measures. We recall the space $N(m_G)$ (cf. [18]). Let $N(m_G) = \{\mu \in M(G) : \phi(h) * \mu \in L^1(G)\ \forall h \in L^1(\mathbb{R})\}$. We have the following theorem, by [17, Corollary 2.1] and [17, Remark 4.1].

Theorem C (cf. [17, Corollary 2.1]) *Let μ be a measure in $N(m_G)$ which is of analytic type. Then $\mu \ll m_G$.*

Theorem 4.2 *Let G be a σ-compact LCA group, and let $\mu \in M(G)$ be a nonzero measure of analytic type. Suppose that $\phi(\nu) * \mu \sim m_G$ for every $\nu \in L^1(\mathbb{R})$ with $\phi(\nu) * \mu \neq 0$ and* supp$(\hat{\nu})$ *compact. Then μ and m_G are mutually absolutely continuous.*

Proof. Since μ is a nonzero measure, there exists $\nu \in M(\mathbb{R})$ such that $\phi(\nu) * \mu \neq 0$ and supp$(\hat{\nu})$ is compact. Then we have $\phi(\nu) * \mu \sim m_G$, by assumption, which implies that

$$m_G \ll \phi(|\nu|) * |\mu|.$$

This, combined with Corollary 2.1, yields

(1) $m_G \ll |\mu|$.

On the other hand, we have, by assumption,

(2) $\phi(\nu) * \mu \in L^1(G)$

for all $\nu \in L^1(\mathbb{R})$ with $\mathrm{supp}(\hat{\nu})$ compact. It follows from [11, 2.6.6 Theorem] that (2) holds for all $\nu \in L^1(\mathbb{R})$. This, together with Theorem C, yields

(3) $\mu \ll m_G$.

It follows from (1) and (3) that $\mu \sim m_G$, and the proof is complete. □

Remark 4.1 We note that Theorem 2.2 follws from Theorem 4.2. In fact, let G be a σ-compact, connected LCA group, and suppose that a nonzero measure $\mu \in M(G)$ satisfies that $\psi^{-1}((-\infty, s]) \cap \mathrm{supp}(\hat{\mu})$ is compact for every $s \in \mathbb{R}$. Since $\psi^{-1}((-\infty, 0]) \cap \mathrm{supp}(\hat{\mu})$ is compact, there exists $\gamma_0 \in \hat{G}$, with $\psi(\gamma_0) < 0$, such that

$$\psi(\psi^{-1}((-\infty, 0]) \cap \mathrm{supp}(\hat{\mu})) \subset [\psi(\gamma_0), \infty).$$

Then $(-\gamma_0)\mu$ is of analytic type. Let ν be a nonzero measure in $L^1(\mathbb{R})$ such that $\phi(\nu) * ((-\gamma_0)\mu) \neq 0$ and $\mathrm{supp}(\hat{\nu})$ is compact. We note that

$$\phi(\nu) * ((-\gamma_0)\mu) = (-\gamma_0)\{\phi(e^{i\psi(\gamma_0)\,\cdot}\,\nu) * \mu\}.$$

Since $\mathrm{supp}(\hat{\nu})$ is compact, there exists a positive real number $s > 0$ such that $\mathrm{supp}((e^{i\psi(\gamma_0)\,\cdot}\,\nu)\hat{}) \subset [-s, s]$. Then

$$\begin{aligned}\mathrm{supp}(\hat{\nu} \circ \psi(\cdot - \gamma_0)\hat{\mu}) &= \mathrm{supp}((\phi(e^{i\psi(\gamma_0)\,\cdot}\,\nu) * \mu)\hat{}) \\ &= \mathrm{supp}((e^{i\psi(\gamma_0)\,\cdot}\,\nu)\hat{} \circ \psi\, \hat{\mu}) \\ &\subset \psi^{-1}((-\infty, s]) \cap \mathrm{supp}(\hat{\mu}),\end{aligned}$$

and $\psi^{-1}((-\infty, s]) \cap \mathrm{supp}(\hat{\mu})$ is compact, by assumption. This implies that $\phi(e^{i\psi(\gamma_0)\,\cdot}\,\nu) * \mu \sim m_G$, by Lemma 3.2. Hence $\phi(\nu) * ((-\gamma_0)\mu) = (-\gamma_0)\{\phi(e^{i\psi(\gamma_0)\,\cdot}\,\nu) * \mu\}$ and m_G are mutually absolutely continuous. It follows from Theorem 4.2 that $(-\gamma_0)\mu \sim m_G$, and the desired result is obtained.

Theorem 4.3 *Let G be a σ-compact LCA group, and let $\mu \in M(G)$ be a nonzero measure of analytic type. Then the following are equivalent*:

(i) $\phi(|\nu|) * |\mu| \sim m_G$ *for every* $\nu \in L^1(\mathbb{R})$ *with* $\phi(\nu) * \mu \neq 0$ *and* $\mathrm{supp}(\hat{\nu})$ *compact.*

(ii) $\mu \sim m_G$.

Proof. We only show that (i) implies (ii), because the converse is trivial. Since μ is a nonzero measure, there exists $\nu \in M(\mathbb{R})$ suth that $\phi(\nu) * \mu \neq 0$ and $\mathrm{supp}(\hat{\nu})$ is compact. Then

$$\phi(|\nu|) * |\mu| \sim m_G,$$

by assumption. This, combined with Corollary 2.1, yields that $|\mu| \sim m_G$, and the proof is complete. □

Remark 4.2 We note that Theorem 2.2 follows from Theorem 4.3. In fact, let G and μ be as in Theorem 2.2. Let ν be a measure in $L^1(\mathbb{R})$ such that $\mathrm{supp}(\hat{\nu})$ is compact and $\phi(\nu)*\mu \neq 0$. Then $\mathrm{supp}((\phi(\nu)*\mu)\hat{})$ is compact, which, together with Lemma 3.2, yields that $\phi(\nu) * \mu \sim m_G$. In particular,

$$m_G \ll \phi(|\nu|) * |\mu|.$$

On the other, Theorem A implies that

$$\phi(|\nu|) * |\mu| \ll m_G.$$

Thus we have that $\phi(|\nu|) * |\mu| \sim m_G$, and the desired result follows from Theorem 4.3.

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