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# A normal family of operator monotone functions

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**Abstract.** We show that the family of all operator monotone functions $f$ on $(-1,1)$ such that $f(0)=0$ and $f'(0)=1$ is a normal family and investigate some properties of odd operator monotone functions. We also characterize the odd operator monotone functions and even operator convex functions on $(-1,1)$. As a consequence, we show that if $f$ is an odd operator monotone function on $(-1,1)$, then $f$ is concave on $(-1,0)$ and convex on $(0,1)$.

*Key words*: Operator monotone function, operator convex function, normal family, integral representation.

## 1. Introduction

Throughout the paper all operators are considered to be in the algebra $\mathbb{B}(\mathscr{H})$ of all bounded linear operators acting on a complex Hilbert space $\mathscr{H}$.

A continuous real valued function $f$ defined on an interval $J$ is called operator monotone if $A \geq B$ implies $f(A) \geq f(B)$ for all self adjoint operators $A, B$ with spectra in $J$. Some structure theorems on operator monotone functions can be found in [4], [9], [5], [8]. A continues function $f$ is called operator convex on $J$ if $f(\alpha A + (1-\alpha)B) \leq \alpha f(A) + (1-\alpha)f(B)$ for all $0 \leq \alpha \leq 1$ and all self adjoint operators $A$ and $B$ with spectra in $J$, see [1], [5], [8], [7] and references therein for several characterizations of the operator convexity. The Löwner theorem says that a function $f$ is operator monotone on an interval $J$ if and only if $f$ has an analytic continuation (denoted by the same $f$) to the upper half plan $\Pi_+$ such that $f$ maps $\Pi_+$ into itself. It is shown [10, Lemma 2.1] that a differentiable function $f$ on an interval $J$ is operator convex if and only if there exists a point $t_0 \in J$ such that the function

$$g(t) = \begin{cases} \dfrac{f(t)-f(t_0)}{t-t_0} & \text{if } t \neq t_0 \\ f'(t_0) & \text{if } t = t_0 \end{cases} \tag{1.1}$$

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is operator monotone on $J$.

If $f(t)$ is an operator monotone function on $(a,b)$, then clearly $f((2t-a-b)/(b-a))$ is operator monotone on $(-1,1)$, so in this paper we study the family of operator monotone functions on $(-1,1)$.

Let $\mathcal{K}$ denote the family of all operator monotone functions on $(-1,1)$ such that $f(0)=0$ and $f'(0)=1$. Hansen and Pedersen [6] showed that $\mathcal{K}$ is a compact convex subset of the space of all functions on $(-1,1)$ with pointwise convergence topology and that the extreme points of $\mathcal{K}$ are of the form $f_\lambda(t)=t/(1-\lambda t)$ with $|\lambda|<1$. They [6] also proved that every $f\in\mathcal{K}$ can be represented as

$$f(t)=\int_{-1}^{1}\frac{t}{1-\lambda t}d\mu(\lambda), \tag{1.2}$$

where $\mu$ is a probability measure on $[-1,1]$, see also [2].

Let $\Omega$ be a open subset of $\mathbb{C}$. A set $\mathcal{F}\subseteq C(\Omega)$ is said to be bounded if for each compact subset $K\subseteq\Omega$, $\sup\{\|f\|_K : f\in\mathcal{F}\}<\infty$. The Montel theorem states that if $\mathcal{F}$ is a bounded subset of the set $A(\Omega)$ of all analytic functions on $\Omega$, then $\mathcal{F}$ is a normal family, i.e, each sequence $\{f_n\}$ in $\mathcal{F}$ has a subsequence $\{f_{n_j}\}$ converging uniformly on each compact subset of $\Omega$.

In this note we show that the family of all operator monotone functions $f$ on $(-1,1)$ such that $f(0)=0$ and $f'(0)=1$ is a normal family and investigate some properties of odd operator monotone functions on the interval $(-1,1)$. We also present the odd operator monotone functions and even operator convex functions on $(-1,1)$ by suitable integrals.

## 2. The results

Throughout this section, let $\Omega=\Pi_+\bigcup\Pi_-\bigcup(-1,1)$, where $\Pi_-$ is the lower half plan.

**Theorem 2.1** *The family $\mathcal{K}$ is bounded in $A(\Omega)$, so it is a normal family.*

*Proof.* Let $S$ be the convex hull of $\{f_\lambda : |\lambda|<1\}$ where $f_\lambda(t)=t/(1-\lambda t)$. By Krein–Millman's theorem, $\mathcal{K}$ is the closed convex hull of it's extreme points, so $\overline{S}=\mathcal{K}$. Fix $K\subseteq\Omega$ as a compact set. Then $h(\lambda,z)=|1-\lambda z|$ is continuous on $[-1,1]\times K$ and so it takes its minimum value. It should be noticed that the minimum value $m$ of $h$ on $[-1,1]\times K$ is nonzero. Put $M_K:=\sup\{|z| : z\in K\}$. Then

$$|f_\lambda(z)| = \frac{|z|}{|1-\lambda z|} \le \frac{M_K}{m}$$

for $(\lambda, z) \in [-1,1] \times K$. If $g = \sum_{i=1}^n c_i f_{\lambda_i} \in \mathcal{S}$, then

$$|g(z)| = \left|\sum_{i=1}^n c_i f_{\lambda_i}(z)\right| \le \sum_{i=1}^n c_i |f_{\lambda_i}(z)| \le \sum_{i=1}^n c_i \frac{M_k}{m} = \frac{M_k}{m},$$

whence $\|g\|_K \le M_K/m$. Now assume that $g \in \mathcal{K}$ is arbitrary. There exists $\{f_n\}$ in $\mathcal{S}$ such that $f_n(t) \to g(t)$ for each $t \in (-1,1)$. Since $S$ is bounded, the sequence $\{f_n\}$ is bounded. By Montel's theorem there exists a subsequence $\{f_{n_j}\}$ converging to a function $h$ in uniform compact convergence topology on $\Omega$. Since $g = h$ on $(-1,1)$, we have $g(z) = h(z)$ for all $z \in \Omega$. Hence

$$|g(z)| = |h(z)| = \lim_{n_j \to \infty} |f_{n_j}(z)| \le \frac{M_K}{m}.$$

Therefore $\mathcal{K}$ is a normal family. □

Let $\mathcal{G}$ denote the family of all operator convex function on $(-1,1)$ that $f(0) = f'(0) = 0$ and $f''(0) = 1$. The next theorem shows that $\mathcal{G}$ is a normal family.

**Proposition 2.2** *Let $f \in \mathcal{K}$ and $f(-1,1) \subseteq (-1,1)$. Then $f(t) = t$ for each $t \in (-1,1)$.*

*Proof.* Since $f(-1,1) \subseteq (-1,1)$, so $f^n = f \circ f \circ \cdots \circ f \in \mathcal{K}$. Hence by Theorem 2.1, $f^n$ has a convergent subsequence that converges to a function $h \in \mathcal{K}$. Assume that $f(t_0) < t_0$ for some $t_0 \in (-1,1)$. Hence $\{f^n(t_0)\}$ is a decreasing sequence converging to $h(t_0)$. Thus

$$h(f(t_0)) = \lim_{n\to\infty} f^n(f(t_0)) = \lim_{n\to\infty} f^{n+1}(t_0) = h(t_0)$$

Since $h$ is one-one, we infer that $f(t_0) = t_0$, which is a contradiction. Therefore we have $f(t_0) \ge t_0$. We similarly get $f(t_0) \le t_0$. Thus $f(t_0) = t_0$. □

**Remark 2.3**

( i ) In Proposition 2.2 the condition "$f$ is operator monotone" is indispensable. Indeed, we have a counterexample: $f(t) = (2/\pi)\sin((\pi/2)t)$

is real analytic and increasing on $(-1,1)$ with $f(0) = 0, f'(0) = 1, |f(t)| < 1$, but $f(t) \neq t$.

(ii) We can prove Proposition 2.2 directly as follows. It follows from

$$f(t) = \int_{-1}^{1} \frac{t}{1-\lambda t} d\mu(\lambda)$$

that

$$-1 < \int_{-1}^{1} \frac{t}{1-\lambda t} d\mu(\lambda) < 1 \quad (-1 < t < 1).$$

Since for each $\lambda$ the integrand $t/(1-\lambda t)$ is positive and increasing on $0 < t < 1$, by Lebesgue's monotone convergence theorem

$$\int_{-1}^{1} \frac{1}{1-\lambda} d\mu(\lambda) = \lim_{t \to 1-0} \int_{-1}^{1} \frac{t}{1-\lambda t} \leq 1.$$

Similarly we have

$$\int_{-1}^{1} \frac{-1}{1+\lambda} d\mu(\lambda) = \lim_{t \to -1+0} \int_{-1}^{1} \frac{t}{1-\lambda t} \geq -1.$$

Thus we have

$$\begin{aligned} \int_{-1}^{1} \frac{1}{1-\lambda^2} d\mu(\lambda) &= \frac{1}{2} \int_{-1}^{1} \left( \frac{1}{1-\lambda} + \frac{1}{1+\lambda} \right) d\mu(\lambda) \\ &\leq 1 = \int_{-1}^{1} 1 d\mu(\lambda). \end{aligned}$$

From this it follows that $1/(1-\lambda^2) = 1$ almost everywhere with respect to $\mu$, Thus $\mu\{0\} = 1$, which implies $f(t) = t$.

**Corollary 2.4** *If $f$ is an odd operator monotone function on $(-1,1)$, then $f(|t|) \geq f'(0)|t|$. Hence $f(|A|) \geq f'(0)|A|$ for $A$ with $\|A\| < 1$*

*Proof.* If $f(t_0) < f'(0)t_0$ for some $t_0 \in (0,1)$, then $f_1(t) = (1/(f'(0)t_0))f(t_0 t) \in \mathcal{K}$ and $f_1(-1,1) \subseteq (-1,1)$, so, by Proposition 2.2, we have $f_1(1) = 1$, which is a contradiction. Hence $f(|t|) \geq f'(0)|t|$

for all $t \in (-1, 1)$. It now follows from the functional calculus that $f(|A|) \geq f'(0)|A|$ for $A$ with $\|A\| < 1$. □

In the sequel we need the following lemma.

**Lemma 2.5** ([2, Lemma 2.4]) *If $f$ is an operator monotone function on an interval $(a, b)$, then $f^{(2p+1)}(t) \geq 0$ for all $p = 0, 1, 2, \dots$ and all $a < t < b$.*

**Theorem 2.6** *Let $f$ be an odd operator monotone function on $(-1, 1)$. Then $f$ is concave on $(-1, 0)$ and convex on $(0, 1)$.*

*Proof.* Without loss of generality we may assume that $f \in \mathcal{K}$. We shall show that $f$ is convex on $(0, 1)$. The proof of Lemma 4.1 of [6] shows that $f'(t) \geq f(t)^2/t^2$. It follows from Corollary 2.4 that $f'(t) \geq 1$ for each $t \in (0, 1)$. Therefore

$$f''(0) = \lim_{t \to 0+} \frac{f'(t) - f'(0)}{t} = \lim_{t \to 0+} \frac{f'(t) - 1}{t} \geq 0.$$

By Lemma 2.5, $f^{(3)}(t) \geq 0$ for all $t \in (-1, 1)$, so $f''(t) \geq 0$ for all $t \in (0, 1)$ since $f''$ is monotone. Hence $f$ is a convex function on (0,1). Since $f$ is an odd function, $f$ is concave on $(-1, 0)$. □

**Example 2.7** The function $f(t) = \tan t$ is well-known as an odd operator monotone function on $(-\pi/2, \pi/2)$. It is actually convex on $(0, \pi/2)$ and concave on $(-\pi/2, 0)$. It follows from Theorem 2.6 that $\sin t$ is not operator monotone on any open interval including $t = 0$, that is a new fact.

**Theorem 2.8** *An odd operator monotone function on $(-1, 1)$ is of the form*

$$f(t) = f'(0) \int_{-1}^{1} \frac{t}{1 - (\lambda t)^2} d\mu(\lambda), \tag{2.1}$$

*where $\mu$ is a probability measure on $[-1, 1]$.*

*Proof.* As before, we may assume that $f \in \mathcal{K}$. The function $f$ can be represented as a power series $f(t) = \sum_{n=1}^{\infty} a_n t^n$, which is convergent for $|t| < 1$, cf. [2]. Since $f$ is odd, $a_{2n} = 0$ for all $n$. It follows from (1.2) that

$$f(t) = \int_{-1}^{1} \frac{t}{1-\lambda t} d\mu(\lambda) = \int_{-1}^{1} \sum_{n=0}^{\infty} t(\lambda t)^n \, d\mu(\lambda) = \sum_{n=0}^{\infty} t^{n+1} \int_{-1}^{1} \lambda^n d\mu(\lambda),$$

in which $\mu$ is a probability measure on $[-1, 1]$. Therefore $a_{2n} = \int_{-1}^{1} \lambda^{2n-1} d\mu(\lambda) = 0$ and so

$$f(t) = \int_{-1}^{1} \sum_{n=0}^{\infty} t(\lambda t)^{2n} \, d\mu(\lambda) = \int_{-1}^{1} \frac{t}{1-(\lambda t)^2} d\mu(\lambda).$$

If $f$ is of the form (2.1), then it is trivially odd. In addition,

$$f(t) = \int_{-1}^{1} \frac{t}{1-(\lambda t)^2} d\mu(\lambda) = \frac{1}{2} \int_{-1}^{1} \frac{t}{1-\lambda t} + \frac{t}{1+\lambda t} d\mu(\lambda) = \frac{1}{2}(g(t) - g(-t)),$$

where $g(t) = \int_{-1}^{1} (t/(1-\lambda t)) d\mu(\lambda)$. Hence $f$ is an odd operator monotone function on $(-1, 1)$. □

**Corollary 2.9** *Any even operator convex function $f$ on $(-1, 1)$ is of the form*

$$f(t) = f(0) + \frac{f''(0)}{2} \int_{-1}^{1} \frac{t^2}{1-(\lambda t)^2} d\mu(\lambda),$$

*where $\mu$ is a probability measure on $[-1, 1]$.*

*Proof.* By (1.1) the function $g(t) = (f(t) - f(0))/t$ is an odd operator monotone function. Now apply Theorem 2.8. □

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