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On the univalence conditions for certain class of analytic functions

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Abstract. A univalence condition for certain class of analytic functions was discussed by D. Yang and S. Owa (Hokkaido Math. J. **32** (2003), 127–136). In the present paper, by discussing some subordination relation, a new univalence condition is deduced.

Key words: Analytic function, univalent function, subordination.

1. Introduction

Let $\mathcal{H}$ denote the class of functions $p(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and a complex number a, let $\mathcal{H}[a, n]$ be the class of functions $p(z) \in \mathcal{H}$ of the form

$$p(z) = a + \sum_{k=n}^{\infty} a_k z^k.$$

Also, let $\mathcal{A}$ be the class of functions $f(z) \in \mathcal{H}$ which are normalized by $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$ is denoted by $\mathcal{S}$.

In 1972, Ozaki and Nunokawa [2] proved a univalence criterion for $f(z) \in \mathcal{A}$ as follows.

Lemma 1.1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 \quad (z \in \mathbb{U}),$$

then $f(z)$ is univalent in $\mathbb{U}$, which means that $f(z) \in \mathcal{S}$.

Let $p(z)$ and $q(z)$ be members of the class $\mathcal{H}$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by $p(z) \prec q(z)$ $(z \in \mathbb{U})$, if there

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exists a function $w(z) \in \mathcal{H}$ with $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$, and such that $p(z) = q(w(z))$ $(z \in \mathbb{U})$. From the definition of the subordinations, it is easy to show that $p(z) \prec q(z)$ $(z \in \mathbb{U})$ implies that

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}). \tag{1.1}$$

In particular, if $q(z)$ is univalent in $\mathbb{U}$, then we see that $p(z) \prec q(z)$ $(z \in \mathbb{U})$ is equivalent to the condition (1.1) by considering the function

$$w(z) = q^{-1}(p(z)) \quad (z \in \mathbb{U}).$$

Let $\mathcal{T}(\lambda, \mu)$ denote the class of functions $f(z) \in \mathcal{A}$ which satisfy $f(z)/z \neq 0$ $(z \in \mathbb{U})$ and the inequality

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - \lambda z^2 \left(\frac{z}{f(z)} \right)'' - 1 \right| < \mu \quad (z \in \mathbb{U}) \tag{1.2}$$

for some real number μ $(\mu > 0)$ and for some complex number λ. Yang and Owa [4] discussed the univalency for $f(z) \in \mathcal{T}(\lambda, \mu)$ as follows.

Lemma 1.2 *Let λ be a complex number with $\operatorname{Re}\lambda \geq 0$. Then the class $\mathcal{T}(\lambda, \mu)$ is a subclass of $\mathcal{S}$ for some real number μ with $0 < \mu \leq |1 + 2\lambda|$.*

To obtain the assertion in Lemma 1.2, Yang and Owa [4] discussed the following subordination relation.

Lemma 1.3 *Let λ be a complex number with $\lambda \neq 0$ and $\operatorname{Re}\lambda \geq 0$. If $p(z) \in \mathcal{H}[1, n]$ satisfies the following subordination*

$$p(z) + \lambda z p'(z) \prec 1 + \mu z \quad (z \in \mathbb{U})$$

for some real number μ $(\mu > 0)$, then

$$p(z) \prec 1 + \frac{\mu}{1 + n\lambda} z \quad (z \in \mathbb{U}).$$

In the present paper, we discuss the subordination relation in Lemma 1.3 for the case that $\operatorname{Re}\lambda$ is negative, and deduce an extension of the assertion in Lemma 1.2.

2. Preliminaries

In order to discuss our main results, we will make use of several lemmas.

A function $L(z,t)$ for $z \in \mathbb{U}$ and $t \geq 0$ is said to be a subordination (or Loewner) chain if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geq 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$, and

$$L(z,s) \prec L(z,t) \quad (z \in \mathbb{U})$$

when $0 \leq s \leq t$ (Pommerenke [3] or Miller and Mocanu [1]). Pommerenke [3] derived a necessary and sufficient condition for $L(z,t)$ to be a subordination chain bellow.

Lemma 2.1 *The function $L(z,t) = \sum_{k=1}^{\infty} a_k(t) z^k$ with $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ for $z \in \mathbb{U}$ and $t \geq 0$ is a subordination chain if and only if*

$$\operatorname{Re}\left\{ \frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}} \right\} > 0$$

for $z \in \mathbb{U}$ and $t \geq 0$.

For $0 < r_0 \leq 1$, we let

$$\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}, \quad \partial\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| = r_0\}$$

and $\overline{\mathbb{U}_{r_0}} = \mathbb{U}_{r_0} \cup \partial\mathbb{U}_{r_0}$. In particular, we write $\mathbb{U}_1 = \mathbb{U}$.

Miller and Mocanu [1] derived the following lemma which is related to the subordination of two functions as follows.

Lemma 2.2 *Let $p(z) \in \mathcal{H}[a,n]$ with $p(z) \not\equiv a$. Also, let $q(z)$ be analytic and univalent on the closed unit disk $\overline{\mathbb{U}}$ except for at most one pole on $\partial\mathbb{U}$ with $q(0) = a$. If $p(z)$ is not subordinate to $q(z)$ in $\mathbb{U}$, then there exist two points $z_0 \in \partial\mathbb{U}_r$ with $0 < r < 1$ and $\zeta_0 \in \partial\mathbb{U}$, and a real number k with $k \geq n$ for which $p(\mathbb{U}_r) \subset q(\mathbb{U})$,*

$$\text{(i)} \quad p(z_0) = q(\zeta_0)$$

and

$$\text{(ii)}\ \ z_0 p'(z_0) = k\zeta_0 q'(\zeta_0).$$

This lemma plays a crucial role in developing the theory of differential subordinations.

3. Main results

By making use of Lemma 2.1 and Lemma 2.2, we first develop the assertion concerned with the differential subordinations bellow.

Theorem 3.1 *Let n be a positive integer, and let λ be a complex number with*

$$\operatorname{Re}\lambda \leq 0 \quad \text{and} \quad \left|\lambda + \frac{1}{2n}\right| > \frac{1}{2n}. \tag{3.1}$$

Also, let $q(z)$ be analytic in $\mathbb{U}$ with $q(0) = a$, $q'(0) \neq 0$ and

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > -\frac{1}{n}\operatorname{Re}\left(\frac{1}{\lambda}\right) \quad (z \in \mathbb{U}). \tag{3.2}$$

If $p(z) \in \mathcal{H}[a, n]$ satisfies the following subordination

$$p(z) + \lambda z p'(z) \prec q(z) + \lambda n z q'(z) \quad (z \in \mathbb{U}), \tag{3.3}$$

then $p(z) \prec q(z) \quad (z \in \mathbb{U})$.

Proof. Noting that $q'(0) \neq 0$ and $\operatorname{Re}\lambda \leq 0$, it follows from the inequality (3.2) that the function $q(z)$ is convex univalent in $\mathbb{U}$. Moreover, if we set

$$h(z) = q(z) + \lambda n z q'(z) \quad (z \in \mathbb{U}), \tag{3.4}$$

then, from the inequality (3.2), we find that

$$\operatorname{Re}\left(\frac{h'(z)}{\lambda q'(z)}\right) = \operatorname{Re}\left\{\frac{1}{\lambda} + n\left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \quad (z \in \mathbb{U}). \tag{3.5}$$

Since the function $\lambda q(z)$ is convex univalent in $\mathbb{U}$, the inequality (3.5) shows that the function $h(z)$ is close-to-convex in $\mathbb{U}$, which implies that $h(z)$ is univalent in $\mathbb{U}$ (cf. [1]).

If we define the function $L(z,t)$ by

$$L(z,t) = q(z) - a + (n+t)\lambda z q'(z) \tag{3.6}$$

for $z \in \mathbb{U}$ and $t \geq 0$, then the function $L(z,t) = a_1(t)z + \cdots$ is analytic in $\mathbb{U}$ for all $t \geq 0$, and continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$. Since $q'(0) \neq 0$, it is clear that

$$a_1(t) = \left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = \{1 + \lambda(n+t)\}q'(0) \neq 0 \quad (t \geq 0)$$

and

$$\lim_{t\to\infty} |a_1(t)| = \lim_{t\to\infty} \left|\{1+\lambda(n+t)\}q'(0)\right| = \infty.$$

From the inequality (3.2), we obtain

$$\begin{aligned} \operatorname{Re}\left\{\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}\right\} &= \operatorname{Re}\left(\frac{1}{\lambda}\right) + (n+t)\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) \\ &\geq \operatorname{Re}\left(\frac{1}{\lambda}\right) + n\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0 \end{aligned}$$

for $z \in \mathbb{U}$ and $t \geq 0$. Then by Lemma 2.1, $L(z,t)$ is subordination chain, and we have $L(z,s) \prec L(z,t)$ $(z \in \mathbb{U})$, when $0 \leq s \leq t$. We now set $\hat{L}(z,t) = L(z,t) + a$. From (3.4) and (3.6), we obtain $h(z) = \hat{L}(z,0) \prec \hat{L}(z,t)$ for $z \in \mathbb{U}$ and $t \geq 0$. Thus, we see that

$$\hat{L}(\zeta,t) \notin h(\mathbb{U}) \tag{3.7}$$

for $|\zeta| = 1$ and $t \geq 0$.

Without loss of generality, we can assume that $q(z)$ is univalent on the closed unit disk $\overline{\mathbb{U}}$. If we assume that $p(z)$ is not subordinate to $q(z)$ in $\mathbb{U}$, then by Lemma 2.1, there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$, and a real number k with $k \geq n$ such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = k\zeta_0 q'(\zeta_0)$. Then from (3.6) and (3.7), we have

$$p(z_0) + \lambda z_0 p'(z_0) = q(\zeta_0) + \lambda k \zeta_0 q'(\zeta_0) = \hat{L}(\zeta_0, k-n) \notin h(\mathbb{U}),$$

where $z_0 \in \mathbb{U}$, $|\zeta_0| = 1$ and $k \geq n$. This contradicts the assumption (3.3) of the theorem, and hence we must have $p(z) \prec q(z)$ $(z \in \mathbb{U})$. This completes the proof of Theorem 3.1. □

Let us consider the function $q(z)$ given by

$$q(z) = 1 + \frac{\mu}{1+n\lambda}z \quad (z \in \mathbb{U})$$

for some real number μ $(\mu > 0)$ and for some complex number λ with the condition (3.1). Then, it is easy to see that

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) = 1 > -\frac{1}{n}\operatorname{Re}\left(\frac{1}{\lambda}\right) \quad (z \in \mathbb{U})$$

and

$$q(z) + \lambda nzq'(z) = 1 + \mu z.$$

Hence by Theorem 3.1, we obtain

Theorem 3.2 *Let n be a positive integer, and let λ be a complex number with the condition* (3.1). *If $p(z) \in \mathcal{H}[1, n]$ satisfies the following subordination*

$$p(z) + \lambda zp'(z) \prec 1 + \mu z \quad (z \in \mathbb{U})$$

for some real number μ $(\mu > 0)$, then

$$p(z) \prec 1 + \frac{\mu}{1+n\lambda}z \quad (z \in \mathbb{U}).$$

By combining Lemma 1.3 and Theorem 3.2, we find the following subordination assertion.

Theorem 3.3 *Let n be a positive integer, and let λ be a complex number with the inequality*

$$\left|\lambda + \frac{1}{2n}\right| > \frac{1}{2n}. \tag{3.8}$$

If $p(z) \in \mathcal{H}[1, n]$ satisfies the following subordination

$$p(z) + \lambda z p'(z) \prec 1 + \mu z \quad (z \in \mathbb{U})$$

for some real number μ $(\mu > 0)$*, then*

$$p(z) \prec 1 + \frac{\mu}{1 + n\lambda} z \quad (z \in \mathbb{U}).$$

For the function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$, we now set

$$p(z) = \frac{z^2 f'(z)}{(f(z))^2} = 1 + (a_3 - a_2{}^2)z^2 + \cdots \quad (z \in \mathbb{U})$$

in Theorem 3.3. Noting that $n = 2$, we derive the following corollary.

Corollary 3.4 *Let* λ *be a complex number with* $|\lambda + 1/4| > 1/4$*. If* $f(z) \in \mathcal{A}$ *satisfies*

$$\frac{z^2 f'(z)}{(f(z))^2} - \lambda z^2 \left(\frac{z}{f(z)} \right)'' \prec 1 + \mu z \quad (z \in \mathbb{U})$$

for some real number μ $(\mu > 0)$*, then*

$$\frac{z^2 f'(z)}{(f(z))^2} \prec 1 + \frac{\mu}{1 + 2\lambda} z \quad (z \in \mathbb{U}).$$

From Corollary 3.4, we find that if $f(z) \in \mathcal{A}$ satisfies the inequality (1.2), then

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \frac{\mu}{|1 + 2\lambda|} \quad (z \in \mathbb{U}) \tag{3.9}$$

for some real number μ $(\mu > 0)$ and for some complex number λ with the inequality (3.8). According to Lemma 1.1, the inequality (3.9) shows that $f(z) \in \mathcal{S}$ if $0 < \mu \leq |1 + 2\lambda|$. Thus, we obtain the following assertion.

Theorem 3.5 *Let* λ *be a complex number with the inequality* (3.8)*. Then the class* $\mathcal{T}(\lambda, \mu)$ *is a subclass of* $\mathcal{S}$ *for some real number* μ *with* $0 < \mu \leq |1 + 2\lambda|$.

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