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Algebraic independence of infinite products generated by Fibonacci and Lucas numbers

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Abstract. The aim of this paper is to give an algebraic independence result for the two infinite products involving the Lucas sequences of the first and second kind. As a consequence, we derive that the two infinite products $\prod_{k=1}^{\infty}(1+1/F_{2^k})$ and $\prod_{k=1}^{\infty}(1+1/L_{2^k})$ are algebraically independent over $\mathbb{Q}$, where $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ are the Fibonacci sequence and its Lucas companion, respectively.

Key words: Infinite products, algebraic independence, Mahler-type functional equation, Fibonacci numbers.

1. Introduction and the results

Throughout this paper, we assume that α and β are algebraic numbers with $|\alpha|>1$ and $\alpha\beta=-1$. Define

$$U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta} \quad \text{and} \quad V_n=\alpha^n+\beta^n \quad (n\geq 0), \tag{1}$$

which are the Lucas sequences of the first and second kind of parameters α and β. When $\alpha=(1+\sqrt{5})/2$, then $U_n=F_n$ and $V_n=L_n$ are the classical Fibonacci and Lucas sequences, respectively. Let $d\geq 2$ be a fixed integer. In [5], the second author gave necessary and sufficient conditions for transcendence of the infinite products

$$\prod_{k=1}^{\infty}\left(1+\frac{a_k}{U_{d^k}}\right) \quad \text{and} \quad \prod_{k=1}^{\infty}\left(1+\frac{a_k}{V_{d^k}}\right),$$

where $\{a_k\}_{k\geq 1}$ is a sequence of algebraic numbers satisfying a certain properties. As applications, both $\prod_{k=1}^{\infty}(1+1/F_{2^k})$ and $\prod_{k=1}^{\infty}(1+1/L_{2^k})$ are transcendental. Necessary and sufficient conditions for the sets of infinite products

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$$\prod_{\substack{k=1\\ U_{d^k}\neq -b_i}}^{\infty}\left(1+\frac{b_i}{U_{d^k}}\right)\quad (1\le i\le m)\quad\text{or}\quad \prod_{\substack{k=1\\ V_{d^k}\neq -b_i}}^{\infty}\left(1+\frac{b_i}{V_{d^k}}\right)\quad (1\le i\le m)$$

to be algebraically independent over $\mathbb{Q}$, where $b_1,\ldots,b_m$ are nonzero integers, were given in [2]. In particular, the two numbers $\prod_{k=1}^{\infty}(1+1/F_{2^k})$ and $\prod_{k=2}^{\infty}(1-1/F_{2^k})$ are algebraically independent over $\mathbb{Q}$.

In this paper, we prove the algebraic independence of the infinite products generated by the two Lucas sequences. Our main results are the following.

Theorem 1 *Let $\{U_n\}_{n\ge 0}$ and $\{V_n\}_{n\ge 0}$ be the sequences defined by* (1)*. Let $d_1, d_2 \ge 2$ be integers and γ_1,γ_2 nonzero algebraic numbers with $(d_2,\gamma_2)\neq (2,-1),(2,2)$. Then the numbers*

$$\prod_{\substack{k=1\\ U_{{d_1}^k}\neq -\gamma_1}}^{\infty}\left(1+\frac{\gamma_1}{U_{{d_1}^k}}\right),\qquad \prod_{\substack{k=1\\ V_{{d_2}^k}\neq -\gamma_2}}^{\infty}\left(1+\frac{\gamma_2}{V_{{d_2}^k}}\right)$$

are algebraically independent over $\mathbb{Q}$.

Remark 1 (cf. [5]) In the cases when $(d_2,\gamma_2)=(2,-1)$ or $(2,2)$, we have

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{V_{2^k}}\right)=\frac{\alpha^4-1}{\alpha^4+\alpha^2+1},\qquad \prod_{k=1}^{\infty}\left(1+\frac{2}{V_{2^k}}\right)=\frac{\alpha^2+1}{\alpha^2-1},$$

by cancellation and using the formula

$$\prod_{k=1}^{\infty}(1+x^{2^k})=\frac{1}{1-x^2}\quad (|x|<1).$$

In particular,

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{L_{2^k}}\right)=\frac{\sqrt{5}}{4},\quad\text{and}\quad \prod_{k=1}^{\infty}\left(1+\frac{2}{L_{2^k}}\right)=\sqrt{5}.$$

Corollary 1 *For any integer $d\ge 2$ and for any nonzero algebraic numbers γ_1,γ_2 with $(d,\gamma_2)\neq(2,-1),(2,2)$, the infinite products*

$$\prod_{\substack{k=1\\ F_{d^k}\neq-\gamma_1}}^{\infty}\left(1+\frac{\gamma_1}{F_{d^k}}\right),\qquad \prod_{\substack{k=1\\ L_{d^k}\neq-\gamma_2}}^{\infty}\left(1+\frac{\gamma_2}{L_{d^k}}\right)$$

are algebraically independent over $\mathbb{Q}$.

Example 1 The numbers

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{F_{2^k}}\right),\qquad \prod_{k=1}^{\infty}\left(1+\frac{1}{L_{2^k}}\right)$$

are algebraically independent over $\mathbb{Q}$.

Example 2 The numbers

$$\prod_{k=1}^{\infty}\left(1+\frac{1+i}{F_{2^k}}\right),\qquad \prod_{k=1}^{\infty}\left(1+\frac{1-i}{L_{2^k}}\right)$$

are algebraically independent over $\mathbb{Q}$.

Example 3 Let $\{P_n\}_{\geq 1}$ and $\{Q_n\}_{n\geq 0}$ be the Pell sequence defined by $P_{n+2} = 2P_{n+1} + P_n$ $(n \geq 0)$, $P_0 = 0$, $P_1 = 1$, and its Pell companion, respectively. Then the numbers

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{P_{2^k}}\right),\qquad \prod_{k=1}^{\infty}\left(1+\frac{1}{Q_{2^k}}\right)$$

are algebraically independent over $\mathbb{Q}$.

2. Mahler-type functional equations

Let $d_1, d_2 \geq 2$ be nonzero integers and γ_1, γ_2 be nonzero algebraic numbers with $(d_2, \gamma_2) \neq (2, -1), (2, 2)$. We put

$$\eta := \prod_{\substack{k=1\\ U_{d_1{}^k}\neq-\gamma_1}}^{\infty}\left(1+\frac{\gamma_1}{U_{d_1{}^k}}\right),\qquad \nu := \prod_{\substack{k=1\\ V_{d_2{}^k}\neq-\gamma_2}}^{\infty}\left(1+\frac{\gamma_2}{V_{d_2{}^k}}\right),$$

where $\{U_n\}_{n\ge 0}$ and $\{V_n\}_{n\ge 0}$ are given in (1). Define $\mathbb{K} = \mathbb{Q}(\alpha, \gamma_1, \gamma_2)$ and

$$\Phi(x) = \prod_{k=0}^{\infty}\left(1+\frac{(\alpha-\beta)\gamma_1 x^{{d_1}^k}}{1-(-1)^{d_1}x^{2{d_1}^k}}\right), \qquad \Psi(x) = \prod_{k=0}^{\infty}\left(1+\frac{\gamma_2 x^{{d_2}^k}}{1+(-1)^{d_2}x^{2{d_2}^k}}\right),$$

which converge in $|x| < 1$ and satisfy the functional equations

$$\Phi(x^{d_1}) = c_1(x)\Phi(x), \qquad \Psi(x^{d_2}) = c_2(x)\Psi(x), \tag{2}$$

with

$$c_1(x) = \frac{1-(-1)^{d_1}x^2}{1+(\alpha-\beta)\gamma_1 x-(-1)^{d_1}x^2}, \qquad c_2(x) = \frac{1+(-1)^{d_2}x^2}{1+\gamma_2 x+(-1)^{d_2}x^2},$$

respectively. Let $\mathbb{K}(x)$ be the field of rational functions over $\mathbb{K}$. Then both the functions $\Phi(x)$ and $\Psi(x)$ are transcendental over $\mathbb{K}(x)$. To see why, suppose on the contrary that $\Phi(x)$ is algebraic over $\mathbb{K}(x)$. Then, by the functional equation (2) and [4, Theorem 1.3] with $C = \overline{\mathbb{Q}}$, we see that $\Phi(x)$ is a rational function over some algebraic number field $\mathbb{L} \supseteq \mathbb{K}$. Hence, at least one of the conditions in [5, Theorem 7] must be satisfied for $\Phi(x)$, which is impossible by the assumptions of α, β, and γ_1. Also in the case of $\Psi(x)$ we get a contradiction by using $(d_2, \gamma_2) \ne (2, -1), (2, 2)$.

By (2), we have for any integers $k_1, k_2 \ge 1$

$$\Phi\big(x^{{d_1}^{k_1}}\big) = \Phi(x)\prod_{i=0}^{k_1-1} c_1\big(x^{{d_1}^i}\big), \qquad \Psi\big(x^{{d_2}^{k_2}}\big) = \Psi(x)\prod_{j=0}^{k_2-1} c_2\big(x^{{d_2}^j}\big). \tag{3}$$

Take an integer N with the property that $\min\{|U_{d^k}|, |V_{d^k}|\} > \max\{|\gamma_1|, |\gamma_2|\}$ for all $k \ge N$. Then, using (3), we get

$$\begin{aligned}\eta &= \Phi\big(\alpha^{-{d_1}^N}\big) \prod_{\substack{k=1\\ U_{d^k}\ne-\gamma_1}}^{N-1}\left(1+\frac{\gamma_1}{U_{{d_1}^k}}\right)\\ &= \Phi(\alpha^{-1})\prod_{i=0}^{N-1} c_1\big(\alpha^{-{d_1}^i}\big) \prod_{\substack{k=1\\ U_{d^k}\ne-\gamma_1}}^{N-1}\left(1+\frac{\gamma_1}{U_{{d_1}^k}}\right),\end{aligned} \tag{4}$$

$$\nu = \Psi\left(\alpha^{-d_2{}^N}\right) \prod_{\substack{k=1 \\ V_{d_2{}^k} \neq -\gamma_2}}^{N-1} \left(1 + \frac{\gamma_2}{V_{d_2{}^k}}\right)$$

$$= \Psi(\alpha^{-1}) \prod_{j=0}^{N-1} c_2\left(\alpha^{-d_2{}^j}\right) \prod_{\substack{k=1 \\ V_{d_2{}^k} \neq -\gamma_2}}^{N-1} \left(1 + \frac{\gamma_2}{V_{d_2{}^k}}\right). \tag{5}$$

In what follows, we distinguish two cases according to whether $\log d_1/\log d_2$ is an irrational number or a rational number, respectively. Let $K[[x]]$ be the ring of formal power series with coefficients in the field K.

3. The case when $\log d_1/\log d_2 \notin \mathbb{Q}$

In this section, under the condition of $\log d_1/\log d_2 \notin \mathbb{Q}$, we prove Theorem 1. We need the following lemma.

Lemma 1 (A special case of Nishioka [3, Theorem 1]) *Let K be an algebraic number field and $d_1, d_2 \geq 2$ integers with $\log d_1/\log d_2 \notin \mathbb{Q}$. Suppose that $f_1(x), f_2(x) \in K[[x]]$ are transcendental over $K(x)$ and satisfy the functional equations*

$$f_i(x^{d_i}) = c_i(x) f_i(x) + b_i(x) \quad (i = 1, 2),$$

where $c_i(x), b_i(x) \in K(x)$, $c_i(0) = 1$. If γ is an algebraic number with $0 < |\gamma| < 1$, $c_i(\gamma^{d_i{}^k}) \neq 0$ $(k \geq 0)$ and $f_i(x)$ converge at $x = \gamma$, then the values $f_1(\gamma)$ and $f_2(\gamma)$ are algebraically independent over $\mathbb{Q}$.

Proof of Theorem 1. Applying Lemma 1 to the transcendental functions $f_1(x) := \Phi(x)$ and $f_2(x) := \Psi(x)$ satisfying the functional equations (3) with $k_1 = k_2 = N$, we can deduce immediately that the values $\Phi(\alpha^{-1})$ and $\Psi(\alpha^{-1})$ are algebraically independent over $\mathbb{Q}$. Hence, by (4) and (5), so are the numbers η and ν, which finishes the proof of Theorem 1. □

4. The case when $\log d_1/\log d_2 \in \mathbb{Q}$

Let K be an algebraic number field. For an integer $d \geq 2$, we define the subgroup H_d of the group $K(x)^\times$ of nonzero elements of $K(x)$ by

$$H_d = \left\{ \frac{g(x^d)}{g(x)} \,\middle|\, g(x) \in K(x)^{\times} \right\}.$$

We use the following lemmas for the proof of Theorem 1.

Lemma 2 (Kubota [1, Corollary 8]) $f_1(x), \ldots, f_m(x) \in K[[x]] \setminus \{0\}$ *satisfy the functional equations*

$$f_i(x^d) = c_i(x) f_i(x), \qquad c_i(x) \in K(x)^{\times} \quad (1 \le i \le m). \tag{6}$$

Then $f_1(x), \ldots, f_m(x)$ *are algebraically independent over* $K(x)$ *if and only if the rational functions* $c_1(x), \ldots, c_m(x)$ *are multiplicatively independent modulo* H_d.

Lemma 3 (Kubota [1], see also Nishioka [4, Theorem 3.6.4]) *Suppose that the functions* $f_1(x), \ldots, f_m(x) \in K[[x]]$ *converge in* $|x| < 1$ *and satisfy the functional equations* (6) *with* $c_i(x)$ *defined and nonzero at* $x = 0$. *Let* γ *be an algebraic number with* $0 < |\gamma| < 1$ *such that* $c_i(\gamma^{d^k})$ *are defined and nonzero for all* $k \ge 0$. *If* $f_1(x), \ldots, f_m(x)$ *are algebraically independent over* $K(x)$, *then the values* $f_1(\gamma), \ldots, f_m(\gamma)$ *are algebraically independent over* $\mathbb{Q}$.

In this section, we assume $\log d_1 / \log d_2 \in \mathbb{Q}$ in Theorem 1. Then there exists a minimal pair of positive integers (ℓ_1, ℓ_2) such that

$$d_1^{\ell_1} = d_2^{\ell_2} := d. \tag{7}$$

By the functional equations (3), we have

$$\Phi(x^d) = \Phi(x) \prod_{i=0}^{\ell_1 - 1} c_1\big(x^{{d_1}^i}\big), \qquad \Psi(x^d) = \Psi(x) \prod_{j=0}^{\ell_2 - 1} c_2\big(x^{{d_2}^j}\big). \tag{8}$$

Suppose to the contrary that η and ν are algebraically dependent over $\mathbb{Q}$. Then, by (4) and (5), so are the values $\Phi(\alpha^{-1})$ and $\Psi(\alpha^{-1})$. Since $\Phi(x)$ and $\Psi(x)$ satisfy the functional equations (8), they are algebraically dependent over $\mathbb{K}(x)$ by Lemma 3. By Lemma 2, the rational functions

$$\prod_{i=0}^{\ell_1 - 1} c_1\big(x^{{d_1}^i}\big), \qquad \prod_{j=0}^{\ell_2 - 1} c_2\big(x^{{d_2}^j}\big)$$

are multiplicatively dependent modulo H_d, namely there exist integers e_1, e_2, not both zero, and $g(x) \in \mathbb{K}(x)^\times$ such that

$$\left(\prod_{i=0}^{\ell_1-1} c_1(x^{d_1^i})\right)^{e_1} \left(\prod_{j=0}^{\ell_2-1} c_2(x^{d_2^j})\right)^{e_2} = \frac{g(x^d)}{g(x)}, \tag{9}$$

where $g(x)$ is defined and nonzero at $x = 0$, since $c_1(0)c_2(0) = 1$.

The remaining part of the paper is dedicated to proving that a relation such as (9) cannot hold.

Noting that $\Phi(x)$ and $\Psi(x)$ are transcendental over $\mathbb{K}(x)$, we deduce that $e_1e_2 \neq 0$. Indeed, if $e_1 = 0$, we then have, by (8) and (9),

$$g(x)\Psi\big(x^{d^k}\big)^{e_2} = \Psi(x)^{e_2} g\big(x^{d^k}\big) \qquad (k \geq 0).$$

Taking the limit as $k \to \infty$, we obtain $g(x) = \Psi(x)^{e_2} g(0)$ $(|x| < 1)$, so that $\Psi(x)$ is algebraic over $\mathbb{K}(x)$. This is a contradiction. A similar contradiction is deduced when $e_2 = 0$.

To simplify the notation, we put $\gamma := (\alpha-\beta)\gamma_1$ and rewrite the equation (9), as

$$\begin{aligned} F(x) = &\left(\prod_{i=0}^{\ell_1-1} \frac{1-(-1)^{d_1}x^{2d_1^i}}{1+\gamma x^{d_1^i}-(-1)^{d_1}x^{2d_1^i}}\right)^{e_1} \\ &\times \left(\prod_{j=0}^{\ell_2-1} \frac{1+(-1)^{d_2}x^{2d_2^j}}{1+\gamma_2 x^{d_2^j}+(-1)^{d_2}x^{2d_2^j}}\right)^{e_2}, \end{aligned} \tag{10}$$

where e_1 and e_2 are nonzero integers and

$$F(x) := \frac{A(x^d)B(x)}{A(x)B(x^d)}, \tag{11}$$

with $A(x)$ and $B(x)$ being polynomials without common roots with complex coefficients obtained from $g(x) = A(x)/B(x)$. We also assume that $e_1 > 0$, otherwise we replace the pair of exponents (e_1, e_2) by the pair $(-e_1, -e_2)$ and interchange $A(x)$ and $B(x)$.

So, in order to derive Theorem 1, we need to show that a relation such as (10) does not hold. We proceed in a sequence of lemmas.

Lemma 4 *Let $d \geq 2$ be the integer defined by* (7)*, and $A(x)$ and $B(x)$ be the polynomials given in* (11)*. Then we have the following properties*:

(i) *The polynomials $A(x)$ and $B(x)$ have the same degree.*
(ii) *The value $x = 1$ is neither a root nor a pole of $A(x^d)B(x)/A(x)B(x^d)$.*
(iii) *d is even.*

Proof. (i). Observe that both $c_1(x)$ and $c_2(x)$ are of degree 0 as rational functions, therefore the left–hand side of (10) is of degree 0 as a rational function. Thus, $F(x)$ is also a degree 0. Since its degree is also $(d-1)(\deg(A) - \deg(B))$, it follows that $\deg(A) = \deg(B)$.

(ii). If $A(1)B(1) \neq 0$, the conclusion is clear. Suppose, without loss of generality, that $A(1) = 0$. Then $A(x) = (x-1)^e C(x)$, where $e \geq 1$ is some positive integer, and $C(x)$ is a polynomial with $C(1) \neq 0$. Then

$$F(x) = \frac{A(x^d)B(x)}{A(x)B(x^d)} = \left(\frac{x^d-1}{x-1}\right)^e \frac{C(x^d)B(x)}{C(x)B(x^d)} = (x^{d-1}+\cdots+1)^e \frac{C(x^d)B(x)}{C(x)B(x^d)}$$

and since $C(1) \neq 0$ and $B(1) \neq 0$ (this last condition holds because $A(x)$ and $B(x)$ do not have common roots), we get that $F(1) = d^e$.

(iii). Assume that d is odd. Then by an argument similar to the one at (ii) above, we conclude that $x = -1$ is neither a root nor a pole of $F(x)$. Indeed, this is clear if $A(-1)B(-1) \neq 0$. Assume say that $A(-1) = 0$ and write $A(x) = (x+1)^e C(x)$ for some positive integer e and some polynomial $C(x)$ with $C(-1) \neq 0$. Then

$$F(x) = \left(\frac{x^d+1}{x+1}\right)^e \frac{C(x^d)B(x)}{C(x)B(x^d)} = (x^{d-1} - x^{d-2} + \cdots + 1)^e \frac{C(x^d)B(x)}{C(x)B(x^d)},$$

so we see that $F(-1) = d^e$ because $C(x^d),\ C(x),\ B(x^d),\ B(x)$ all evaluate to either $C(-1) \neq 0$, or to $B(-1) \neq 0$, when $x = -1$. Thus, $F(-1) \neq 0$.

Since d is odd, both d_1 and d_2 are odd. By the equation (10), since $x = -1$ is a root of all the polynomials $1 - x^{2d_2^j}$ for $j = 0, \ldots, \ell_1 - 1$, but of neither one of the polynomials $1 + x^{2d_1^i}$ for $i = 0, \ldots, \ell_1 - 1$ or $1 + \gamma_2 x^{d_2^j} - x^{2d_2^j}$ for $j = 0, \ldots, \ell_2 - 1$ (because $\gamma_2 \neq 0$), we get that in fact $x = -1$ should be a root of $1 + \gamma x^{d_1^i} + x^{2d_1^i}$ for some $i = 0, \ldots, \ell_1 - 1$. Hence, we have $\gamma = 2$. However, in this case, from (10), we deduce that $x = 1$ is either a root or a pole of $F(x)$ with multiplicity $|e_2|\ell_2$, which is impossible by (ii). □

Lemma 5 *$\gamma_2 = -2$ and $e_1\ell_1 = 2e_2\ell_2$. In particular, $e_2 > 0$.*

Proof. We know from Lemma 4 (iii) that d_1 and d_2 are both even. Hence, in the equation (10), the family of polynomials $1 - x^{2d_1^i}$, $i = 0, \ldots, \ell_1 - 1$ all have $x = 1$ as a root. Since $x = 1$ is not a zero or a pole of the rational function appearing in the right–hand side of (10) by Lemma 4 (ii), it follows at least one (hence, all of them) of the ℓ_2 polynomials $1 + \gamma_2 x^{d_2^j} + x^{2d_2^j}$, $j = 0, \ldots, \ell_2 - 1$ must have $x = 1$ as a root. Thus, $\gamma_2 = -2$. So, in this case the formula (10) becomes

$$F(x) = \left(\prod_{i=0}^{\ell_1 - 1} \frac{1 - x^{2d_1^i}}{1 + \gamma x^{d_1^i} - x^{2d_1^i}} \right)^{e_1} \left(\prod_{j=0}^{\ell_2 - 1} \frac{1 + x^{2d_2^j}}{(1 - x^{d_2^j})^2} \right)^{e_2}. \tag{12}$$

The multiplicity of $x = 1$ in the first factor in the right–hand side of (12) is $e_1\ell_1$ and in the second factor is $-2e_2\ell_2$. Thus, $e_1\ell_1 = 2e_2\ell_2$ again by Lemma 4 (ii). This finishes the proof of the lemma. □

Lemma 6 *All roots of $A(x)$ and $B(x)$ are roots of unity.*

Proof. We first deal with the roots of $A(x)$. Say ζ is a root of $A(x)$. Clearly, $\zeta \neq 0$, for if $A(x) = x^e C(x)$ for some positive integer e with $C(x)$ a polynomial such that $C(0) \neq 0$, then $x = 0$ is a root of $A(x^d)B(x)/A(x)B(x^d)$ with multiplicity $e(d-1) > 0$, which is not possible by the equation (12) since its left–hand side evaluates to 1 when $x = 0$. Assume now that ζ is not a root of unity. Then all the numbers

$$\zeta, \zeta^{1/d}, \zeta^{1/d^2}, \ldots$$

are distinct for any choice of the d^ith power roots, since a relation of the type $\zeta^{1/d^u} = \zeta^{1/d^v}$ for some nonnegative integers $u \neq v$ implies that $\zeta^{d^u - d^v} = 1$, and $d^u - d^v \neq 0$, so ζ is a root of unity. Choose a nonnegative integer ℓ maximal such that $\zeta_1 = \zeta^{1/d^\ell}$ is a root of $A(x)$. Thus, $\zeta_1^{1/d} = \zeta^{1/d^{\ell+1}}$ is not a root of $A(x)$. Then $F(x)$ has $\zeta_2 = \zeta_1^{1/d}$ as a root because $A(\zeta_2^d) = A(\zeta_1) = 0$, but $A(\zeta_2) \neq 0$ and also $B(\zeta_2^d) = B(\zeta_1) \neq 0$ because $A(x)$ and $B(x)$ do not have roots in common. By the equation (12), ζ_2 is a root of unity, therefore $\zeta = \zeta_2^{d^{\ell+1}}$ is also a root of unity. Thus, all the roots of $A(x)$ are roots of unity.

A similar argument works for the roots of $B(x)$. Let ζ be any root of $B(x)$ and look at the sequence

$$\zeta, \zeta^d, \zeta^{d^2}, \ldots$$

If it is a finite sequence, then ζ is a root of unity. If it is infinite, then there exists a nonnegative integer l maximal such that $\zeta_1 = \zeta^{d^l}$ is a root of $B(x)$. Thus, $\zeta_1^d = \zeta^{d^{l+1}}$ is not a root of $B(x)$. But then ζ_1 is a root of $F(x)$ because $B(\zeta_1) = 0$, $B(\zeta_1^d) \neq 0$ and $A(\zeta_1) \neq 0$. Thus, by (12), ζ_1 is a root of 1, therefore ζ is a root of unity. □

Lemma 7 *The roots of $1 + \gamma x - x^2$ are complex nonreal roots of unity. Furthermore, γ^2 is a negative real number.*

Proof. In the equation (12) any root ζ of $1 + \gamma x - x^2$ is either a root of $x^{2d_2^i} + 1$ for some $i = 0, \ldots, \ell_2 - 1$, or a root of $x^{2d_1^j} - 1$ for some $j = 0, \ldots, \ell_1 - 1$, or a root of $A(x)$ or of $B(x^d)$, and whichever may be the case, it is a root of unity. We cannot have $\zeta = \pm 1$ since it would lead to $\gamma = 0$, so ζ is complex nonreal. Let ζ and η be the two possibly equal roots of $1 + \gamma x - x^2$. By the Vieté relations, $\eta = -\zeta^{-1}$ and $\gamma = \zeta + \eta$. Writing $\zeta = e^{2\pi i u/m}$ for some integers $m \geq 3$ and $u \in \{1, \ldots, m-1\}$ coprime to m, we get that

$$\gamma = e^{2\pi i u/m} - e^{-2\pi i u/m} = 2i \sin(2\pi u/m),$$

so $\gamma^2 = -4\sin^2(2\pi u/m)$ is a negative real number. □

Lemma 8 *Let $\zeta = e^{2\pi i u/m}$ with u and m coprime be a primitive root of unity of order m. Then there are at least $\phi(d)$ primitive roots of unity of order md which are roots of the polynomial $x^d - \zeta$.*

Proof. All roots of $x^d - \zeta$ are of the form

$$e^{2\pi i(u/dm + v/d)}, \tag{13}$$

where $v \in \{0, 1, \ldots, d-1\}$. A number of the form (13) is a primitive root of unity precisely when $u + mv$ is coprime to md. Let p be a prime factor of d and let α_p be such that $p^{\alpha_p} \| d$. If p divides m, then u is already coprime to p, therefore $u + pv \equiv u \pmod{p}$ is coprime to p^{α_p}. If p does not divide m, then m is invertible modulo p^{α_p} and we can choose v modulo p^{α_p} such that $v \equiv (u_i - u)m^{-1} \pmod{p^{\alpha_p}}$, where $u_1, \ldots, u_{\phi(p^{\alpha_p})}$ are all the residue classes modulo p^{α_p} coprime to p^{α_p}. By the Chinese Remainder Theorem to deal with all the prime powers p^{α_p} exactly dividing d, we get that the number of congruence classes v modulo d such that the number shown at (13) is a primitive root of unity of order md is

$$\prod_{\substack{p|d \\ p|m}} p^{\alpha_p} \prod_{\substack{p|d \\ p\nmid m}} \phi(p^{\alpha_p}) \geq \phi(d). \qquad \square$$

Lemma 9 *Let m_A be the maximal order of the roots of unity which are roots of $A(x)$. Then either $m_A = 1$ and $d_1 = 2$, or $m_A \leq 2$ and $d_2 \leq 4$.*

Proof. Let ζ be a root of order m_A of $A(x)$. Then $A(x^d)$ is a multiple of the polynomial $x^d - \zeta$, which has $\phi(d) \geq 1$ primitive roots of unity of order $m_A d > m_A$ by Lemma 8. These roots are not roots of $A(x)$ or of $B(x^d)$, so they are all roots of $F(x)$. Looking in the right–hand side of (12), we get that

$$m_A d \leq \max\left\{2d_1^{\ell_1-1}, 4d_2^{\ell_2-1}\right\}.$$

If the maximum above is $2d_1^{\ell_1-1}$, we then get $m_A d = (m_A d_1)d_1^{\ell_1-1} \leq 2d_1^{\ell_1-1}$, so $m_A d_1 \leq 2$, giving $m_A = 1$ and $d_1 = 2$. If the maximum above is $4d_2^{\ell_2-1}$, we then get $m_A d = (m_A d_2)d_2^{\ell_2-1} \leq 4d_2^{\ell_2-1}$, so $m_A d_2 \leq 4$, therefore $m_A \leq 2$ and $d_2 \leq 4$. These are the desired conclusions. $\square$

Lemma 10 *The polynomial $B(x)$ is nonconstant and has at least one complex nonreal root.*

Proof. If $B(x)$ is constant, then so is $A(x)$ by Lemma 4 (i). In this case, $F(x) = 1$. If $B(x)$ is not constant but has only real roots, then since $m_A \leq 2$ by Lemma 9, it follows that all the roots of $A(x)B(x)$ are in $\{-1, 1\}$. Hence, $\{A(x), B(x)\} = \{a_0(x-1)^e, b_0(x+1)^e\}$ holds with some positive integer e and some nonzero complex numbers a_0 and b_0. In either case, $F(x) \in \mathbb{R}(x)$ is a rational function with real coefficients. Separating out the denominator of the first product in the right–hand side of (12) and using the fact that all other components of the right–hand side of (12) have real coefficients, we get that

$$\left(\prod_{i=0}^{\ell_1-1}(1+\gamma x^{d_1^j}-x^{2d_1^j})\right)^{e_1}\in\mathbb{R}(x). \tag{14}$$

Since the rational function appearing in the left–hand side of containment (14) is in fact a polynomial, it follows that this polynomial is in $\mathbb{R}[x]$. But it is easy to see that

$$\prod_{i=0}^{\ell_1-1}\left(1+\gamma x^{d_1^j}-x^{2d_1^j}\right)=(-1)^{\ell_1}\big(x^{2+2d_1+\cdots+2d_1^{\ell_1-1}}-\gamma x^{1+2d_1+\cdots+2d_1^{\ell_1-1}} + \text{smaller degree monomials}\big),$$

therefore the polynomial appearing in the left–hand side of containment of (14) is a real polynomial of the form

$$(-1)^{\ell_1 e_1}\big(x^{2e_1(1+d_1+\cdots+d_1^{\ell_1-1})}-e_1\gamma x^{2e_1(1+d_1+\cdots+d_1^{\ell_1-1})-1} + \text{smaller degree monomials}\big),$$

showing that γ is real, which contradicts Lemma 7. □

Using the previous Lemmas 11 and 12, we prove $d_1=2$.

Lemma 11 *In the equation* (12), *we have* $d_1\in\{2,4,6\}$.

Proof. Let ζ be some root of $B(x)$ with maximal order $m\geq 3$. Then $B(x^d)$ is divisible by the polynomial $x^d-\zeta$, so it has at least $\phi(d)$ distinct roots of order md by Lemma 8. By (12), such roots must be among the roots of

$$\prod_{j=1}^{\ell_1-1}\left(1+\gamma x^{d_1^i}-x^{2d_1^i}\right),\ (0\leq i\leq \ell_1-1),\quad \text{or}\quad x^{d_2^j}-1,\ (0\leq j\leq \ell_2-1).$$

The second polynomials have only roots of unity of order at most $d_2^{\ell_2-1}<d$. So, we look at the roots of the first polynomials. Let $\zeta_1,\ \zeta_2$ be such that $1+\gamma x-x^2=-(x-\zeta_1)(x-\zeta_2)$. Let

$$u_1,\ldots,u_{\phi(d)}$$

be distinct integers coprime to md in $\{1,\ldots,md\}$ such that $e^{2\pi i u_k/md}$ for $k=1,\ldots,\phi(d)$ are primitive roots of unity of order md which are also roots of $B(x^d)$. If $e^{2\pi i u_k/md}$ is a root of $1+\gamma x^{d_1^j}-x^{2d_1^j}$ for some $j=0,1,\ldots,\ell_1-1$, then

$$e^{2\pi i u_k/m(d/d_1^j)}\in\{\zeta_1,\zeta_2\}.$$

In particular,

$$md_1^{\ell_1-j}=m(d/d_1^j)\in\{\mathrm{ord}(\zeta_1),\mathrm{ord}(\zeta_2)\}.$$

Here, for a root of unity ζ we write $\mathrm{ord}(\zeta)$ for its order. The above containment shows that there are at most two possible values for $j\in\{0,1,\ldots,\ell_1-1\}$; namely, we write $\mathrm{ord}(\zeta_s)=md_1^{\ell_1-j_s}$ for $s=1,2$, and then $j\in\{j_1,j_2\}$. Thus, our $\phi(d)$ distinct roots are to be found among the roots of

$$\left(x^{d_1^{j_1}}-\zeta_1\right)\left(x^{d_1^{j_2}}-\zeta_2\right),$$

which is a polynomial of degree $d_1^{j_1}+d_1^{j_2}\le 2d_1^{\ell_1-1}$. We have thus arrived at the inequality

$$\phi(d)\le 2d_1^{\ell_1-1}.$$

Since $\phi(d)=\phi(d_1^{\ell_1})=d_1^{\ell_1-1}\phi(d_1)$, we get $d_1^{\ell_1-1}\phi(d_1)=\phi(d)\le 2d_1^{\ell_1-1}$, so $\phi(d_1)\le 2$, which leads to $d_1\in\{2,4,6\}$. □

Lemma 12 *In the equation* (12)*, we have* $d_1\ne d_2$*.*

Proof. Assume that $d_1=d_2$. Then $\ell_1=\ell_2=1$ and in (12) we have $e_1=2e_2$. Thus, the equation (12) is

$$F(x)=\left(\frac{(x^2+1)(x+1)^2}{(1+\gamma x-x^2)^2}\right)^{e_2}. \tag{15}$$

Let t be the number of distinct complex nonreal roots of $B(x)$ and let these roots be $\zeta_1,\ldots,\zeta_t$. Then the polynomial $B(x^d)$ is divisible by the polynomial $C(x)=\prod_{i=1}^t(x^d-\zeta_i)$, whose roots are all complex nonreal and simple. Indeed, it is clear that all roots of $C(x)$ are nonreal, and they are simple because $x^d-\zeta_i$ has only simple roots for all $i=1,\ldots,t$, and if $i\ne j$, then $x^d-\zeta_i$

and $x^d - \zeta_j$ cannot have a common root ζ_0, since the existence of such a root will imply that $\zeta_i = \zeta_0^d = \zeta_j$, a contradiction. Thus, $B(x^d)$ has at least td distinct complex nonreal roots, showing that $F(x) = A(x^d)B(x)/A(x)B(x^d)$ has at least $td - t = t(d-1)$ complex nonreal distinct poles (namely all of the roots of $C(x)$ with the possible exception of $\zeta_1, \ldots, \zeta_t$, which might get cancelled in $F(x)$). Comparing this with the number of distinct complex nonreal poles of the function appearing in the right–hand side of the formula (15), we deduce that $t(d-1) \le 2$. Since $t \ge 1$ by Lemma 10, we have that $d \le 3$. Thus, $d = 2$ by Lemma 11.

Then $t \le 2$. If $t = 2$, then $B(x^d)$ is divisible by $C(x) = (x^2 - \zeta_1)(x^2 - \zeta_2)$. Since the function appearing in the right–hand side of (15) has at most two distinct complex poles, we conclude that both ζ_1 and ζ_2 are zeros of $C(x)$. Thus, either $\zeta_1^2 = \zeta_1$, or $\zeta_2^2 = \zeta_2$, or both $\zeta_1^2 = \zeta_2$ and $\zeta_2^2 = \zeta_1$. The first two situations give $\zeta_1 = 1$, or $\zeta_2 = 1$, which are not acceptable. The last situation gives that $\zeta_1^4 = (\zeta_1^2)^2 = \zeta_2^2 = \zeta_1$, so $\zeta_1^3 = 1$ and similarly $\zeta_2^3 = 1$. So, ζ_1 and ζ_2 are the two complex cubic nonreal roots of unity and $C(x) = x^2 + x + 1$. But then $x^2 + \gamma x - 1$ is associated to $C(x^2)/C(x) = x^2 - x + 1$, which is impossible.

Finally, if $t = 1$, then $C(x) = x^2 - \zeta$, and $C(x)$ does not have ζ as a root, otherwise we would get $\zeta^2 = \zeta$, so $\zeta = 1$, which is not acceptable. But then $x^2 - \zeta = x^2 + \gamma x - 1$, which is also impossible. □

Lemma 13 *In the equation* (12)*, we have* $d_1 = 2$*.*

Proof. We let again $\zeta_1, \ldots, \zeta_t$ be all the complex nonreal roots of $B(x)$ and look at $C(x) = \prod_{i=1}^{t}(x^d - \zeta_i)$. As we have seen in the proof of Lemma 12, $F(x)$ has at least $t(d-1)$ distinct nonreal poles. On the other hand, by (12), all such poles are either roots of $x^{d_2^{\ell_2 - 1}} - 1$, or of $1 + \gamma x^{d_1^i} - x^{2d_1^i}$ for some $i = 0, \ldots, \ell_1 - 1$. Thus, we get

$$t(d-1) \le d_2^{\ell_2 - 1} + 2\sum_{i=0}^{\ell_1 - 1} d_1^i = \frac{d}{d_2} + \frac{2(d-1)}{d_1 - 1}.$$

Thus,

$$t \le \frac{d}{d_2(d-1)} + \frac{2}{d_1 - 1}.$$

If $d_1 = 6$, then $d_2 \geq 36$, and we get

$$t \leq \frac{36}{36(36-1)} + \frac{2}{5} < 1,$$

a contradiction. If $d_1 = 4$ and $d_2 \geq 8$, and we get

$$t \leq \frac{8}{8(8-1)} + \frac{2}{3} < 1,$$

a contradiction. If $d_1 = 4$ and $d_2 = 2$, then $\ell_1 = 1,\ \ell_2 = 2,\ e_1 = 4e_2$, so the formula (12) becomes

$$F(x) = \left(\frac{(x^2+1)(x^4+1)(1+x)^2}{(1+\gamma x - x^2)^4} \right)^{e_2},$$

and a contradiction can now be reached by counting again the complex nonreal poles of $F(x)$ as in the proof of Lemma 12. Hence, by Lemma 11, we conclude that $d_1 = 2$. $\square$

Since now we know that $d_1 = 2$, it follows that $d_2 = 2^{\ell_1}$ for some $\ell_1 \geq 2$, and $\ell_2 = 1$. Further, the equation (12) becomes

$$F(x) = \frac{(x+1)^{\ell_1 e_1}(x^2+1)^{(\ell_1-1)e_1+e_2}(x^4+1)^{(\ell_1-2)e_1} \cdots (x^{2^{\ell_1-1}}+1)^{e_1}}{(1+\gamma x - x^2)^{e_1} \cdots (1+\gamma x^{2^{\ell_1-1}} - x^{2^{\ell_1}})^{e_1}}, \tag{16}$$

where $e_1 = 2e_2/\ell_1$. Next, we prove that $\ell_1 = 2$ by using Lemma 14.

Lemma 14 *Let $A(x)$ and $B(x)$ be the polynomials given in* (11)*. Then the following properties hold*:

(i) *There exist positive integer e and nonzero complex number a_0 such that $A(x) = a_0(x-1)^e$.*

(ii) *We have $B(-1) \neq 0$.*

Proof. (i). If this would not be so, then, by Lemma 9, it would follow that -1 is a root of $A(x)$. Thus, $x^d + 1 = x^{2^{\ell_1}} + 1$ is a divisor of $A(x^d)$ and since d is even, $x^d + 1$ does not have any roots in common neither with $A(x)$ (whose only roots are -1 or 1), nor with $B(x^d)$, since $A(x)$ and $B(x)$ are coprime. Thus, $F(x)$ has roots which are primitive roots of unity of order

$2^{\ell_1+1} = 2d$. However, this is not possible by the formula (16).

(ii). Assume that $x+1$ divides $B(x)$. Then x^d+1 divides $B(x^d)$ and it is coprime to $A(x)$. Let $\zeta_1, \ldots, \zeta_t$ be the $t \geq 1$ complex nonreal roots of $B(x)$ and look at the polynomial $(x^d+1)C(x) = (x^d+1)\prod_{i=1}^{t}(x^d-\zeta_i)$. Since d is even, all the roots of this polynomial are complex nonreal and it is easy to see that they are also simple. Thus, $B(x^d)$ has at least $d(t+1)$ distinct complex nonreal roots. Thus, $F(x)$ has at least $d(t+1)-t = t(d-1)+d \geq 2d-1$ distinct poles. Comparing this observation with the number of poles of the function appearing in the right–hand side of (16), we get

$$2d-1 \leq 2+4+\cdots+2^{\ell_1} = 2^{\ell_1+1}-2 = 2d-2,$$

a contradiction. □

Lemma 15 *In equation* (16)*, we have* $\ell_1 = 2$.

Proof. By (7), (16), and Lemma 14, we have

$$F(x) = \left(\frac{x^d-1}{x-1}\right)^e \frac{B(x)}{B(x^d)},$$

where all the t distinct roots of $B(x)$ are complex. It follows that all the roots of $B(x^d)$ are also complex. So, identifying the multiplicity of $x+1$ in (16), we get that $e = \ell_1 e_1$. If $\ell_1 \geq 3$, then $e^{2\pi i/8}$, which is a root of x^4+1 appears with multiplicity at least as large as $e = \ell_1 e_1$ as a root of $F(x)$ (because $A(x^d)$ and $B(x^d)$ are coprime). However, (16) tells us that this multiplicity is at most $(\ell_1-2)e_1 < e$, a contradiction. Thus, $\ell_1 \leq 2$. If $\ell_1 = 1$, then $d_1 = d_2$, which contradicts Lemma 12. Therefore we obtain $\ell_1 = 2$. □

Now $d_1 = 2$, $d_2 = 4$, $d = 4$, $\ell_1 = 2$, $\ell_2 = 1$, so the formula (16) becomes

$$F(x) = \left(\frac{(1+x)^2(1+x^2)^2}{(1+\gamma x - x^2)(1+\gamma x^2 - x^4)}\right)^{e_2}. \tag{17}$$

Finally, we prove Theorem 1 in the case of $\log d_1/\log d_2 \in \mathbb{Q}$.

Proof of Theorem 1. We know, by Lemmas 6 and 14, that $A(x) = a_0(x-1)^e$ and that $B(x)$ has no real roots. So, we get, by (11) and (17),

$$\left(\frac{x^4-1}{x-1}\right)^e \frac{B(x)}{B(x^4)} = \left(\frac{(1+x)^2(1+x^2)^2}{(1+\gamma x - x^2)(1+\gamma x^2 - x^4)}\right)^{e_2}.$$

Since e is the exact multiplicity as a root of $x=-1$ in $F(x)$ by Lemma 14 (ii), we get that $e = 2e_2$. Hence,

$$\frac{B(x^4)}{B(x)} = ((1+\gamma x - x^2)(1+\gamma x^2 - x^4))^{e_2}. \tag{18}$$

The above equation tells us that $B(x^4)/B(x)$ is a polynomial. If we write $\zeta_1, \ldots, \zeta_t$ for all the roots of $B(x)$, we then again have that $C(x) = \prod_{i=1}^t (x^4 - \zeta_i)$ is a divisor of $B(x^4)$ and it has $4t$ distinct roots. Thus, $B(x^4)/B(x)$ has exactly $4t - t = 3t$ distinct roots. Comparing this with the right–hand side of (18), we get $3t \leq 6$, so $t \leq 2$. If $t = 1$, then $B(x) = b_0(x - \zeta_1)^{e_0}$, so

$$\frac{B(x^4)}{B(x)} = \left(\frac{x^4 - \zeta_1}{x - \zeta_1}\right)^{e_0}.$$

Since ζ_1 is a root of $B(x^4)$, we get that $\zeta_1^4 = \zeta_1$, therefore $\zeta_1^3 = 1$. Since $(x^2 - \gamma x - 1)(x^4 - \gamma x^2 - 1)$ has a totality of 3 distinct roots, it follows, in particular, that $x^4 - \gamma x^2 - 1$ has double roots. Such a root ζ satisfies

$$4\zeta^3 - 2\zeta\gamma = 0, \quad \text{therefore} \quad \zeta^2 = \gamma/2;$$

so

$$0 = \zeta^4 - \gamma\zeta^2 - 1 = (\gamma/2)^2 - \gamma(\gamma/2) - 1,$$

therefore $\gamma^2 = -4$. Thus, $\gamma = \pm 2i$. But in this case, $x^2 - x\gamma - 1 = (x \pm i)^2$ has a double root which is a root of unity of order 4 and this number cannot be a root of $B(x^4) = b_0(x^4 - \zeta_1)^{e_0}$.

Finally, assume that $t = 2$. Then

$$B(x) = b_0(x - \zeta_1)^{f_1}(x - \zeta_2)^{f_2}.$$

So,

$$\frac{B(x^4)}{B(x)} = \frac{(x^4 - \zeta_1)^{f_1}(x^4 - \zeta_2)^{f_2}}{(x - \zeta_1)^{f_1}(x - \zeta_2)^{f_2}}.$$

From what we have seen, $B(x^4)$ has exactly 8 distinct roots and two of them are ζ_1 and ζ_2. Thus, $B(x^4)/B(x)$ has exactly 6 distinct roots showing that $(x^2-\gamma x-1)(x^4-\gamma x^2-1)$ has only distinct roots. Now $x^4-\zeta_1$ has four distinct roots, two of which might be ζ_1 and/or ζ_2, but the other two roots appear with multiplicity precisely f_1 in $B(x^4)/B(x)$. So, $f_1=e_2$. A similar argument shows that $f_2=e_2$, so in fact $e_2=f_1=f_2$, and

$$\frac{(x^4-\zeta_1)(x^4-\zeta_2)}{(x-\zeta_1)(x-\zeta_2)}=(x^2-\gamma x-1)(x^4-\gamma x^2-1). \tag{19}$$

To rule out this last possibility, we deal with various cases.

Case 1. $\zeta_1^4=\zeta_1$ and $\zeta_2^4=\zeta_2$.
In this case, $\zeta_1^3=\zeta_2^3=1$ and

$$\begin{aligned}\frac{B(x^4)}{B(x)}&=\frac{(x^4-\zeta_1^4)(x^4-\zeta_2^4)}{(x-\zeta_1)(x-\zeta_2)}=(x^3+\zeta_1x^2+\cdots)(x^3+\zeta_2x^2+\cdots)\\&=x^6+(\zeta_1+\zeta_2)x^5+\cdots.\end{aligned}$$

Identifying coefficients, we get $\gamma=-(\zeta_1+\zeta_2)=1\in\mathbb{R}$, contradiction. Here, we used that fact that ζ_1 and ζ_2 are the two complex roots of unity of order 3.

Case 2. $\zeta_1^4=\zeta_2$ and $\zeta_2^4=\zeta_1$.
In this case, $\zeta_1^{16}=(\zeta_1^4)^4=\zeta_2^4=\zeta_1$, so $\zeta_1^{15}=1$ and a similar argument shows that $\zeta_2^{15}=1$. So,

$$\begin{aligned}\frac{B(x^4)}{B(x)}&=\frac{(x^4-\zeta_1^{16})(x^4-\zeta_2^{16})}{(x-\zeta_1^4)(x-\zeta_2^4)}=(x^3+\zeta_1^4x^2+\cdots)(x^3+\zeta_2^4x^2+\cdots)\\&=x^6+(\zeta_1+\zeta_2)x^5+\cdots\end{aligned}$$

Identifying coefficients, we get $\gamma=-(\zeta_1+\zeta_2)$. Writing $\zeta_1=e^{2\pi iu_1/15}$, $\zeta_2=e^{2\pi iu_2/15}$, we get that the real part of γ is

$$-(\cos(2\pi u_1/15)+\cos(2\pi u_2/15))=-2\cos(\pi(u_1-u_2)/15)\cos(\pi(u_1+u_2)/15).$$

This is never zero for any choices of u_1 and u_2 in $\{1,\ldots,15\}$, contradicting Lemma 7.

Case 3. $\zeta_1^4 = \zeta_1$ and $\zeta_2^4 = \zeta_1$.

In this case, $\zeta_1^3 = 1$ and $\zeta_2 \in \{-\zeta_1,\ i\zeta_1,\ -i\zeta_1\}$. Rewriting our formula (19) as

$$(x^4 - \zeta_1)(x^4 - \zeta_2) = (x - \zeta_1)(x - \zeta_2)(x^2 - \gamma x - 1)(x^4 - \gamma x^2 - 1),$$

and identifying the coefficient of x^7 we get

$$\gamma = -(\zeta_1 + \zeta_2) \in \{0, -(1+i)\zeta_1,\ -(1-i)\zeta_1\}.$$

The case $\gamma = 0$ is not convenient and the remaining cases yield values for γ whose real part is nonzero, contradicting Lemma 7. □

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