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An estimate of the spread of trajectories for Kähler magnetic fields

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Abstract. On a Kähler manifold we consider trajectories under the influence of Kähler magnetic fields. They are smooth curves which are parameterized by their arclengths and whose velocities and normal vectors form complex lines. In this paper we study how trajectories spread, and give an estimate of norms of magnetic Jacobi fields from below and an estimate of area elements of trajectory-spheres.

Key words: Kähler magnetic fields, magnetic Jacobi fields, trajectory-spheres, area elements, comparison theorems.

1. Introduction

On a Kähler manifold M with complex structure J, we consider Kähler magnetic fields which are constant multiples of the Kähler form $\mathbb{B}_J$ on M. A trajectory γ for a Kähler magnetic field $\mathbb{B}_\kappa = \kappa\mathbb{B}_J$ ($\kappa \in \mathbb{R}$) is a smooth curve parameterized by its arclength satisfying the equation $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$ (see [1], [6]). Since the velocity vector and the normal of a trajectory γ span a complex line in the tangent space $T_{\gamma(t)}M$ at each point $\gamma(t)$, it is likely that properties of trajectories and properties of the base Kähler manifold M are closely related with each other. For a trivial magnetic field $\mathbb{B}_0$, which is a null 2-form, trajectories are geodesics. It is needless to say that geodesics play quite an important role in the study of Riemannian manifolds. The authors hence consider that trajectories may play the similar role of geodesics when they investigate Kähler manifolds from the viewpoint of Riemannian geometry. If we say a bit more on trajectories, when $\mathbb{B}_J$ is exact, that is $\mathbb{B}_J = d\mathbb{A}_J$ with a 1-form $\mathbb{A}_J$, trajectories (or trajectory segments) for $\mathbb{B}_\kappa$ of constant speed are stationary curves with respect to the energy functional $\int_a^b \{(1/2)\|c'(t)\|^2 + \kappa\mathbb{A}_J(c'(t))\}\,dt$ for smooth curves. But

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such a 1-form does not necessarily exist, it is not easy to investigate them from this energy functional point of view.

In order to study the behavior of trajectories, it is a basic problem to investigate how they spread on a Kähler manifold. Just like the study on geodesics, we consider variations of trajectories and define magnetic Jacobi fields. In the preceding paper [2], we studied the components of magnetic Jacobi fields which are orthognal to trajectories, and give their estimates. Though it corresponds to Rauch's comparison theorem on Jacobi fields along geodesics, as there is an interaction between components of magnetic Jacobi fields which are orthogonal to trajectories and those which are parallel to trajectories, the situation for magnetic Jacobi fields is different from that of Jacobi fields. In this paper, by adding terms into the magnetic index form which control components of fields parallel to trajectories, we estimate norms of whole components of magnetic Jacobi fields from below. As an application, we study area elements of trajectory-spheres, each of which are formed by trajectory-segments of the same length and emanating from a same point.

2. Magnetic Jacobi fields

Let M be a complete Kähler manifold of complex dimension n with complex structure J and Riemannian metric $\langle\ ,\ \rangle$. We take a Kähler magnetic field $\mathbb{B}_\kappa = \kappa\mathbb{B}_J$, where $\kappa \in \mathbb{R}$ and $\mathbb{B}_J$ denotes the Kähler form on M. Given a point $p \in M$, we define a *magnetic exponential map* $\mathbb{B}_\kappa \exp_p : T_pM \to M$ for $\mathbb{B}_\kappa$ of the tangent space T_pM at p by

$$\mathbb{B}_\kappa \exp_p(w) = \begin{cases} \gamma_{w/\|w\|}\big(\|w\|\big), & \text{if } w \neq 0_p, \\ p, & \text{if } w = 0_p. \end{cases}$$

Here, for a unit tangent vector $u \in U_pM$ we denote by γ_u the trajectory for $\mathbb{B}_\kappa$ of initial vector u. That is, a smooth curve parameterized by its arclength satisfying $\nabla_{\dot\gamma_u}\dot\gamma_u = \kappa J\dot\gamma_u$ and $\dot\gamma_u(0) = u$.

In order to investigate magnetic exponential maps for $\mathbb{B}_\kappa$, we need to study their differentials, hence need to study variations of trajectories for $\mathbb{B}_\kappa$. A vector field Y along a trajectory γ for $\mathbb{B}_\kappa$ is said to be a *normal magnetic Jacobi field* if it satisfies

$$\begin{cases} \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y - \kappa J\nabla_{\dot\gamma}Y + R(Y,\dot\gamma)\dot\gamma = 0, \\ \langle \nabla_{\dot\gamma}Y, \dot\gamma\rangle = 0. \end{cases} \tag{MJ}$$

It is known that normal magnetic Jacobi fields correspond one-to-one to variations of trajectories (see [2]). Being different from variations of geodesics, as we suppose trajectories are parameterized by their arclength, we need the second equality. For a vector field X along a trajectory γ for $\mathbb{B}_\kappa$ we decompose it into three components and denote it as $X = f_X\dot\gamma + g_X J\dot\gamma + X^\perp$ with functions f_X, g_X and a vector field $X^\perp$ which is orthogonal to both $\dot\gamma$ and $J\dot\gamma$ at each point. We set $X^\sharp = g_X J\dot\gamma + X^\perp$ and $X^\top = f_X\dot\gamma + g_X J\dot\gamma$. By use of this representation we see equalities (MJ) turn to the following:

$$\begin{cases} f_Y' = \kappa g_Y, \\ (g_Y'' + \kappa^2 g_Y)J\dot\gamma + \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y^\perp - \kappa J(\nabla_{\dot\gamma}Y)^\perp + R(Y^\sharp,\dot\gamma)\dot\gamma = 0. \end{cases}$$

We should hence note that there is an interaction between the component of a magnetic Jacobi field which is parallel to a trajectory and that orthogonal to the trajectory.

We say a point $\gamma(t_0)$ to be a *spherical magnetic conjugate point* of $\gamma(0)$ along a trajectory γ if there is a non-trivial normal magnetic Jacobi field Y along γ satisfying $Y(0) = 0$ and $Y(t_0) = 0$. We call this t_0 a spherical magnetic conjugate value of $\gamma(0)$ along γ. We denote by $c^s_\gamma(\gamma(0))$ the minimum positive spherical magnetic conjugate value of $\gamma(0)$ along γ if it exists. We set $c^s_\gamma(\gamma(0)) = \infty$ when there are no positive spherical magnetic conjugate values along γ. If we define a map $\Phi^\kappa_{p,r} : U_pM \to M$ by $\Phi^\kappa_{p,r}(u) = \mathbb{B}_\kappa \exp_p(ru)$, then its differential $(d\Phi^\kappa_{p,r})_u : T_u(U_pM) \to T_{\gamma_u(r)}M$ is singular if and only if r is a spherical magnetic conjugate value of p along γ_u. We recall that a point $\gamma(t_1)$ is called a magnetic conjugate point of $\gamma(0)$ along a trajectory γ if there is a non-trivial normal magnetic Jacobi field Y along γ satisfying $Y(0) = 0$ and $Y^\sharp(t_1) = 0$. We call such t_1 a magnetic conjugate value of $\gamma(0)$ along γ. We then have that the differential $(d\mathbb{B}_\kappa \exp_p)_{ru} : T_{ru}(T_pM) \to T_{\gamma_u(r)}M$ is singular if and only if r is a magnetic conjugate value of $\gamma(0)$ along γ. As there is an interaction between the component parallel to a trajectory and that orthogonal to that trajectory, the first magnetic conjugate value $c_\gamma(\gamma(0))$ does not coincide with $c^s_\gamma(\gamma(0))$, in general, and satisfies $c_\gamma(\gamma(0)) \le c^s_\gamma(\gamma(0))$.

We here make mention of spherical magnetic conjugate values on a complex space form $\mathbb{C}M^n(c)$ of constant holomorphic sectional curvature c, which is one of a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n$ and a complex hyperbolic space $\mathbb{C}H^n(c)$ according as c is positive, zero or negative. On $\mathbb{C}M^n(c)$ a normal magnetic Jacobi field Y along a trajectory γ for $\mathbb{B}_\kappa$ ($\kappa \neq 0$) satisfying $Y(0) = 0$ is of the form

$$Y(t) = \begin{cases} \dfrac{a\kappa}{\sqrt{|c|-\kappa^2}}\big(\cosh\sqrt{|c|-\kappa^2}\,t - 1\big)\dot\gamma(t) & \\ \qquad + a\sinh\sqrt{|c|-\kappa^2}\,t\,J\dot\gamma(t) & \text{if } \kappa^2 + c < 0, \\ \qquad + e^{\sqrt{-1}\kappa t/2}\sinh\dfrac{1}{2}\sqrt{|c|-\kappa^2}\,t\,E(t), & \\ a\Big\{\dfrac{|c|t^2}{2}\dot\gamma(t) + \kappa t J\dot\gamma(t)\Big\} + te^{\sqrt{-1}\kappa t/2}E(t), & \text{if } \kappa^2 + c = 0, \\ \dfrac{a\kappa}{\sqrt{\kappa^2+c}}\big(1 - \cos\sqrt{\kappa^2+c}\,t\big)\dot\gamma(t) & \\ \qquad + a\sin\sqrt{\kappa^2+c}\,t\,J\dot\gamma(t) & \text{if } \kappa^2 + c > 0. \\ \qquad + e^{\sqrt{-1}\kappa t/2}\sin\dfrac{1}{2}\sqrt{\kappa^2+c}\,t\,E(t), & \end{cases}$$

with a constant a and a parallel vector field E along γ (see [2]). Here, we denote by $e^{\sqrt{-1}\alpha t}E(t)$ the vector field $\cos\alpha t\,E(t) + \sin\alpha t\,JE(t)$.

We define a function $\mathfrak{s}_\kappa(\cdot\,; c) : [0, 2\pi/\sqrt{\kappa^2+c}) \to \mathbb{R}$ by

$$\mathfrak{s}_\kappa(t;c) = \begin{cases} \dfrac{2}{\sqrt{|c|-\kappa^2}}\sinh\big(\sqrt{|c|-\kappa^2}\,t/2\big), & \text{if } \kappa^2 + c < 0, \\ t, & \text{if } \kappa^2 + c = 0, \\ \dfrac{2}{\sqrt{\kappa^2+c}}\sin\big(\sqrt{\kappa^2+c}\,t/2\big), & \text{if } \kappa^2 + c > 0, \end{cases}$$

and put $\mathfrak{c}_\kappa(t;c) = (d/dt)\mathfrak{s}_\kappa(t;c)$ and $\mu_\kappa(\cdot\,;c) = \{(\kappa^2/4)\mathfrak{s}_\kappa(t;c)^2 + \mathfrak{c}_\kappa(t;c)^2\}^{1/2}$. Here, we regard $2\pi/\sqrt{\kappa^2+c}$ infinity when $\kappa^2 + c \leq 0$. We use such a convention without notice in the following. If we write down explicitly, we have

$$\mu_\kappa(t;c) = \begin{cases} \sqrt{\dfrac{|c|\cosh^2\big(\sqrt{|c|-\kappa^2}\,t/2\big)-\kappa^2}{|c|-\kappa^2}}, & \text{if } \kappa^2+c<0, \\ \sqrt{|c|t^2+4}/2, & \text{if } \kappa^2+c=0, \\ \sqrt{\dfrac{\kappa^2+c\cos^2\big(\sqrt{\kappa^2+c}\,t/2\big)}{\kappa^2+c}}, & \text{if } \kappa^2+c>0. \end{cases}$$

These functions have the following properties:

i) If $c_1 < c_2$, then $\mathfrak{s}_\kappa(t;c_1) > \mathfrak{s}_\kappa(t;c_2)$ and $\mu_\kappa(t;c_1) > \mu_\kappa(t;c_2)$;
ii) $\mu_\kappa(\cdot;c)$ satisfies $\mu_\kappa(t;c) > 1$, $\mu_\kappa(t;c) = 1$ and $\mu_\kappa(t;c) < 1$ according as $c < 0$, $c = 0$ and $c > 0$;
iii) For sufficiently small t (> 0), we have $\mathfrak{s}_\kappa(t;c)\mu_\kappa(t;c) > \mathfrak{s}_\kappa(t;4c)$ when $c > 0$, and $\mathfrak{s}_\kappa(t;c)\mu_\kappa(t;c) < \mathfrak{s}_\kappa(t;4c)$ when $c < 0$.

The authors consider that $\mathfrak{s}_\kappa(t;c)\mu_\kappa(t;c) > \mathfrak{s}_\kappa(t;4c)$ when $c > 0$ and $\mathfrak{s}_\kappa(t;c)\mu_\kappa(t;c) < \mathfrak{s}_\kappa(t;4c)$ when $c < 0$ for $0 < t < \pi/\sqrt{\kappa^2+c}$. Still they can not make clear this and can only say this in a neighborhood of the origin.

By the expression of magnetic Jacobi fields on $\mathbb{C}M^n(c)$ we have the following.

Proposition 1 *On $\mathbb{C}M^n(c)$, a normal magnetic Jacobi field Y along a trajectory γ for a non-trivial Kähler magnetic field $\mathbb{B}_\kappa$ satisfies*

$$\|Y^\top(t)\| = \big\|\nabla_{\dot\gamma}Y^\top(0)\big\|\mathfrak{s}_\kappa(t;c)\mu_\kappa(t;c), \quad \|Y^\perp(t)\| = \big\|\nabla_{\dot\gamma}Y^\perp(0)\big\|\mathfrak{s}_\kappa(t;c),$$

for $0 \le t < 2\pi/\sqrt{\kappa^2+c}$. Thus the first spherical conjugate value does not depend on the choice of trajectories and initial points. If we denote it by $\mathsf{c}^s(\kappa,c)$, it is given as $\mathsf{c}^s(\kappa,c) = 2\pi/\sqrt{\kappa^2+c}$ when $\kappa^2+c>0$ and $\mathsf{c}^s(\kappa,c)=\infty$ when $\kappa^2+c\le 0$.

We here note that on $\mathbb{C}M^n(c)$ the first magnetic conjugate values along trajectories for $\mathbb{B}_\kappa$ also do not depend on the choice of trajectories and initial points. If we denote this value by $\mathsf{c}(\kappa,c)$, it is equal to $\pi/\sqrt{\kappa^2+c} = \mathsf{c}^s(\kappa,c)/2$ (see [2]).

3. Comparison theorems on magnetic Jacobi fields

In this section we study a comparison theorem on magnetic Jacobi fields. As we mentioned in Section 1, we studied comparison theorems on components of magnetic Jacobi fields which are orthogonal to trajectories in the preceding papers [2], [3]. Since their components parallel to trajectories and orthogonal to trajectories are interacted to each other, we here study the whole components of magnetic Jacobi fields.

For a vector field $X = f_X\dot\gamma + g_X J\dot\gamma + X^{\perp}$ along a trajectory γ for a Kähler magnetic field $\mathbb{B}_\kappa$ on a Kähler manifold M, we set

$$\widetilde{\mathcal{I}}_0^T(X) = \kappa\{f_X(T)g_X(T)-f_X(0)g_X(0)\}+\int_0^T\{\kappa g_X(t)-f_X'(t)\}^2dt+\mathcal{I}_0^T(X^{\sharp}),$$

and call this the modified index of X. Here, $\mathcal{I}_0^T(X^{\sharp})$ denotes the index of $X^{\sharp} = g_X J\dot\gamma + X^{\perp}$ which is given by

$$\mathcal{I}_0^T(X^{\sharp}) = \int_0^T \bigl\{g_X'^2 - \kappa^2 g_X^2 + \langle \nabla_{\dot\gamma}X^{\perp} - \kappa JX^{\perp}, \nabla_{\dot\gamma}X^{\perp}\rangle - \langle R(X,\dot\gamma)\dot\gamma, X\rangle\bigr\}dt.$$

As we see in [2], indices of vector fields along trajectories which are orthogonal to their velocity vectors satisfy the following properties which correspond to properties of Jacobi fields along geodesics (see [7]).

1) For a normal magnetic Jacobi field Y along a trajectory γ, we have

$$\mathcal{I}_0^T(Y^{\sharp}) = \langle(\nabla_{\dot\gamma}Y^{\sharp})(T), Y^{\sharp}(T)\rangle - \langle\nabla_{\dot\gamma}Y^{\sharp}(0), Y^{\sharp}(0)\rangle.$$

2) Suppose $0 < T < c_\gamma(\gamma(0))$. If a vector field X along γ satisfies $X^{\sharp} = X$, $X(0) = 0$ and $X(T) = Y^{\sharp}(T)$ with some normal magnetic Jacobi field Y satisfying $Y^{\sharp}(0) = 0$, then it satisfies $\mathcal{I}_0^T(X) \geq \mathcal{I}_0^T(Y^{\sharp})$, and the equality holds if and only if $X \equiv Y^{\sharp}$.

We here show that the modified indices satisfy the similar properties.

Lemma 1 *The modified index of a normal magnetic Jacobi field Y for $\mathbb{B}_\kappa$ is given as*

$$\widetilde{\mathcal{I}}_0^T(Y) = \langle(\nabla_{\dot\gamma}Y)(T), Y(T)\rangle - \langle\nabla_{\dot\gamma}Y(0), Y(0)\rangle.$$

Proof. Since Y is a normal magnetic Jacobi field for $\mathbb{B}_\kappa$, we have

$$\begin{aligned}
&\langle \nabla_{\dot\gamma} Y(T), Y(T)\rangle - \langle \nabla_{\dot\gamma} Y(0), Y(0)\rangle \\
&= \int_0^T \frac{d}{dt}\langle \nabla_{\dot\gamma} Y(t), Y(t)\rangle dt = \int_0^T \bigl\{\langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} Y(t), Y(t)\rangle + \|\nabla_{\dot\gamma} Y(t)\|^2\bigr\} dt \\
&= \int_0^T \bigl\{\kappa\langle J\nabla_{\dot\gamma} Y(t), Y(t)\rangle + \|\nabla_{\dot\gamma} Y(t)\|^2 - \langle R(Y(t),\dot\gamma(t))\dot\gamma(t), Y(t)\rangle\bigr\} dt.
\end{aligned}$$

As we have $\nabla_{\dot\gamma} Y = (g_Y' + \kappa f_Y)J\dot\gamma + \nabla_{\dot\gamma} Y^\perp$, we find

$$\begin{aligned}
&= \int_0^T \bigl\{-\kappa(g_Y'(t) + \kappa f_Y(t)) f_Y(t) + (g_Y'(t) + \kappa f_Y(t))^2 \\
&\qquad + \kappa\langle J\nabla_{\dot\gamma} Y^\perp(t), Y^\perp(t)\rangle + \|\nabla_{\dot\gamma} Y^\perp(t)\|^2 \\
&\qquad - \langle R(Y^\sharp(t),\dot\gamma(t))\dot\gamma(t), Y^\sharp(t)\rangle\bigr\} dt \\
&= \int_0^T \bigl\{\kappa f_Y(t) g_Y'(t) + g_Y'(t)^2 + \langle \nabla_{\dot\gamma} Y^\perp(t) - \kappa J Y^\perp(t), \nabla_{\dot\gamma} Y^\perp(t)\rangle \\
&\qquad - \langle R(Y^\sharp(t),\dot\gamma(t))\dot\gamma(t), Y^\sharp(t)\rangle\bigr\} dt \\
&= \kappa \int_0^T \bigl\{f_Y(t) g_Y'(t) + \kappa g_Y(t)^2\bigr\} dt + \mathcal{I}_0^T(Y^\sharp) \\
&= \kappa\bigl[f_Y(t) g_Y(t)\bigr]_0^T + \kappa \int_0^T g_Y(t)\bigl\{\kappa g_Y(t) - f_Y'(t)\bigr\} dt + \mathcal{I}_0^T(Y^\sharp).
\end{aligned}$$

This completes the proof because $f_Y' \equiv \kappa g_Y$. $\square$

Lemma 2 *Let Y be a normal magnetic Jacobi field along a trajectory γ for $\mathbb{B}_\kappa$ satisfying $Y^\sharp(0) = 0$. If $0 < T < \mathsf{c}_\gamma(\gamma(0))$ and a vector field X along γ satisfyies $X^\sharp(0) = 0$ and $X(T) = Y(T)$, then it satisfies $\widetilde{\mathcal{I}}_0^T(X) \geq \widetilde{\mathcal{I}}_0^T(Y)$. The equality holds if and only if $X \equiv Y$.*

Proof. Since we have $\mathcal{I}_0^T(X^\sharp) \geq \mathcal{I}_0^T(Y^\sharp)$, we get

$$\begin{aligned}
\widetilde{\mathcal{I}}_0^T(X) &= \kappa f_X(T) g_X(T) + \int_0^T \bigl\{\kappa g_X(t) - f_X'(t)\bigr\}^2 dt + \mathcal{I}_0^T(X^\sharp) \\
&\geq \kappa f_Y(T) g_Y(T) + \mathcal{I}_0^T(Y^\sharp) = \widetilde{\mathcal{I}}_0^T(Y).
\end{aligned}$$

This also shows that the equality holds if and only if both $\mathcal{I}_0^T(X^\sharp) = \mathcal{I}_0^T(Y^\sharp)$ and $\int_0^T(\kappa g_X - f_X')^2 dt = 0$ hold. As we have $X(T) = Y(T)$, which shows $X^\sharp(T) = Y^\sharp(T)$ and $f_X(T) = f_Y(T)$, the first equality $\mathcal{I}_0^T(X^\sharp) = \mathcal{I}_0^T(Y^\sharp)$ leads us to $X^\sharp \equiv Y^\sharp$. The second equality shows $\kappa g_X \equiv f_X'$, hence we have $f_Y' \equiv \kappa g_Y \equiv \kappa g_X \equiv f_X'$. As $f_X(T) = f_Y(T)$, we find $f_X \equiv f_Y$. Thus we get $X \equiv Y$. $\square$

In order to show the inequality $\mathcal{I}_0^T(X^\sharp) \geq \mathcal{I}_0^T(Y^\sharp)$ on indices of vector fields orthognal to γ, we need $T \leq \mathsf{c}_\gamma(\gamma(0))$. We hence need this assumption in Lemma 2.

We here consider indices of normal magnetic Jacobi fields on a complex space form $\mathbb{C}M^n(c)$. We define functions $\mathfrak{t}_\kappa(\cdot\,;c) : (0, 2\pi/\sqrt{\kappa^2+c}) \to \mathbb{R}$ and $\nu_\kappa(\cdot\,;c) : \mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{t}_\kappa(t;c) = \frac{\mathfrak{c}_\kappa(t;c)}{\mathfrak{s}_\kappa(t;c)} = \begin{cases} \big(\sqrt{|c|-\kappa^2}/2\big)\coth\big(\sqrt{|c|-\kappa^2}\,t/2\big), & \text{if } \kappa^2+c<0,\\ 1/t, & \text{if } \kappa^2+c=0,\\ \big(\sqrt{\kappa^2+c}/2\big)\cot\big(\sqrt{\kappa^2+c}\,t/2\big), & \text{if } \kappa^2+c>0,\end{cases}$$

$$\nu_\kappa(t;c) = \begin{cases} \dfrac{|c|\cosh\sqrt{|c|-\kappa^2}\,t-\kappa^2}{|c|\cosh^2\big(\sqrt{|c|-\kappa^2}\,t/2\big)-\kappa^2}, & \text{if } \kappa^2+c<0,\\ (2|c|t^2+4)/(|c|t^2+4), & \text{if } \kappa^2+c=0,\\ \dfrac{\kappa^2+c\cos\sqrt{\kappa^2+c}\,t}{\kappa^2+c\cos^2\big(\sqrt{\kappa^2+c}\,t/2\big)}, & \text{if } \kappa^2+c>0.\end{cases}$$

These functions satisfy the following properties:

i) If $c_1+\kappa_1^2 < c_2+\kappa_2^2$, then $\mathfrak{t}_{\kappa_1}(t;c_1) > \mathfrak{t}_{\kappa_2}(t;c_2)$ for $0<t<2\pi/\sqrt{\kappa_2^2+c_2}$;

ii) $\nu_\kappa(t;c)>1$, $\nu_\kappa(t;c)=1$ or $\nu_\kappa(t;c)<1$ according as $c<0$, $c=0$ or $c>0$;

iii) When $c<0$, the function $\nu_\kappa(\cdot\,;c) : \big[0,\pi/\sqrt{\kappa^2+c}\,\big] \to \mathbb{R}$ is monotone increasing and satisfies $\nu_\kappa(t;c)<2$;

iv) When $c>0$, the function $\nu_\kappa(\cdot\,;c) : \big[0,\pi/\sqrt{\kappa^2+c}\,\big] \to \mathbb{R}$ is monotone decreasing and satisfies $\nu_\kappa(t;c) > 1-(c/\kappa^2)$;

v) $\mathfrak{t}_\kappa(t;c)\nu_\kappa(t;c) > 2\mathfrak{t}_\kappa(2t;c)$ for $0<t<\pi/\sqrt{\kappa^2+c}$.

We also note that the functions $\mathfrak{s}_\kappa(t;c)\mu_\kappa(t;c)$ and $\mathfrak{t}_\kappa(t;c)\nu_\kappa(t;c)$ are related to each other as

$$\frac{d}{dt}\log(\mathfrak{s}_\kappa(t;c)\mu_\kappa(t;c)) = \mathfrak{t}_\kappa(t;c)\nu_\kappa(t;c).$$

We here show logarithmic derivatives of norms of normal magnetic Jacobi fields on a complex space form.

Proposition 2 *Let γ be a trajectory for a non-trivial Kähler magnetic field $\mathbb{B}_\kappa$ on $\mathbb{C}M^n(c)$. A normal magnetic Jacobi field Y along γ with $Y(0) = 0$ satisfies*

$$\begin{aligned}
\big\langle \nabla_{\dot\gamma} Y^\top(t), Y^\top(t)\big\rangle &= \|Y^\top(t)\|^2\, \mathfrak{t}_\kappa(t;c)\nu_\kappa(t;c),\\
\big\langle \nabla_{\dot\gamma} Y^\perp(t), Y^\perp(t)\big\rangle &= \|Y^\perp(t)\|^2\, \mathfrak{t}_\kappa(t;c),
\end{aligned}$$

for $0 < t < 2\pi/\sqrt{\kappa^2 + c}$.

We now study magnetic Jacobi fields on a general Kähler manifold M. By using modified indices for vector fields along trajectories, we give an estimate on derivatives of norms of normal magnetic Jacobi fields from below.

Theorem 1 *Let γ be a trajectory for a non-trivial Kähler magnetic field $\mathbb{B}_\kappa$ on a Kähler manifold M. Suppose sectional curvatures satisfy $\max\{\mathrm{Riem}(v, \dot\gamma(t)) \mid v \in U_{\gamma(t)}M,\ v \perp \dot\gamma(t)\} \le c$ for some constant c for $0 \le t \le \pi/\sqrt{\kappa^2 + c}$. Then, for a normal magnetic Jacobi field Y along γ satisfying $Y(0) = 0$, we have*

$$\langle \nabla_{\dot\gamma} Y(t), Y(t)\rangle \ge \|Y^\top(t)\|^2 \mathfrak{t}_\kappa(t;c)\nu_\kappa(t;c) + \|Y^\perp(t)\|^2 \mathfrak{t}_\kappa(t;4c)$$

for $0 < t < \pi/\sqrt{\kappa^2 + c}$. If the equality holds at some t_0 with $0 < t_0 < \pi/\sqrt{\kappa^2 + c}$, then it holds at every t with $0 \le t \le t_0$. In this case, the normal magnetic Jacobi field Y is of the form

$$\begin{aligned}
Y(t) = \big\|\nabla_{\dot\gamma} Y^\top(0)\big\| \mathfrak{s}_\kappa(t;c)\big\{(\kappa/2)\mathfrak{s}_\kappa(t;c)\dot\gamma(t) + \mathfrak{c}_\kappa(t;c)J\dot\gamma(t)\big\}&\\
+ \big\|\nabla_{\dot\gamma} Y^\perp(0)\big\| \mathfrak{s}_\kappa(t;4c)\big\{\cos(\kappa t/2)E(t) + \sin(\kappa t/2)JE(t)\big\}& \qquad (3.1)
\end{aligned}$$

with a parallel vector field E along γ satisfying $E(0) = \nabla_{\dot\gamma} Y^\perp(0)/\|\nabla_{\dot\gamma} Y^\perp(0)\|$, and the curvature tensor satisfies $R(Y^\sharp, \dot\gamma)\dot\gamma \equiv cY^\sharp$ for $0 \le t \le t_0$. Here, when $\nabla_{\dot\gamma} Y^\perp(0) = 0$ we take E as a null vector field.

Proof. We note that $\mathsf{c}^s_\gamma(\gamma(0)) \geq \mathsf{c}_\gamma(\gamma(0)) \geq \pi/\sqrt{\kappa^2+c}$ (see [3]). We take an arbitrary T with $0 < T < \pi/\sqrt{\kappa^2+c}$. When the complex dimension of M is n, we take a complex space form $\widehat{M} = \mathbb{C}M^n(4c)$ of constant holomorphic sectional curvature $4c$. Let $P^t_\gamma : T_{\gamma(t)}M \to T_{\gamma(0)}M$ and $\widehat{P}^t_{\hat\gamma} : T_{\hat\gamma(0)}\widehat{M} \to T_{\hat\gamma(t)}\widehat{M}$ denote parallel translations along γ and along a trajectory $\hat\gamma$ for $\mathbb{B}_\kappa$ on $\widehat{M}$, respectively. We take a holomorphic linear isometry $I : T_{\gamma(0)}M \to T_{\hat\gamma(0)}\widehat{M}$ satisfying $I(\dot\gamma(0)) = \dot{\hat\gamma}(0)$, and define a vector field $\widehat{X}$ along $\hat\gamma$ by $\widehat{X}(t) = \widehat{P}^t_{\hat\gamma} \circ I \circ P^t_\gamma(Y^\perp(t))$. We also take a trajectory $\tilde\gamma$ for $\mathbb{B}_\kappa$ on $\mathbb{C}M^n(c)$ and choose a normal magnetic Jacobi field $\tilde f \dot{\tilde\gamma} + \tilde g J\dot{\tilde\gamma}$ along $\tilde\gamma$ satisfying $\tilde g(0) = 0$, $\tilde g(T) = g_Y(T)$ and $\tilde f(T) = f_Y(T)$. We note that $\tilde f(0)$ may not be zero. We also take a normal magnetic Jacobi field $\widehat{Y}$ along $\hat\gamma$ satisfying $\widehat{Y}(0) = 0$ and $\widehat{Y}(T) = \widehat{X}(T)$. Clearly, this satisfies $\widehat{Y}^\perp = \widehat{Y}$. Since $\langle R(Y,\dot\gamma)\dot\gamma, Y\rangle \leq c\|Y^\sharp\|^2$, we have

$$
\begin{aligned}
\big\langle \nabla_{\dot\gamma} Y(T), Y(T)\big\rangle &= \widetilde{I}^T_0(Y) \\
&= \kappa f_Y(T) g_Y(T) \\
&\quad + \int_0^T \big\{ g_Y'^2 - \kappa^2 g_Y^2 + \langle \nabla_{\dot\gamma} Y^\perp - \kappa J Y^\perp, \nabla_{\dot\gamma} Y^\perp\rangle - \langle R(Y,\dot\gamma)\dot\gamma, Y\rangle \big\} dt \\
&\geq \kappa f_Y(T) g_Y(T) + \int_0^T \big(g_Y'^2 - \kappa^2 g_Y^2 - c g_Y^2\big) dt \\
&\quad + \int_0^T \big\{ \langle \nabla_{\dot\gamma} Y^\perp - \kappa J Y^\perp, \nabla_{\dot\gamma} Y^\perp\rangle - c\|Y^\perp\|^2 \big\} dt \\
&= \kappa f_Y(T) g_Y(T) + \int_0^T \big(g_Y'^2 - \kappa^2 g_Y^2 - c g_Y^2\big) dt \\
&\quad + \int_0^T \big\{ \langle \nabla_{\dot{\hat\gamma}} \widehat{X} - \kappa J \widehat{X}, \nabla_{\dot{\hat\gamma}} \widehat{X}\rangle - c\|\widehat{X}\|^2 \big\} dt \\
&= \widetilde{I}^T_0\big(f_Y \dot{\tilde\gamma} + g_Y J\dot{\tilde\gamma}\big) + \widetilde{I}^T_0(\widehat{X}) \\
&\geq \widetilde{I}^T_0\big(\tilde f \dot{\tilde\gamma} + \tilde g J\dot{\tilde\gamma}\big) + \widetilde{I}^T_0(\widehat{Y})
\end{aligned}
$$

Therefore we get

$$\begin{aligned}\langle \nabla_{\dot\gamma} Y(T), Y(T)\rangle &\geq \big\{\tilde f(T)^2 + \tilde g(T)^2\big\}\mathfrak{t}_\kappa(T;c)\nu_\kappa(T;c) + \|\widehat{Y}(T)\|^2\mathfrak{t}_\kappa(T;4c)\\ &= \|Y^{\top}(T)\|^2\,\mathfrak{t}_\kappa(T;c)\nu_\kappa(T;c) + \|Y^{\perp}(T)\|^2\,\mathfrak{t}_\kappa(T;4c).\end{aligned}$$

The equality holds if and only if the following two conditions hold:

i) $f_Y \equiv \tilde f$, $g_Y \equiv \tilde g$ and $\widehat{X} \equiv \widehat{Y}$ for $0 \leq t \leq T$,
ii) $\langle R(Y^\sharp, \dot\gamma)\dot\gamma, Y^\sharp\rangle \equiv c\|Y^\sharp\|^2$ for $0 < t \leq T$.

These show that if the equality holds at T then it holds for $0 \leq t \leq T$. By the expression of normal magnetic Jacobi fields on complex space forms, the above condition i) shows that Y is of the form (3.1) for $0 \leq t \leq T$. The expression of Y, the first equality in (MJ) and the above condition ii) show that $R(Y^\sharp, \dot\gamma)\dot\gamma = cY^\sharp$. This completes the proof. □

For the sake of completeness, we here give an estimate from above.

Theorem 2 *Let γ be a trajectory for a non-trivial Kähler magnetic field $\mathbb{B}_\kappa$ on a Kähler manifold M. Suppose sectional curvatures satisfy* $\min\{\mathrm{Riem}(v, \dot\gamma(t)) \mid v \in U_{\gamma(t)}M,\ v \perp \dot\gamma(t)\} \geq c$ *for some constant c for $0 \leq t \leq \mathsf{c}_\gamma(\gamma(0))$. Then, for a normal magnetic Jacobi field Y along γ satisfying $Y(0) = 0$, we have*

$$\langle \nabla_{\dot\gamma} Y(t), Y(t)\rangle \leq \|Y^{\top}(t)\|^2\mathfrak{t}_\kappa(t;c)\nu_\kappa(t;c) + \|Y^{\perp}(t)\|^2\mathfrak{t}_\kappa(t;4c)$$

for $0 < t < \mathsf{c}_\gamma(\gamma(0))$. If the equality holds at some t_0 with $0 < t_0 < \mathsf{c}_\gamma(\gamma(0))$, then the equality holds at every t with $0 \leq t \leq t_0$. In this case, the normal magnetic Jacobi field Y is of the form

$$\begin{aligned}Y(t) = \big\|\nabla_{\dot\gamma}Y^{\top}(0)\big\|\mathfrak{s}_\kappa(t;c)\big\{(\kappa/2)\mathfrak{s}_\kappa(t;c)\dot\gamma(t) + \mathfrak{c}_\kappa(t;c)J\dot\gamma(t)\big\}\\ + \big\|\nabla_{\dot\gamma}Y^{\perp}(0)\big\|\mathfrak{s}_\kappa(t;4c)\big\{\cos(\kappa t/2)E(t) + \sin(\kappa t/2)JE(t)\big\},\end{aligned}$$

with a parallel vector field E along γ satisfying $E(0) = \nabla_{\dot\gamma}Y^{\perp}(0)/\|\nabla_{\dot\gamma}Y^{\perp}(0)\|$, the curvature tensor satisfies $R(Y^\sharp, \dot\gamma)\dot\gamma \equiv c\,Y^\sharp$ for $0 \leq t \leq t_0$.

Proof. We take a trajectory $\tilde\gamma$ for $\mathbb{B}_\kappa$ on $\mathbb{C}M^n(c)$ and a trajectory $\hat\gamma$ for $\mathbb{B}_\kappa$ on $\mathbb{C}M^n(4c)$. For an arbitrary potitive T with $T < \mathsf{c}_\gamma(\gamma(0))$ we choose a normal magnetic Jacobi field $\widehat{Y}$ along $\hat\gamma$ satisfying $\widehat{Y}(0) = 0$ and $\widehat{Y}(T) = \widehat{P}^T_{\hat\gamma} \circ I \circ P^T_\gamma(Y^{\perp}(T))$, where $\widehat{P}^T_{\hat\gamma}$, I, P^T_γ are as in the proof of Theorem 1. We also choose a normal magnetic Jacobi field $\tilde f\dot{\tilde\gamma} + \tilde g J\dot{\tilde\gamma}$ along $\tilde\gamma$ satisfying

$\tilde{g}(0) = 0$, $\tilde{f}(T) = f_Y(T)$, $\tilde{g}(T) = g_Y(T)$, and define a vector field X along γ by $X(t) = \tilde{f}(t)\dot{\gamma}(t) + \tilde{g}(t)J\dot{\gamma}(t) + (\widehat{P}^T_{\tilde{\gamma}} \circ I \circ P^T_{\gamma})^{-1}(\widehat{Y}(t))$. We are enough to show that $\widetilde{I}^T_0(\tilde{f}\dot{\tilde{\gamma}} + \tilde{g}J\dot{\tilde{\gamma}}) + \widetilde{I}^T_0(\widehat{Y}) \geq \langle \nabla_{\dot{\gamma}} Y(t), Y(t)\rangle$. Like the calculation in the proof of Theorem 1 we have

$$\begin{aligned}
&\widetilde{I}^T_0\big(\tilde{f}\dot{\tilde{\gamma}} + \tilde{g}J\dot{\tilde{\gamma}}\big) + \widetilde{I}^T_0(\widehat{Y}) \\
&\quad = \kappa\tilde{f}(T)\tilde{g}(T) + \int_0^T \big(\tilde{g}'^2 - \kappa^2\tilde{g}^2 - c\tilde{g}^2\big)dt \\
&\qquad + \int_0^T \big\{\langle \nabla_{\dot{\gamma}} X^{\perp} - \kappa J X^{\perp}, \nabla_{\dot{\gamma}} X^{\perp}\rangle - c\|X^{\perp}\|^2\big\}dt \\
&\quad \geq \kappa\tilde{f}(T)\tilde{g}(T) \\
&\qquad + \int_0^T \big\{\tilde{g}'^2 - \kappa^2\tilde{g}^2 + \langle \nabla_{\dot{\gamma}} X^{\perp} - \kappa J X^{\perp}, \nabla_{\dot{\gamma}} X^{\perp}\rangle - \langle R(X,\dot{\gamma})\dot{\gamma}, X\rangle\big\}dt \\
&\quad = \widetilde{I}^T_0(X) \geq \widetilde{I}^T_0(Y) = \langle \nabla_{\dot{\gamma}} Y(T), Y(T)\rangle.
\end{aligned}$$

Thus we get our conclusion along the same lines as in the proof of Theorem 1. □

4. Trajectory-spheres

We shall apply our result to a problem on estimates of area-elements of trajectory-spheres. We consider a Kähler magnetic field $\mathbb{B}_\kappa$ on a Kähler manifold M. For a positive r, we put $S^\kappa_p(r) = \{\mathbb{B}_\kappa \exp_p(ru) \mid u \in U_pM\}$ and call it a *trajectory-sphere* of arc-radius r centered at p. When $\kappa = 0$, the magnetic field $\mathbb{B}_0$ is the trivial magnetic field *0*, and its magnetic exponential map is the usual exponential map. Hence trajectory-spheres for the trivial magnetic field are nothing but geodesic spheres. Therefore we may consider trajectory-spheres as deformations of geodesic spheres under actions of magnetic fields.

For a unit tangent vector $u \in U_pM$ at a point $p \in M$, we choose unit tangent vectors $e_2, \ldots, e_{2n} \in T_pM$ so that $\{u, e_2, \ldots, e_{2n}\}$ is an orthonomal basis of T_pM. For each j we take a normal magnetic Jacobi field Y_j along a trajectory γ_u for $\mathbb{B}_\kappa$ satisfying $Y_j(0) = 0$, $(\nabla_{\dot{\gamma}_u} Y_j)(0) = e_j$. As we have $Y_j(t) = (d\mathbb{B}_\kappa \exp_p)_{tu}(te_j)$, we set $\alpha_\kappa(r, u)$ for r with $0 < r < \boldsymbol{c}^s_\gamma(\gamma(0))$ as

$$\alpha_\kappa(r,u) = \begin{vmatrix} \langle Y_2, Y_2\rangle & \cdots & \langle Y_2, Y_{2n}\rangle \\ \vdots & & \vdots \\ \langle Y_{2n}, Y_2\rangle & \cdots & \langle Y_{2n}, Y_{2n}\rangle \end{vmatrix}^{1/2},$$

which is the Jacobian of $\Phi^\kappa_{p,r}$ at u. With the standard volume element $d\omega$ of $S^{2n-1} = U_pM$, we can define a area element of $S^\kappa_p(r)$ by use of $\alpha_\kappa(r,u)d\omega$ through $\Phi^\kappa_{p,r}$. We shall call $\alpha_\kappa(r,u)$ the density function of the area element of $S^\kappa_p(r)$.

In a complex space form $\mathbb{C}M^n(c)$, we know that every trajectory-sphere coincides with some geodesic sphere (see [4]). We can hence get the area-element of a trajectory-sphere of arc-radius r in $\mathbb{C}M^n(c)$, which is denoted by $\alpha_\kappa(r;c,n)$ because it does not depend on the choice of u and p. As we have expressions of normal magnetic Jacobi fields, we can also compute it directly.

Proposition 3 *In a complex space form $\mathbb{C}M^n(c)$, the density function of the area-element of a trajectory-sphere of arc-radius r is given as $\alpha_\kappa(r;c,n) = \mu_\kappa(r;c)(\mathfrak{s}_\kappa(r;c))^{2n-1}$.*

In order to estimate density functions of area-forms on general Kähler manifolds, we shall study the behavior of them with respect to arc-radii. For a trajectory γ for $\mathbb{B}_\kappa$ we denote by $\mathcal{J}_\gamma$ the set of all normal magnetic Jacobi fields along γ whose initial values are null. It is a vector space of dimension $(2n-1)$. The real subspace $T_{\gamma_u(t)}S^\kappa_t(p) = \{Z(t) \in T_{\gamma_u(t)}M \mid Z \in \mathcal{J}_{\gamma_u}\}$ of $T_{\gamma_u(t)}M$ is also $(2n-1)$ dimensional when $0 < t < \mathfrak{c}^s_{\gamma_u}(p)$.

Lemma 3 *For an arbitrary r with $0 < r < \mathfrak{c}^s_{\gamma_u}(p)$, we take normal magnetic Jacobi fields $W_2, \dots, W_{2n}$ along γ_u for $\mathbb{B}_\kappa$ satisfying the following conditions:*

i) *$W_j(0) = 0$ for $j = 2, \dots, 2n$;*
ii) *$\{W_2(r), \dots, W_{2n}(r)\}$ is an orthonormal basis of $T_{\gamma(r)}S^\kappa_r(p)$.*

We then have

$$\frac{1}{\alpha_\kappa(r,u)} \frac{\partial}{\partial t}\alpha_\kappa(t,u)\Big|_{t=r} = \sum_{j=2}^{2n} \langle (\nabla_{\dot\gamma_u} W_j)(r), W_j(r)\rangle.$$

Proof. Since we see $\{W_2(r), \dots, W_{2n}(r)\}$ is a basis of $\mathcal{J}_\gamma$, we can write the normal magnetic Jacobi field $Y_j \in \mathcal{J}_{\gamma_u}$ with $(\nabla_{\dot\gamma_u} Y_j)(0) = e_j$ as $Y_j = \sum_{\ell=2}^{2n} a_{j\ell} W_\ell$ with some real numbers $a_{j\ell}$ ($\ell = 2, \dots, 2n$) for $j = 2, \dots, 2n$. We define a matrix A by $A = (a_{ij})$ and a matrix-valued function W by $W(t) = (\langle W_k(t), W_\ell(t)\rangle)$. As we have $\langle Y_i, Y_j\rangle = \sum_{k,\ell} a_{ik} a_{j\ell} \langle W_k, W_\ell\rangle$, we obtain

$$\alpha_\kappa(t,u)^2 = \det(\langle Y_i(t), Y_j(t)\rangle) = \det(A\, W(t)\, {}^t\!A) = \det(A)^2 \det(W(t)).$$

We hence have

$$\begin{aligned} &\frac{1}{\alpha_\kappa(r,u)} \frac{\partial}{\partial t} \alpha_\kappa(t,u)\Big|_{t=r} \\ &= \frac{\partial}{\partial t} \log \alpha_\kappa(t,u)\Big|_{t=r} = \frac{1}{2}\frac{\partial}{\partial t} \log\big(\det(A)^2 |\det(W(t))|\big)\big|_{t=r} \\ &= \frac{1}{2}\frac{\partial}{\partial t} \log\big(|\det(W(t))|\big)\big|_{t=r} = \frac{1}{2}\mathrm{trace}(W'(r)W(r)^{-1}), \end{aligned}$$

where we set $W'(r) = ((d/dt)\langle W_k(t), W_\ell(t)\rangle)$. Since we choose normal magnetic fields W_i so that $\{W_2(r), \dots, W_{2n}(r)\}$ is orthonormal, we have $W(r)$ is the identity matrix. Thus we obtain

$$\frac{1}{\alpha_\kappa(r,u)} \frac{\partial}{\partial t} \alpha_\kappa(t,u)\Big|_{t=r} = \frac{1}{2}\mathrm{trace}(W'(r)) = \mathrm{trace}\big(\langle(\nabla_{\dot\gamma_u} W_i)(r), W_j(r)\rangle\big),$$

and get the conclusion. □

5. Estimates on density functions of area elements

We now give estimates on the density function $\alpha_\kappa(r,u)$ of the area-element of a trajectory-sphere for $\mathbb{B}_\kappa$ on a Kähler manifold M. First we give an estimate of the density function from below. We set

$$\beta_\kappa(r;c,n) = \mathfrak{s}_\kappa(r;c)\mu_\kappa(r;c)\big\{\min\{\mathfrak{s}_\kappa(r;c)\mu_\kappa(r;c), \mathfrak{s}_\kappa(r;4c)\}\big\}^{2(n-1)}.$$

It satisfies $\beta_\kappa(r;c,n) > \mathfrak{s}_\kappa(r;c)\mu_\kappa(r;c)(\mathfrak{s}_{2\kappa}(r;4c))^{2(n-1)}$ for $0 < t < \pi/\sqrt{\kappa^2 + c}$. If we compare this with the density function $\alpha_\kappa(r;c,n) = \mu_\kappa(r;c)(\mathfrak{s}_\kappa(r;c))^{2n-1}$ of trajectory-spheres in a complex space form $\mathbb{C}M^n(c)$, we have

$$\begin{cases} \alpha_\kappa(r;c,n) > \beta_\kappa(r;c,n), & \text{when } c > 0, \\ \alpha_\kappa(r;0,n) = \beta_\kappa(r;0,n), & \text{when } c = 0, \\ \alpha_\kappa(r;4c,n) > \beta_\kappa(r;c,n) > \alpha_\kappa(r;c,n), & \text{when } c < 0, \end{cases}$$

for $0 < r \leq \pi/\sqrt{\kappa^2 + c}$. When we study some geometric objects, it is natural to compare them with those in a standard Riemannian manifold. In our case, standard Kähler manifolds are complex space forms. But, unfortunately, our comparison theorems on magnetic Jacobi fields need assumptions on sectional curvatures. We therefore use $\beta_\kappa(r;c,n)$ to estimate density functions of area elements. We also note that in a real space form $\mathbb{R}M^m(c)$ of constant sectional curvature c, which is one of a standard sphere S^m, a Euclidean space $\mathbb{R}^m$ and a real hyperbolic space $\mathbb{R}H^m$, the area element of a geodesic sphere of radius r is given by $(\mathfrak{s}(r;c))^{m-1}d\omega$.

Theorem 3 *Let M be a Kähler manifold of complex dimension n, and $u \in U_pM$ be an arbitrary unit tangent vector at an arbitrary point $p \in M$. If sectional curvatures satisfy* $\max\{\mathrm{Riem}(v, \dot\gamma_u(t)) \mid v \in U_{\gamma_u(t)}M,\ v \perp \dot\gamma_u(t)\} \leq c$ *with some constant c for $0 \leq t < \pi/\sqrt{\kappa^2 + c}$, then we have the following properties.*

(1) $\alpha_\kappa(t,u) \geq \beta_\kappa(t;c,n)$ *for* $0 < t \leq \pi/\sqrt{\kappa^2 + c}$.
(2) *If $\alpha_\kappa(t_0,u) = \beta_\kappa(t_0;c,n)$ holds at some t_0 with $0 < t_0 < \pi/\sqrt{\kappa^2 + c}$, then $c \geq 0$ and on the interval $[0,t_0]$ we have $\alpha_\kappa(t,u) = \beta_\kappa(t;c,n)$ and $R(v,\dot\gamma_u(t))\dot\gamma_u(t) = cv$ for every $v \in T_{\gamma_u(t)}M$ which is orthogonal to $\dot\gamma_u(t)$.*

Proof. We take an arbitrary r with $0 < r < \pi/\sqrt{\kappa^2 + c}$ and choose normal magnetic Jacobi fields $W_2, \ldots, W_{2n} \in \mathcal{J}_{\gamma_u}$ so that $W_2(r)^\top = W_2(r)$ and that $W_2(r), \ldots, W_{2n}(r)$ are orthonormal. By Lemma 3 and Theorem 1, we have

$$\begin{aligned} \frac{\partial}{\partial t} \log \alpha_\kappa(t,u)\Big|_{t=r} &= \sum_{j=2}^{2n} \big\langle (\nabla_{\dot\gamma_u} W_j)(r), W_j(r) \big\rangle \\ &\geq \sum_{j=2}^{2n} \big\| W_j^\top(r) \big\|^2 \mathfrak{t}_\kappa(r;c)\nu_\kappa(r;c) + \sum_{j=2}^{2n} \big\| W_j^\perp(r) \big\|^2 \mathfrak{t}_\kappa(r;4c). \end{aligned}$$

By putting $a = \sum_{j=3}^{2n} \|W_j^\top(r)\|^2$ and $b = \sum_{j=3}^{2n} \|W_j^\perp(r)\|^2$ (> 0), we define a function $\epsilon(t)$ by $\epsilon(t) = \{\mathfrak{s}_\kappa(t;c)\mu_\kappa(t;c)\}^{1+a}\mathfrak{s}_\kappa(t;c)^b$. Then the above inequality is rewritten as

$$\frac{\partial}{\partial t}\log \alpha_\kappa(t,u)\Big|_{t=r} \geq \frac{d}{dt}\log \epsilon(t)\Big|_{t=r}.$$

Thus we find that the function $\alpha_\kappa(t,u)/\epsilon(t)$ is monotone increasing. As we have $\lim_{t\downarrow 0}\alpha_\kappa(t,u)/\epsilon(t) = 1$ and $\epsilon(t) \geq \beta_\kappa(t;c,n)$ because $a+b = 2(n-1)$, we get $\alpha_\kappa(t,u) \geq \beta_\kappa(t;c,n)$.

We now suppose $\alpha_\kappa(t_0,u) = \beta_\kappa(t_0;c,n)$. We take the above normal magnetic Jacobi field $W_2,\ldots,W_{2n}$ for $r = t_0$. Theorem 1 and the above first inequality show that $R(W_j^\sharp,\dot\gamma_u)\dot\gamma_u = cW_j^\sharp$ for $0 < t \leq t_0$. Since $\{W_2(t),\ldots,W_{2n}(t)\}$ is a basis of $(T_{\gamma(t)}M)^\sharp = \{v \in T_{\gamma(t)}M \mid v \perp \dot\gamma(t)\}$, we have $R(v,\dot\gamma_u(t))\dot\gamma_u(t) = cv$ for all $v \in (T_{\gamma(t)}M)^\sharp$ and $0 < t \leq t_0$. Also the inequality $\epsilon(t) \geq \beta_\kappa(t;c,n)$ and the properties of $\mathfrak{s}_\kappa(r;c), \mu_\kappa(r;c)$ show that $c \geq 0$. □

Given a point $p \in M$ on a Kähler manifold and a constant κ, we set $\iota_\kappa(p) = \sup\{r > 0 \mid \mathbb{B}_\kappa \exp_\kappa|_{B_r(o_p)}$ is a diffeomorphism$\}$ and call it the $\mathbb{B}_\kappa$-injectivity radius at p. We put $\iota_\kappa(M) = \inf\{\iota_\kappa(p) \mid p \in M\}$. When M is simply connected and its sectional curvatures satisfy $\mathrm{Riem}_M \leq c < 0$, it is known that $\iota_\kappa(M) = \infty$ for κ with $|\kappa| \leq \sqrt{|c|}$ (see [4]). If $0 < r < \iota_\kappa(p)$, the area of a trajectory-sphere $S_p^\kappa(r)$ in M is given by $\int_{U_pM}\alpha_\kappa(r,u)dS(u)$. In particular, in $\mathbb{C}M^n(c)$ it is given by $\alpha_\kappa(r;c,n)\omega_{2n}$ with the volume ω_{2n} of the unit sphere in a Euclidean space $\mathbb{R}^{2n}$. As a consequence of Theorem 3 we have the following.

Corollary 1 *Let M be a Kähler manifold of complex dimension n and whose sectional curvatures satisfy* $\mathrm{Riem}_M \leq c$ *for some constant c. At an arbitrary point p, the area of a trajectory-sphere is estimated from below as area$(S_p^\kappa(r)) > \beta_\kappa(r;c,n)\omega_{2n}$ if $0 < r < \iota_\kappa(p)$.*

Next we give an estimate of the density function from above. We set

$$\delta_\kappa(r;c,n) = \mathfrak{s}_\kappa(r;c)\mu_\kappa(r;c)\big\{\max\{\mathfrak{s}_\kappa(r;c)\mu_\kappa(r;c),\ \mathfrak{s}_\kappa(r;4c)\}\big\}^{2(n-1)}.$$

This satisfies

$$\begin{cases} \alpha_\kappa(r;c,n) > \delta_\kappa(r;c,n) > \alpha_\kappa(r;4c,n), & \text{when } c>0, \\ \alpha_\kappa(r;0,n) = \delta_\kappa(r;0,n), & \text{when } c=0, \\ \delta_\kappa(r;c,n) > \alpha_\kappa(r;c,n), & \text{when } c<0, \end{cases}$$

for $0 < r \le \pi/\sqrt{\kappa^2+c}$.

Theorem 4 *Let M be a Kähler manifold of complex dimension n, and $u \in U_pM$ be an arbitrary unit tangent vector at an arbitrary point $p \in M$. If sectional curvatures satisfy* $\min\{\mathrm{Riem}(v, \dot\gamma_u(t)) \mid v \in U_{\gamma_u(t)}M,\ v \perp \dot\gamma_u(t)\} \ge c$ *with some constant c for $0 \le t < \mathsf{c}_{\gamma_u}(\gamma_u(0))$, then we have the following properties.*

(1) *$\alpha_\kappa(t,u) \le \delta_\kappa(t;c,n)$ for $0 < t \le \mathsf{c}_{\gamma_u}(\gamma_u(0))$.*
(2) *If $\alpha_\kappa(t_0,u) = \delta_\kappa(t_0;c,n)$ holds at some t_0 with $0 < t_0 < \mathsf{c}_{\gamma_u}(\gamma_u(0))$, then $c \le 0$ and on the interval $[0,t_0]$ we have $\alpha_\kappa(t,u) \equiv \delta_\kappa(t;c,n)$ and $R(v,\dot\gamma(t))\dot\gamma(t) = cv$ for all $v \in T_{\gamma(t)}M$ which are orthogonal to $\dot\gamma(t)$.*

Proof. We take an arbitrary r with $0 < r < \mathsf{c}_{\gamma_u}(\gamma_u(0))$, and choose normal magnetic Jacobi fields $W_2, \dots, W_{2n} \in \mathcal{J}_{\gamma_u}$ so that $W_2(r)^\top = W_2(r)$ and $W_2(r), \dots, W_{2n}(r)$ are orthonormal. By Lemma 3 and Theorem 2, we have

$$\begin{aligned} \frac{\partial}{\partial t}\log\alpha_\kappa(t;u)\Big|_{t=r} &= \sum_{j=2}^{2n}\big\langle(\nabla_{\dot\gamma_u}W_j)(r), W_j(r)\big\rangle \\ &\le \sum_{j=2}^{2n}\big\|W_j^\top(r)\big\|^2\mathfrak{t}_\kappa(r;c)\nu_\kappa(r;c) + \sum_{j=3}^{2n}\big\|W_j^\perp(r)\big\|^2\mathfrak{t}_\kappa(r;4c) \\ &\le \frac{d}{dt}\log\epsilon(t)\Big|_{t=r}, \end{aligned}$$

where $\epsilon(t) = \{\mathfrak{s}_\kappa(t;c)\mu_\kappa(t;c)\}^{1+a}\mathfrak{s}_\kappa(t;c)^b$ with $a = \sum_{j=3}^{2n}\|W_j^\top(r)\|^2$ and $b = \sum_{j=3}^{2n}\|W_j^\perp(r)\|^2$ (>0). Since we have $\epsilon(t) \le \delta_\kappa(t;c,n)$, along the same lines as in the proof of Theorem 3 we get the conclusion. □

Corollary 2 *Let M be a Kähler manifold of complex dimension n and whose sectional curvatures satisfy* $\mathrm{Riem}_M \ge c$ *for some constant c. At an arbitrary point p, the area of a trajectory-sphere is estimated from above as $area(S_p^\kappa(r)) \le \beta_\kappa(r;c,n)\omega_{2n}$ if $0 < r < \inf\{\mathsf{c}_{\gamma_u} \mid u \in U_pM\}$.*

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