Hokkaido Mathematical Journal Vol. 42 (2013) p. 397–416

The Lie algebra of rooted planar trees

Tomohiko Ishida and Nariya Kawazumi

(Received November 7, 2011; Revised August 30, 2012)

Abstract. We study a natural Lie algebra structure on the free vector space generated by all rooted planar trees as the associated Lie algebra of the nonsymmetric operad (non-Σ operad, preoperad) of rooted planar trees. We determine whether the Lie algebra and some related Lie algebras are finitely generated or not, and prove that a natural surjection called the augmentation homomorphism onto the Lie algebra of polynomial vector fields on the line has no splitting preserving the units.

Key words: nonsymmetric operad, polynomial vector field.

1. Introduction

The Lie algebra of polynomial vector fields on the line, $W_1 = \mathbb{Q}[x](d/dx)$, and its Lie subalgebras $L_0 = x\mathbb{Q}[x](d/dx)$ and $L_1 = x^2\mathbb{Q}[x](d/dx)$ have been studied in the context of Gel'fand-Fuks theory. In particular, Goncharova [2] computed the cohomology group $H^*(L_1)$ completely. Based on her monumental work, various studies including [1], [12] and [14] have been developed. See also [3], [5] and [6]. On the other hand, Kuno and the second author [7] discovered a Lie algebra structure on the free $\mathbb{Q}$-vector space generated by the set of all linear chord diagrams, $\mathcal{LC}$, and a surjective homomorphism $\kappa : \mathcal{LC} \to L_0$. The Lie algebra $\mathcal{LC}$ is purely combinatorial and comes from the derivation Lie algebra of the tensor algebra of a symplectic vector space. So it seems to have no relation with Gel'fand-Fuks theory.

The link between the linear chord diagrams and the vector fields on the line is the notion of a nonsymmetric operad, or equivalently a non-Σ operad or a preoperad. Kapranov and Manin [4] introduced a Lie algebra $\Lambda(\mathcal{P})$ associated to a nonsymmetric operad of $\mathbb{Q}$-vector spaces $\mathcal{P}$. To understand the homomorphism κ, we introduce the augmentation homomorphism of the Lie algebra induced from a nonsymmetric operad of sets. We denote $\mathcal{P} = \mathbb{Q}\mathcal{C}$, if $\mathcal{P}((m))$, $m \geq 0$, is the free $\mathbb{Q}$-vector space of $\mathcal{C}((m))$ for a

---

2000 Mathematics Subject Classification : Primary 18D50; Secondary 57R32.

nonsymmetric operad of sets $\mathcal{C}$. The augmentation maps $\mathbb{Q}\mathcal{C}((m)) \to \mathbb{Q}$ induce a natural homomorphism of Lie algebras $\varepsilon : \Lambda(\mathbb{Q}\mathcal{C}) \to W_1$, which we call *the augmentation homomorphism*. The Lie algebra $\mathcal{LC}$ is regarded as the Lie algebra induced from an operad of sets, and the homomorphism $\kappa : \mathcal{LC} \to L_0$ is derived from the augmentation homomorphism.

In this paper we study two fundamental problems for some nonsymmetric operad of sets $\mathcal{C}$;

(i) Is the Lie algebra $\Lambda(\mathbb{Q}\mathcal{C})$ finitely generated?
(ii) Does the augmentation homomorphism have a splitting preserving the units $1 \in \mathcal{C}((1))$ and $x(d/dx) \in L_0$?

As typical examples of nonsymmetric operads of sets, we have the nonsymmetric operad of rooted planar trees $\underline{\mathsf{Tree}}$ and its nonsymmetric suboperad of binary planar trees $\underline{\mathsf{Tree}}_2$. We prove both of the questions for both of the nonsymmetric operads have negative answers (Theorems 5.1, 6.5 and 7.1). In order to prove Theorem 7.1, we introduce the nonsymmetric operad of partitions $\underline{\mathsf{Par}}$ and its nonsymmetric suboperad $\underline{\mathsf{Par}}_2$ of binary partitions. The answers of (i) and (ii) for $\underline{\mathsf{Par}}$ and that of (ii) for $\underline{\mathsf{Par}}_2$ are negative (Theorems 6.4 and 7.1), while that of (i) for $\underline{\mathsf{Par}}_2$ is affirmative (Theorem 6.1). Here it should be remarked Loday and Ronco [9] have already studied algebraic structures on binary rooted planar trees in a different way from ours. The answer of the question (i) for the Lie algebra $\mathcal{LC}$ is negative [7], while that of (ii) is still open.

In this paper we work over the rationals $\mathbb{Q}$, but all the results hold true over any field of characteristic zero. An operad without assuming the symmetric group action has various names; a non-Σ operad [11], a preoperad [8], an asymmetric operad, and a nonsymmetric operad [10]. For details, see [8]. As will be shown in this paper, the notion of an operad without assuming the symmetric group action is quite fundamental. In this paper we adopt *a nonsymmetric operad* following [10].

2. The Lie algebra associated to a nonsymmetric operad

We begin by recalling the definition of a *nonsymmetric operad* or a *non-Σ operad* or a *preoperad* of $\mathbb{Q}$-vector spaces.

Definition 2.1 A sequence of $\mathbb{Q}$-vector spaces $\mathcal{P} = \{\mathcal{P}((m))\}_{m\geq 0}$ is a nonsymmetric operad of $\mathbb{Q}$-vector spaces, if it admits an element $1 \in \mathcal{P}((1))$

called *the unit* and $\mathbb{Q}$-linear maps called *the composition maps*

$$\gamma = \gamma^{\mathcal{P}} : \mathcal{P}((k)) \otimes \mathcal{P}((j_1)) \otimes \cdots \otimes \mathcal{P}((j_k)) \to \mathcal{P}\left(\left(\sum_{s=1}^{k} j_s\right)\right), \quad k \geq 1,\, j_s \geq 0,$$

which satisfy the following two conditions.

(1) (Associativity) For any $c \in \mathcal{P}((k))$, $d_s \in \mathcal{P}((j_s))$, $1 \leq s \leq k$, and $e_t \in \mathcal{P}((i_t))$, $1 \leq t \leq j = \sum_{s=1}^{k} j_s$, we have

$$\gamma(\gamma(c \otimes d_1 \otimes \cdots \otimes d_k) \otimes e_1 \otimes \cdots \otimes e_j) = \gamma(c \otimes f_1 \otimes \cdots \otimes f_k),$$

where $f_s = \gamma(d_s \otimes e_{j_1+\cdots+j_{s-1}+1} \otimes \cdots \otimes e_{j_1+\cdots+j_{s-1}+j_s})$.

(2) (Unit) We have $\gamma(1 \otimes d) = d$ and $\gamma(c \otimes 1^{\otimes k}) = c$ for any $d \in \mathcal{P}((j))$ and $c \in \mathcal{P}((k))$, $k \geq 1$.

As usual, we denote

$$c \circ_s d_s := \gamma\big(c \otimes 1^{\otimes(s-1)} \otimes d_s \otimes 1^{\otimes(k-s)}\big) \in \mathcal{P}((k + j_s - 1)), \quad 1 \leq s \leq k.$$

A nonsymmetric operad of sets $\mathcal{C}$ is defined in a similar way. For any $c \in \mathcal{C}((k))$, $d_s \in \mathcal{C}((j_s))$, $1 \leq s \leq k$, we denote the composition by $\gamma(c; d_1, \ldots, d_s) \in \mathcal{C}((\sum_{s=1}^{k} j_s))$. Then we denote by $\mathbb{Q}\mathcal{C}$ the nonsymmetric operad of $\mathbb{Q}$-vector spaces defined by

$$(\mathbb{Q}\mathcal{C})((m)) := \mathbb{Q}(\mathcal{C}((m))),$$

the free $\mathbb{Q}$-vector space generated by the set $\mathcal{C}((m))$, $m \geq 0$.

For any $\mathbb{Q}$-vector space V, the endomorphism operad $\mathcal{E}_V$ is defined by

$$\mathcal{E}_V(m) := \mathrm{Hom}(V^{\otimes m}, V)$$

with the obvious unit and composition maps. The augmentation maps of the free $\mathbb{Q}$-vector space $\mathbb{Q}\mathcal{C}((m))$, $\varepsilon : \mathbb{Q}\mathcal{C}((m)) \to \mathbb{Q} = \mathrm{Hom}(\mathbb{Q}^{\otimes m}, \mathbb{Q}) = \mathcal{E}_{\mathbb{Q}}(m)$, $\sum_{x \in \mathcal{C}((m))} a_x x \mapsto \sum_{x \in \mathcal{C}((m))} a_x$, define a homomorphism of nonsymmetric operads of $\mathbb{Q}$-vector spaces

$$\varepsilon : \mathbb{Q}\mathcal{C} \to \mathcal{E}_{\mathbb{Q}},$$

which we call the augmentation homomorphism.

Kapranov and Manin [4] define two Lie algebras associated to an operad of $\mathbb{Q}$-vector spaces. One requires the symmetric group action, but the other denoted by

$$\Lambda(\mathcal{P}) := \bigoplus_{m=0}^{\infty} \mathcal{P}((m))$$

can be defined for any nonsymmetric operad of $\mathbb{Q}$-vector spaces, $\mathcal{P}$. See also [10, 5.3.16 and 5.8.17]. The Lie bracket $[c,d]$, $c \in \mathcal{P}((k))$, $d \in \mathcal{P}((j))$, is defined by

$$[c,d] := \sum_{t=1}^{j} d \circ_t c - \sum_{s=1}^{k} c \circ_s d \in \mathcal{P}((k+j-1)).$$

Here it should be remarked our sign convention is different from that in [4], in order to make the bijection $\Lambda(\mathcal{E}_{\mathbb{Q}}) \xrightarrow{\cong} W_1 := \mathbb{Q}[x](d/dx)$ stated below an isomorphism of Lie algebras.

To check the Jacobi identity of $\Lambda(\mathcal{P})$, we write simply

$$c(d) := \sum_{t=1}^{j} d \circ_t c.$$

Then the map

$$\delta : \Lambda(\mathcal{P}) \to \mathrm{End}(\Lambda(\mathcal{P})), \quad c \mapsto (\delta_c : d \mapsto c(d))$$

is injective since $\delta_c(1) = c$. One computes $\delta_{[c,d]} = [\delta_c, \delta_d] \in \mathrm{End}(\Lambda(\mathcal{P}))$. $\Lambda(\mathcal{P})$ inherits the Jacobi identity from the Lie algebra $\mathrm{End}(\Lambda(\mathcal{P}))$ by the injection δ. Here we remark the Lie algebra $\Lambda(\mathcal{P})$ has a finer structure, a *pre-Lie algebra*. For details, see [4, 1.7] and [10, 5.8.17].

For a finite dimensional $\mathbb{Q}$-vector space V, we have a natural isomorphism of Lie algebras onto the derivation Lie algebra of $T(V^*)$

$$\Lambda(\mathcal{E}_V) = \mathrm{Der}(T(V^*)), \tag{2.1}$$

where $T(V^*) = \bigoplus_{m=0}^{\infty}(V^*)^{\otimes m}$ is the tensor algebra of the dual space $V^* =$

$\mathrm{Hom}(V, \mathbb{Q})$. In order to describe the isomorphism (2.1) explicitly for the case $V = \mathbb{Q}$, we denote the element corresponding to $1 \in \mathbb{Q} = \mathrm{Hom}(\mathbb{Q}^{\otimes m}, \mathbb{Q})$ by $1_m \in \mathcal{E}_{\mathbb{Q}}(m)$, $m \geq 0$. Then we have

$$[1_m, 1_n] = (n - m)1_{m+n-1}.$$

This means the map given by

$$1_m \in \Lambda(\mathcal{E}_{\mathbb{Q}}) \mapsto x^m \frac{d}{dx} \in W_1$$

is an isomorphism onto the Lie algebra of polynomial vector fields on the line, $W_1 = \mathbb{Q}[x](d/dx)$, For the rest of this paper we identify $\Lambda(\mathcal{E}_{\mathbb{Q}}) = W_1$ through this isomorphism.

Thus, for any nonsymmetric operad of sets $\mathcal{C}$, the augmentation homomorphism $\varepsilon : \mathbb{Q}\mathcal{C} \to \mathcal{E}_{\mathbb{Q}}$ induces a natural homomorphism of Lie algebras

$$\varepsilon : \Lambda(\mathbb{Q}\mathcal{C}) \to \Lambda(\mathcal{E}_{\mathbb{Q}}) = W_1,$$

which we call also *the augmentation homomorphism.* If $\mathcal{C}((m)) \neq \emptyset$ for each $m \geq 0$, it is surjective. It is natural to ask whether it does split or not. The answer to this question should describe the complexity of the given nonsymmetric operad $\mathcal{C}$.

As usual, we denote $L_k := x^{k+1}\mathbb{Q}[x](d/dx)$ for any $k \geq -1$, which is a Lie subalgebra of W_1. Similarly we denote

$$\Lambda_k(\mathcal{P}) := \bigoplus_{m=k+1}^{\infty} \mathcal{P}((m)),$$

which is also a Lie subalgebra of $\Lambda(\mathcal{P})$. The augmentation homomorphism induces a homomorphism of Lie algebras

$$\varepsilon : \Lambda_k(\mathbb{Q}\mathcal{C}) \to L_k$$

for each $k \geq -1$.

We denote by $e_0 = {e_0}^{\mathcal{P}} \in \Lambda(\mathcal{P})$ the unit $1 \in \mathcal{P}((1))$ regarded as an element of the Lie algebra $\Lambda(\mathcal{P})$. When $\mathcal{P} = \mathcal{E}_{\mathbb{Q}}$, we have ${e_0}^{\mathcal{E}_{\mathbb{Q}}} = x(d/dx) \in W_1$. For any $m \geq 0$, the subspace $\mathcal{P}((m)) \subset \Lambda(\mathcal{P})$ is exactly the $(m-1)$-eigenspace of the adjoint action of the unit, $\mathrm{ad}e_0$. Hence, for any nonsymmetric operads of $\mathbb{Q}$-vector spaces $\mathcal{P}$ and $\mathcal{P}'$, if a homomorphism of Lie algebras $\varphi : \Lambda(\mathcal{P}) \to \Lambda(\mathcal{P}')$, which is not necessarily the induced homomorphism of a homomorphism of nonsymmetric operads, preserves the units $\varphi({e_0}^{\mathcal{P}}) = {e_0}^{\mathcal{P}'}$, then we have $\varphi(\mathcal{P}((m))) \subset \mathcal{P}'((m))$ for any $m \geq 0$.

In view of the action of e_0 we find out the center $Z(\Lambda(\mathcal{P}))$ satisfies

$$Z(\Lambda(\mathcal{P})) \subset Z(\Lambda_0(\mathcal{P})) \subset Z(\mathcal{P}((1))). \tag{2.2}$$

Here we regard $\mathcal{P}((1))$ as a Lie subalgebra of $\Lambda(\mathcal{P})$. The standard chain complex $C_*(\Lambda(\mathcal{P}))$ of the Lie algebra $\Lambda_k(\mathcal{P})$, $k \geq -1$, is decomposed into the eigenspaces of the adjoint action $\mathrm{ad} e_0$. The l-eigenspace of $\mathrm{ad} e_0$, $C_*(\Lambda_k(\mathcal{P}))_{(l)}$ is a subcomplex of $C_*(\Lambda_k(\mathcal{P}))$. We denote

$$H_*(\Lambda_k(\mathcal{P}))_{(l)} := H_*(C_*(\Lambda_k(\mathcal{P}))_{(l)}).$$

Clearly we have

$$H_*(\Lambda_k(\mathcal{P})) = \bigoplus_{l=k}^{\infty} H_*(\Lambda_k(\mathcal{P}))_{(l)}.$$

The formula $\mathrm{ad} e_0 = d \circ (e_0 \wedge) + (e_0 \wedge) \circ d$ on the standard chain complex implies

$$H_*(\Lambda_k(\mathcal{P})) = H_*(\Lambda_k(\mathcal{P}))_{(0)}$$

for $k = -1$ or 0. In particular, if a nonsymmetric operad of sets $\mathcal{C}$ satisfies the condition $\sharp\mathcal{C}((0)) = \sharp\mathcal{C}((1)) = \sharp\mathcal{C}((2)) = 1$, then we have $C_*(\Lambda(\mathbb{Q}\mathcal{C}))_{(0)} = C_*(W_1)_{(0)} = C_*(sl_2(\mathbb{Q}))_{(0)}$, so that

$$H_*(\Lambda(\mathbb{Q}\mathcal{C})) = H_*(W_1) = H_*(sl_2(\mathbb{Q})) = \begin{cases} \mathbb{Q}, & \text{if } * = 0, 3, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Similarly, if $\sharp\mathcal{C}((1)) = 1$, then

$$H_*(\Lambda_0(\mathbb{Q}\mathcal{C})) = H_*(L_0) = \begin{cases} \mathbb{Q}, & \text{if } * = 0, 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

We conclude this section by a comment on the Lie algebra of linear chord diagrams, $\mathcal{LC}$, introduced by Kuno and the second author [7]. We denote the free $\mathbb{Q}$-vector space generated by the set of linear chord diagrams of m chords, $m \geq 1$, by $\mathsf{lcd}(2m-1)$, while we define $\mathsf{lcd}(2m) = 0$. The j-th amalgamation of two linear chord diagrams C and C', $C *_j C'$, defined in [7], gives a composition map on $\mathsf{lcd} = \{\mathsf{lcd}(n)\}_{n\geq 0}$. In a similar way to [7], we can prove lcd is an anticyclic operad. The Lie algebra $\mathcal{LC}$ is exactly $\Lambda(\mathsf{lcd})$. What we have stated in this section is a straight-forward generalization of some of observations in [7]. As a nonsymmetric operad, we have $\mathsf{lcd} = \mathbb{Q}\overline{\mathsf{lcd}}$, where $\overline{\mathsf{lcd}}$ is the operad of sets consisting of all linear chord diagrams. The homomorphism $\kappa : \mathcal{LC} \to L_0$ in [7] is the composite of the augmentation homomorphism and the homomorphism

$$x\mathbb{Q}[x^2]\frac{d}{dx} \to L_0 = x\mathbb{Q}[x]\frac{d}{dx}, \quad x^{2n+1}\frac{d}{dx} \mapsto 2x^{n+1}\frac{d}{dx}.$$

For an anticyclic operad $\mathcal{P}$, we denote by $\Lambda^+(\mathcal{P})$ the cyclic invariants in $\Lambda(\mathcal{P})$. One can prove $\Lambda^+(\mathcal{P})$ is a Lie subalgebra of $\Lambda(\mathcal{P})$. If $\mathcal{P} = \mathsf{lcd}$, the Lie algebra $\Lambda^+(\mathcal{P})$ is exactly the Lie algebra of (circular) chord diagrams $\mathcal{C}$ introduced in [7].

3. The nonsymmetric operad of rooted planar trees

We recall the definition of the nonsymmetric operad of rooted planar trees, $\underline{\mathsf{Tree}}$, following Markl, Shnider and Stasheff [11, I.1.5]. Let $\underline{\mathsf{Tree}}((m))$ be the set of planar trees with 1 root at the bottom and m leaves at the top, regarded as labeled from left to right; 1 through m. For $S \in \underline{\mathsf{Tree}}((m))$, $T \in \underline{\mathsf{Tree}}((n))$ and $1 \leq i \leq m$, $S \circ_i T$ is defined to be the tree obtained by grafting the root of T to the i-th leaf of S. This operation makes the sequence $\underline{\mathsf{Tree}} := \{\underline{\mathsf{Tree}}((m))\}_{m\geq 1}$ a nonsymmetric operad of sets, which we call *the nonsymmetric operad of rooted planar trees.* It is known $\underline{\mathsf{Tree}}$ is a free nonsymmetric operad. See [10, 5.8.6] and [11, II.1.9].

For $n \geq 2$, we denote by $\underline{\mathsf{Tree}}_n((m))$ the subset of $\underline{\mathsf{Tree}}((m))$ consisting of trees all of whose vertices are of valency $\leq n+1$. The sequence $\underline{\mathsf{Tree}}_n := \{\underline{\mathsf{Tree}}_n((m))\}_{m\geq 1}$ is a nonsymmetric suboperad of $\underline{\mathsf{Tree}}$. We call it *the nonsymmetric operad of n-ary rooted planar trees.* As is known, each element of the set $\underline{\mathsf{Tree}}((m))$ corresponds to a meaningful way of inserting one set of parentheses into the word $12\cdots m$, that is, a cell of the Stasheff associahedron K_m [13]. For example, $\underline{\mathsf{Tree}}((1)) = \{1\}$, $\underline{\mathsf{Tree}}((2)) = \{(12)\}$, and $\underline{\mathsf{Tree}}((3)) = \{((12)3), (1(23)), (123)\}$. The set $\underline{\mathsf{Tree}}_2((m))$ corresponds exactly to the vertices of the associahedron K_m. From (2.4) we have

$$H_*(\Lambda(\mathbb{Q}\underline{\mathsf{Tree}})) = H_*(\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}_n)) = H_*(L_0).$$

Now we introduce an enhancement of the nonsymmetric operad $\underline{\mathsf{Tree}}$. To do this, we consider the i-th face $\partial_i c$ of $c \in \underline{\mathsf{Tree}}((m))$ for $m \geq 2$ and $1 \leq i \leq m$, defined by erasing the i-th leaf of c. For example, $\partial_i((12)3) = \partial_i(1(23)) = (12)$, $\partial_i((1(23))4) = ((12)3)$ for $1 \leq i \leq 3$, and $\partial_4((1(23))4) = (1(23))$. Let $\underline{\mathsf{Tree}}^-((0))$ be a singleton, whose unique element we denote by ∇. We define $\underline{\mathsf{Tree}}^-((m)) := \underline{\mathsf{Tree}}((m))$ for $m \geq 1$, $\partial_1 1 := \nabla$, and $c \circ_i \nabla := \partial_i c$ for $c \in \underline{\mathsf{Tree}}((m))$, $m \geq 1$ and $1 \leq i \leq m$. Then $\underline{\mathsf{Tree}}^- := \{\underline{\mathsf{Tree}}^-((m))\}_{m \geq 0}$ forms a nonsymmetric operad of sets. For $n \geq 2$, the sequence $\underline{\mathsf{Tree}}_n^- := \{\underline{\mathsf{Tree}}_n^-((m))\}_{m\geq 0}$, given by $\underline{\mathsf{Tree}}_n^-((0)) = \underline{\mathsf{Tree}}^-((0))$ and $\underline{\mathsf{Tree}}_n^-((m)) = \underline{\mathsf{Tree}}_n((m))$ for $m \geq 1$, is a nonsymmetric suboperad of $\underline{\mathsf{Tree}}^-$. From (2.3) we have

$$H_*(\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}^-)) = H_*(\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}_n^-)) = H_*(W_1).$$

Clearly we have $\partial_i \partial_j c = \partial_{j-1}\partial_i c$ if $i < j$. Hence the linear map

$$\partial := \sum_{i=1}^{m} \partial_i : \mathbb{Q}\underline{\mathsf{Tree}}^-((m)) \to \mathbb{Q}\underline{\mathsf{Tree}}^-((m-1))$$

satisfies $\partial\partial = 0$, so that $\mathbb{Q}\underline{\mathsf{Tree}}^-((*)) = \{\mathbb{Q}\underline{\mathsf{Tree}}^-((m)), \partial\}_{m\geq 0}$ is a chain complex, and $\mathbb{Q}\underline{\mathsf{Tree}}_n^-((*)) = \{\mathbb{Q}\underline{\mathsf{Tree}}_n^-((m)), \partial\}_{m\geq 0}$ a subcomplex. Consider the tree $(12) \in \underline{\mathsf{Tree}}_2^-((2))$. Then we have $\partial((12)\circ_2 c) = c - (12)\circ_2 \partial c$ for any $c \in \underline{\mathsf{Tree}}^-((m))$, $m \geq 0$. This implies the vanishing of the homology groups

$$H_*(\mathbb{Q}\underline{\mathsf{Tree}}^-((*))) = H_*(\mathbb{Q}\underline{\mathsf{Tree}}_n^-((*))) = 0.$$

4. The nonsymmetric operad of partitions

In this section we introduce the nonsymmetric operad of partitions, $\underline{\mathsf{Par}}$.

Let $\mathbb{Q}[x_i; i \geq 1]$ be the rational polynomial ring in infinitely many indeterminates $\{x_i\}_{i\geq 1}$. A monomial $\prod_{i=1}^{N} {x_i}^{a_i}$ with $N \geq 2$, $a_i \geq 1$

$(1 \le \forall i \le N)$, corresponds to the nontrivial order-preserving partition of the set $\{1, 2, \dots, m\}$ with $m = \sum_{i=1}^{N} a_i$ given by

$$\{1, 2, \dots, m\} = \coprod_{i=1}^{N} \Big\{ \sum_{j=1}^{i-1} a_j + 1, \sum_{j=1}^{i-1} a_j + 2, \dots, \sum_{j=1}^{i-1} a_j + a_i \Big\}.$$

We define

$$\underline{\mathsf{Par}}((m)) := \Big\{ \prod_{i=1}^{N} {x_i}^{a_i}; N \ge 2,\ a_i \ge 1\ (1 \le \forall i \le N),\ \text{and} \sum a_i = m \Big\},$$

which is regarded as the set of nontrivial order-preserving partitions of the set $\{1, 2, \dots, m\}$ for $m \ge 2$, and $\underline{\mathsf{Par}}((1))$ to be a singleton, whose unique element we denote by 1. The composition map is defined by

$$\gamma\Big(\prod_{i=1}^{N} {x_i}^{a_i}; \prod_{k=1}^{N_1} {x_k}^{a_{1k}}, \dots, \prod_{k=1}^{N_m} {x_k}^{a_{mk}} \Big) := \prod_{i=1}^{N} {x_i}^{b_i},$$

where

$$b_i = \sum_{j=a_1+\cdots+a_{i-1}+1}^{a_1+\cdots+a_{i-1}+a_i} \Big(\sum_{k=1}^{N_j} a_{jk} \Big).$$

In other words, we define

$$\Big(\prod_{i=1}^{N} {x_i}^{a_i} \Big) \circ_s \Big(\prod_{k=1}^{N_s} {x_k}^{a_{sk}} \Big) := {x_l}^{a_l - 1 + \sum_{k=1}^{N_s} a_{sk}} \prod_{i \ne l} {x_i}^{a_i}$$

if $a_1 + \cdots + a_{l-1} + 1 \le s \le a_1 + \cdots + a_{l-1} + a_l$. For the unit 1, we define

$$\Big(\prod_{i=1}^{N} {x_i}^{a_i} \Big) \circ_j 1 = 1 \circ_1 \Big(\prod_{i=1}^{N} {x_i}^{a_i} \Big) := \prod_{i=1}^{N} {x_i}^{a_i}.$$

It is easy to check this composition satisfies the axiom of associativity. The reason why we defined $\underline{\mathsf{Par}}((1))$ as above is that there does not exist a unit in the polynomial ring $\mathbb{Q}[x_i; i \ge 1]$. In fact, $\big(\prod_{i=1}^{N} {x_i}^{a_i} \big) \circ_j x_1 = \prod_{i=1}^{N} {x_i}^{a_i}$, but

$x_1 \circ_1 \left(\prod_{i=1}^N {x_i}^{a_i}\right) = {x_1}^{\sum a_i}$. Then $\underline{\mathsf{Par}} := \{\underline{\mathsf{Par}}((m))\}_{m\geq 1}$ forms a nonsymmetric operad of sets, which we call *the nonsymmetric operad of partitions.* For $n \geq 2$, we denote $\underline{\mathsf{Par}}_n((1)) := \underline{\mathsf{Par}}((1)) = \{1\}$ and

$$\underline{\mathsf{Par}}_n((m)) := \underline{\mathsf{Par}}((m)) \cap \mathbb{Q}[x_1, x_2, \dots, x_n]$$

for $m \geq 2$. Then $\underline{\mathsf{Par}}_n := \{\underline{\mathsf{Par}}_n((m))\}_{m\geq 1}$ is a nonsymmetric suboperad of $\underline{\mathsf{Par}}$. We call it *the nonsymmetric operad of n-ary partitions.*

The reason why we introduce the nonsymmetric operad $\underline{\mathsf{Par}}$ is to simplify the Lie algebra $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}})$ by using the following homomorphism $\nu : \underline{\mathsf{Tree}} \to \underline{\mathsf{Par}}$.

Let c be a rooted planar tree in $\underline{\mathsf{Par}}((m))$, $m \geq 2$. Look at the nearest vertex to the root. Each edge except the one attached to the root has the set of leaves sitting above itself. Hence the tree c gives a nontrivial order-preserving partition of the set of leaves $\{1, 2, \dots, m\}$, which we denote by $\nu(c) \in \underline{\mathsf{Par}}((m))$. For example, $\nu((1(23))4) = {x_1}^3 x_2$, $\nu((12)(34)) = {x_1}^2 {x_2}^2$. Further we define $\nu(1) := 1 \in \underline{\mathsf{Par}}((1))$. Then the maps $\nu : \underline{\mathsf{Tree}}((m)) \to \underline{\mathsf{Par}}((m))$, $m \geq 1$, form a homomorphism of nonsymmetric operads

$$\nu : \underline{\mathsf{Tree}} \to \underline{\mathsf{Par}}$$

from the definition of the composition maps in $\underline{\mathsf{Par}}$. Clearly it induces a homomorphism of nonsymmetric operads

$$\nu : \underline{\mathsf{Tree}}_n \to \underline{\mathsf{Par}}_n$$

for each $n \geq 2$.

To compute the Lie bracket on $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})$, we regard the polynomial ring $\mathbb{Q}[x_i; i \geq 1]$ as an L_0-module by the diagonal action. More precisely, $\xi(x)(d/dx) \in L_0$ acts on $f(x_1, x_2, \dots) \in \mathbb{Q}[x_i; i \geq 1]$ by

$$\left(\xi(x)\frac{d}{dx}\right)(f(x_1, x_2, \dots)) = \sum_{i=1}^{\infty} \xi(x_i)\frac{\partial}{\partial x_i} f(x_1, x_2, \dots).$$

Then it is easy to prove the following.

Lemma 4.1 *For $c, d \in \Lambda_1(\mathbb{Q}\underline{\mathsf{Par}}) \subset \mathbb{Q}[x_i; i \geq 1]$ we have*

$$[c, d] = \varepsilon(c)(d) - \varepsilon(d)(c) \in \Lambda_1(\mathbb{Q}\underline{\mathsf{Par}}) \subset \mathbb{Q}[x_i; i \geq 1].$$

As a corollary, we obtain

Corollary 4.2 *The kernel of the augmentation homomorphism* ε : $\Lambda(\mathbb{Q}\underline{\mathsf{Par}}) = \Lambda_0(\mathbb{Q}\underline{\mathsf{Par}}) \to \Lambda_0(\mathcal{E}_{\mathbb{Q}}) = L_0$ *is abelian.*

The Lie bracket on $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})$ extends to the Laurent polynomial ring in infinitely many indeterminates $\mathbb{Q}[{x_i}^{\pm 1}; i \geq 1]$, and makes it a Lie algebra.

5. The Lie algebra $\boldsymbol{\Lambda}(\mathbb{Q}\underline{\mathsf{Tree}}_2^-)$ is not finitely generated

In this section, we prove the following theorem.

Theorem 5.1 *The Lie algebra* $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}_2^-)$ *is not finitely generated.*

As a preliminary of the proof of Theorem 5.1, we show Lemma 5.2 and Lemma 5.4.

Lemma 5.2 *For any* $m \geq 2$,

$$H_1(\Lambda_1(\mathbb{Q}\underline{\mathsf{Tree}}_2))_{(m)} \neq 0.$$

Proof. The cardinality of $\underline{\mathsf{Tree}}_2((m+1))$ is the m-th Catalan number

$$c(m) = \frac{1}{m+1}\binom{2m}{m},$$

and it coincides with the dimension of $C_1(\Lambda_1\underline{\mathsf{Tree}}_2)_{(m)}$. Let $c'(m)$ denote the dimension of the second chain complex $C_2(\Lambda_1\underline{\mathsf{Tree}}_2)_{(m)}$.

We prove that $c'(m) < c(m)$ for any $m \geq 1$. Since $c'(m)$ can be computed from the equation

$$c'(m) = \begin{cases} \displaystyle\sum_{l=1}^{k} c(l)c(m-l) & (\text{if } m = 2k+1) \\ \displaystyle\sum_{l=1}^{k-1} c(l)c(m-l) + \binom{c(k)}{2} & (\text{if } m = 2k) \end{cases},$$

we have the inequality

$$c'(m) \leq \frac{1}{2}\sum_{l=1}^{m-1} c(l)c(m-l).$$

By the well-known recurrence equation

$$c(m) = \sum_{l=0}^{m-1} c(l)c(m-l-1),$$

we have

$$c'(m) + c(m) \le \frac{1}{2}\sum_{l=0}^{m} c(l)c(m-l) = \frac{1}{2}c(m+1).$$

Since the ratio of consecutive Catalan numbers is described as

$$\frac{c(m+1)}{c(m)} = \frac{2(2m+1)}{m+2},$$

we obtain finally

$$c'(m) \le \frac{m-1}{m+2}c(m) < c(m).$$

Therefore, $H_1(\Lambda_1(\mathbb{Q}\underline{\mathsf{Tree}}_2))_{(m)}$ does not vanish for any $m \ge 1$. □

Corollary 5.3 *The Lie subalgebra $\Lambda_1(\mathbb{Q}\underline{\mathsf{Tree}}_2)$ of $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}_2^-)$ is not finitely generated.*

To prove Theorem 5.1, we define $\mathfrak{h}_m$ to be the Lie subalgebra of $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}_2^-)$ generated by $\bigcup_{j=2}^{m} \underline{\mathsf{Tree}}_2((j))$.

Lemma 5.4 *The vector subspace $\mathbb{Q}\underline{\mathsf{Tree}}_2^-((0)) \oplus \mathbb{Q}\underline{\mathsf{Tree}}_2^-((1)) \oplus \mathfrak{h}_m$ is a Lie subalgebra of $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}_2^-)$.*

Proof. It is obvious that the subspace $\mathbb{Q}\underline{\mathsf{Tree}}_2^-((0)) \oplus \mathbb{Q}\underline{\mathsf{Tree}}_2^-((1))$ is a Lie subalgebra of $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}_2^1)$. Hence it is sufficient to prove that the inclusions

$$(\mathrm{ad}1)(\mathfrak{h}_m),\ (\mathrm{ad}\nabla)(\mathfrak{h}_m) \subset \mathbb{Q}\underline{\mathsf{Tree}}_2^-((1)) \oplus \mathfrak{h}_m$$

hold.

Now we consider an arbitrary Lie algebra $\mathfrak{g}$. For any n elements $u_1, u_2, \ldots, u_n$ of $\mathfrak{g}$ and a binary tree c which belongs to $\underline{\mathsf{Tree}}_2((n))$, we define $f_c(u_1, u_2, \ldots, u_n)$ to be an element of $\mathfrak{g}$ obtained from $u_1, u_2, \ldots, u_n$ by the Lie bracket following the parentheses corresponding to c. For example, $f_{(1)}(u_1) = u_1, f_{((12)(34))}(u_1, u_2, u_3, u_4) = [[u_1, u_2], [u_3, u_4]]$. By the Jacobi's identity, the equation

$$(\mathrm{ad}v)f_c(u_1, u_2, \ldots, u_n) = \sum_{i=1}^{n} f_c(u_1, u_2, \ldots, u_{i-1}, (\mathrm{ad}v)u_i, u_{i+1}, \ldots, u_n) \tag{5.1}$$

holds for any $v \in \mathfrak{g}$.

Under these settings, the Lie algebra $\mathfrak{h}_m$ is the vector subspace spanned by the set

$$\{f_c(u_1, u_2, \ldots, u_n); n \geq 1, c \in \underline{\mathsf{Tree}}_2((n)), u_i \in \underline{\mathsf{Tree}}_2((j_i)), 2 \leq j_i \leq m\}.$$

Hence the assertion holds if we prove that $f_c(u_1, u_2, \ldots, u_{i-1}, (\mathrm{ad}v)u_i, u_{i+1}, \ldots, u_n)$ is in $\mathbb{Q}\underline{\mathsf{Tree}}_2((1)) \oplus \mathfrak{h}_m$ for any u_i's and $v = 1$ and ∇.

Since $(\mathrm{ad}1)(u) = ju$ for any $u \in \underline{\mathsf{Tree}}_2((j))$, the claim holds true for $v = 1$.

In the case $n = 1$, $f_c(u_1) = u_1$ and $(\mathrm{ad}\nabla)(u_1)$ is in $\mathbb{Q}\underline{\mathsf{Tree}}_2((j_1 - 1))$. In the case $n \geq 2$, if $j_i \geq 3$, then $(\mathrm{ad}\nabla)(u_i)$ is in $\mathbb{Q}\underline{\mathsf{Tree}}_2((j_i - 1))$. If $j_i = 2$, then $u_i = (12)$ and $(\mathrm{ad}\nabla)((12)) = 2 \cdot 1$ and

$$\begin{aligned} &f_c(u_1, u_2, \ldots, u_{i-1}, (\mathrm{ad}\nabla)(12), u_{i+1}, \ldots, u_n) \\ &\quad = 2f_c(u_1, u_2, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_n) \\ &\quad = Cf_{\partial_i c}(u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n) \end{aligned}$$

for some integer C. Here $\partial_i c$ is the i-th face of c defined in Section 3. The claim holds true also for $v = \nabla$. □

Proof of Theorem 5.1. Assume that $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}_2^-)$ is finitely generated. Then there exists a sufficiently large $m \geq 2$ so that $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}_2^-)$ is generated by $\bigoplus_{j=0}^{m} \mathbb{Q}\underline{\mathsf{Tree}}_2((j))$. In other words, $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}_2^-)$ has the decomposition

$$\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}_2^-) = \mathbb{Q}\underline{\mathsf{Tree}}_2((0)) \oplus \mathbb{Q}\underline{\mathsf{Tree}}_2((1)) \oplus \mathfrak{h}_m.$$

In particular, if $l > m$, the inclusion

$$\mathfrak{h}_m \cap \mathbb{Q}\underline{\mathsf{Tree}}_2((l)) \subset [\Lambda_1(\mathbb{Q}\underline{\mathsf{Tree}}_2^-), \Lambda_1(\mathbb{Q}\underline{\mathsf{Tree}}_2^-)]$$

holds. This implies that $H_1(\Lambda_1(\mathbb{Q}\underline{\mathsf{Tree}}_2))_{(l-1)} = 0$ and it contradicts Lemma 5.2. This concludes the proof of Theorem 5.1. $\square$

6. The Lie algebra $\Lambda(\mathbb{Q}\underline{\mathsf{Par}}_2)$ is finitely generated

In this section, we prove the following theorem.

Theorem 6.1 *The Lie algebra $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}}_2)$ is generated by $x_1x_2, x_1^2x_2, x_1x_2^2$, and $x_1^3x_2 + x_1x_2^3$.*

Proof. It should be remarked $\dim C_1(\Lambda(\mathbb{Q}\underline{\mathsf{Par}}_2))_{(m-1)} = \dim \mathbb{Q}\underline{\mathsf{Par}}_2((m)) = m-1$. It is obvious that $x_1x_2, x_1^2x_2$ and $x_1x_2^2$ are not in the derived ideal $[\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}}_2), \Lambda_1(\mathbb{Q}\underline{\mathsf{Par}}_2)]$. Next we compute brackets which take values in $\mathbb{Q}\underline{\mathsf{Par}}_2((4))$. We obtain

$$[x_1x_2, x_1^2x_2] = x_1^3x_2 + x_1^2x_2^2 - x_1x_2^3,$$
$$[x_1x_2, x_1x_2^2] = -x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3.$$

Hence $x_1^3x_2 + x_1x_2^3$ is not in the derived ideal.

On the other hand, if $m \geq 5$, any elements of $\mathbb{Q}\underline{\mathsf{Par}}_2((m-1))$ can be obtained by repetition of Lie brackets. In fact, if $m = 5$, then

$$[x_1x_2, x_1^3x_2] = 2x_1^4x_2 + x_1^3x_2^2 - x_1x_2^4,$$
$$[x_1x_2, x_1^2x_2^2] = -x_1^4x_2 + 2x_1^3x_2^2 + 2x_1^2x_2^3 - x_1x_2^4,$$
$$[x_1x_2, x_1x_2^3] = -x_1^4x_2 + x_1^2x_2^3 + 2x_1x_2^4,$$
$$[x_1^2x_2, x_1x_2^2] = -2x_1^4x_2 + x_1^3x_2^2 - x_1^2x_2^3 + 2x_1x_2^4,$$

and thus the boundary map

$$\delta_{1,5} \colon C_2(\Lambda(\mathbb{Q}\underline{\mathsf{Par}}_2))_{(4)} \to C_1(\Lambda(\mathbb{Q}\underline{\mathsf{Par}}_2))_{(4)}$$

can be represented by the matrix

$$A_5 = \begin{pmatrix} 2 & 1 & 0 & -1 \\ -1 & 2 & 2 & -1 \\ -1 & 0 & 1 & 2 \\ -2 & 1 & 1 & 2 \end{pmatrix}.$$

Hence $\det A_5 \neq 0$. Further if $m \geq 6$, then

$$\begin{aligned}
[x_1x_2, x_1^{m-k-1}x_2^k] &= -x_1^{m-1}x_2 + (m-k-1)x_1^{m-k}x_2^k \\
&\qquad + kx_1^{m-k-1}x_2^{k+2} - x_1x_2^{m-1} \quad (\text{for any } 2 \leq k \leq m-3), \\
[x_1x_2, x_1x_2^{m-2}] &= -x_1^{m-1}x_2 + x_1^2x_2^{m-2} + (m-3)x_1x_2^{m-1}, \\
[x_1^2x_2, x_1x_2^{m-3}] &= -2x_1^{m-1}x_2 + x_1^3x_2^{m-3} - x_1^2x_2^{m-2} + (m-3)x_1x_2^{m-1}, \\
[x_1^2x_2, x_1^2x_2^{m-4}] &= -2x_1^{m-1}x_2 + 2x_1^4x_2^{m-4} + (m-5)x_1^2x_2^{m-2}
\end{aligned}$$

and thus a matrix representation A_m of the boundary map

$$\delta_{1,m} \colon C_2(\Lambda(\mathbb{Q}\underline{\mathsf{Par}}_2))_{(m-1)} \to C_1(\Lambda(\mathbb{Q}\underline{\mathsf{Par}}_2))_{(m-1)}$$

has the $(m-1) \times (m-1)$ submatrix

$$A'_m = \begin{pmatrix}
-1 & m-3 & 2 & & & & & & -1 \\
-1 & & m-4 & 3 & & & & O & -1 \\
\vdots & & & \ddots & \ddots & & & & \vdots \\
-1 & & & & 4 & m-5 & & & -1 \\
-1 & & & & & 3 & m-4 & 0 & -1 \\
-1 & & & & & 0 & 2 & m-3 & -1 \\
-1 & O & & & & 0 & 0 & 1 & m-3 \\
-2 & & & & & 0 & 1 & -1 & m-3 \\
-2 & & & & & 2 & 0 & m-5 & 0
\end{pmatrix}.$$

Hence

$$\begin{aligned}
\det A'_m &= \frac{1}{6}(-1)^m(m-3)! \cdot \det \begin{pmatrix}
-1 & 3 & m-4 & 0 & -1 \\
-1 & 0 & 2 & m-3 & -1 \\
-1 & 0 & 0 & 1 & m-3 \\
-2 & 0 & 1 & -1 & m-3 \\
-2 & 2 & 0 & m-5 & 0
\end{pmatrix} \\
&= \frac{1}{6}(-1)^{m+1}(m-3)! \cdot m(m-1)(2m-7) \\
&\neq 0.
\end{aligned}$$

Consequently, the boundary map $\delta_{1,m}$ is surjective if $m \geq 5$ and this concludes the proof of Theorem 6.1. $\square$

Corollary 6.2

$$H_1(\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}}_2);\mathbb{Q}) \cong \mathbb{Q}^4.$$

Corollary 6.3 *The Lie algebra $\Lambda(\mathbb{Q}\underline{\mathsf{Par}}_2)$ is generated by 1, $x_1x_2, x_1^2x_2$, $x_1x_2^2$, and $x_1^3x_2 + x_1x_2^3$.*

In contrast, the following proposition holds.

Proposition 6.4 *The Lie algebra $\Lambda(\mathbb{Q}\underline{\mathsf{Par}})$ is not finitely generated.*

Proof. Assume that $\Lambda(\mathbb{Q}\underline{\mathsf{Par}})$ is finitely generated. Then there exists a sufficiently large $m \geq 2$ so that $\Lambda(\mathbb{Q}\underline{\mathsf{Par}})$ is generated by $\bigoplus_{j=0}^{m} \mathbb{Q}\underline{\mathsf{Par}}((j))$. However, the Lie subalgebra $\Lambda(\mathbb{Q}\underline{\mathsf{Par}}_m)$ of $\Lambda(\mathbb{Q}\underline{\mathsf{Par}})$ contains $\bigoplus_{j=0}^{m} \mathbb{Q}\underline{\mathsf{Par}}((j))$ although it doesn't generate $\Lambda(\mathbb{Q}\underline{\mathsf{Par}})$. Thus we have a contradiction. $\square$

Since $\nu\colon \underline{\mathsf{Tree}} \to \underline{\mathsf{Par}}$ induces a surjective homomorphism of Lie algebras, we directly have the first half of the following corollary. The rest is proved by an argument similar to the proof of Theorem 5.1.

Corollary 6.5 *The Lie algebra $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}})$ is not finitely generated. Furthermore, neither the Lie algebra $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}^-)$ is.*

7. The augmentation homomorphism on $\boldsymbol{\Lambda}(\mathbb{Q}\underline{\mathsf{Tree}})$ has no splitting

Let ε and ε_1 denote the augmentation homomorphism from $\Lambda(\mathbb{Q}\underline{\mathsf{Par}})$ and $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}})$ to L_0, respectively. In this section, we prove the following theorem.

Theorem 7.1 *The augmentation homomorphism $\varepsilon_1\colon \Lambda(\mathbb{Q}\underline{\mathsf{Tree}}) \to L_0$ has no splitting preserving the units.*

In the proof of Theorem 7.1, the nonsymmetric operad $\underline{\mathsf{Par}}$ plays an important role. We denote by $\nu : \Lambda(\mathbb{Q}\underline{\mathsf{Tree}}) \to \Lambda(\mathbb{Q}\underline{\mathsf{Par}})$ the homomorphism of Lie algebras induced by $\nu : \underline{\mathsf{Tree}} \to \underline{\mathsf{Par}}$. Then we have $\varepsilon_1 = \varepsilon \circ \nu$: $\Lambda(\mathbb{Q}\underline{\mathsf{Tree}}) \to \Lambda(\mathbb{Q}\underline{\mathsf{Par}}) \to L_0$. Hence it suffices to prove that $\varepsilon\colon \Lambda(\mathbb{Q}\underline{\mathsf{Par}}) \to L_0$ has no splitting preserving the units. If such a splitting would exist, it must map $\mathbb{Q}x^m(d/dx)$ to $\mathbb{Q}\underline{\mathsf{Par}}((m))$ for each $m \geq 2$. In fact, both of them are the $(m-1)$-eigenspaces of $\mathrm{ad}{e_0}^{\mathcal{E}_{\mathbb{Q}}}$ and $\mathrm{ad}{e_0}^{\mathbb{Q}\underline{\mathsf{Par}}}$, respectively.

Let $\iota\colon \underline{\mathsf{Par}} \to \underline{\mathsf{Par}}$ be the involution defined by

$$\iota(x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}) = x_1^{a_n}x_2^{a_{n-1}}\dots x_n^{a_1}.$$

Then it is obvious that ι induces an automorphism of the Lie algebra $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})$. If we denote by $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{\pm}$ the (± 1)-eigenspace of the involution

$$\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{\pm} = \{u \in \Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{\pm}; \iota(u) = \pm u\},$$

then $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{+}$ is a Lie subalgebra and $[\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{+}, \Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{-}] \subset \Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{-}$. Since the kernel of the augmentation homomorphism ε includes $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{-}$, we have $[\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{-}, \Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{-}] = 0$. Hence $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})$ is the semi-direct product of $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{+}$ and $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{-}$

$$\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}}) = \Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{-} \rtimes \Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{+}. \tag{7.1}$$

Since $\varepsilon(\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{-}) = 0$, we have a factorization

$$\varepsilon\mid_{\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})} = \varepsilon_2 \circ p : \Lambda_1(\mathbb{Q}\underline{\mathsf{Par}}) \xrightarrow{p} \Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{+} \xrightarrow{\varepsilon_2} L_0,$$

where p is the second projection in (7.1) and ε_2 is the restriction of the augmentation homomorphism to $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{+}$. Therefore, in order to establish Theorem 7.1, it suffices to prove the following proposition.

Proposition 7.2 *The augmentation homomorphism $\varepsilon_2\colon \Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^{+} \to L_1$ has no splitting which maps $\mathbb{Q}x^m(d/dx)$ to $\mathbb{Q}\underline{\mathsf{Par}}((m))$ for each $m \geq 2$.*

Proof. We assume that there exists a splitting s which maps $\mathbb{Q}x^m(d/dx)$ to $\mathbb{Q}\underline{\mathsf{Par}}((m))$ for each $m \geq 2$. We denote

$$e_i = x^{i+1}\frac{d}{dx} \in L_1$$

for $i \geq 1$. Recall that L_1 is generated by e_1 and e_2. In fact, e_n is obtained from e_1 and e_2 by

$$e_n = \frac{1}{(n-2)!}(\mathrm{ad}e_1)^{n-2}(e_2),$$

for $n \geq 3$. Therefore the splitting s is uniquely determined by its values of e_1 and e_2. Since $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^+ \cap \mathbb{Q}\underline{\mathsf{Par}}((2))$ is generated by x_1x_2 and $\Lambda_1(\mathbb{Q}\underline{\mathsf{Par}})^+ \cap \mathbb{Q}\underline{\mathsf{Par}}((3))$ by $x_1^2x_2 + x_1x_2^2$ and $x_1x_2x_3$, the value u_1 of e_1 by s must be x_1x_2 and u_2 of e_2 must have the form

$$u_2 = \frac{t}{2}\bigl(x_1^2x_2 + x_1x_2^2\bigr) + (1-t)x_1x_2x_3.$$

If we define u_n by

$$u_n = \frac{1}{(n-2)!}(\mathrm{ad}u_1)^{n-2}(u_2)$$

for $n \geq 3$, then the equation $u_n = s(e_n)$ must hold also for $n \geq 3$. In particular, u_5 must coincide with $[u_2, u_3]$ since $e_5 = [e_2, e_3]$. To prove that $u_5 \neq [u_2, u_3]$, we compute u_3, u_4, and u_5 explicitly. Then we obtain

$$\begin{aligned}
u_3 &= \left[x_1x_2, \frac{t}{2}(x_1^2x_2 + x_1x_2^2) + (1-t)x_1x_2x_3\right] \\
&= tx_1^2x_2^2 - (1-t)\bigl(x_1^3x_2 + x_1x_2^3\bigr) + (1-t)x_1x_2^2x_3 \\
&\quad + (1-t)\bigl(x_1^2x_2x_3 + x_1x_2x_3^2\bigr),
\end{aligned}$$

$$\begin{aligned}
u_4 &= \frac{1}{2}[x_1x_2, u_3] \\
&= \frac{-1+3t}{2}\bigl(x_1^3x_2^2 + x_1^2x_2^3\bigr) + \frac{-4+3t}{2}\bigl(x_1^4x_2 + x_1x_2^4\bigr) \\
&\quad + (1-t)\bigl\{x_1x_2^3x_3 + x_1^2x_2x_3^2 + (x_1^2x_2^2x_3 + x_1x_2^2x_3^2) + (x_1^3x_2x_3 + x_1x_2x_3^3)\bigr\},
\end{aligned}$$

and

$$\begin{aligned}
u_5 &= \frac{1}{6}[x_1x_2, u_4] \\
&= (2t-3)x_1^3x_2^3 + \left(2t - \frac{7}{6}\right)\bigl(x_1^4x_2^2 + x_1^2x_2^4\bigr) + (2t-3)\bigl(x_1^5x_2 + x_1x_2^5\bigr) \\
&\quad + \frac{1-t}{3}\bigl\{(x_1x_2^3x_3^2 + x_1^2x_2^3x_3) + 3x_1x_2^4x_3 + 3x_1^2x_2^2x_3^2 \\
&\qquad + 2(x_1^2x_2^3x_3 + x_1x_2^3x_3^2) + 3(x_1^3x_2x_3^2 + x_1^2x_2x_3^3) \\
&\qquad + 3(x_1^3x_2^2x_3 + x_1x_2^2x_3^3) + 3(x_1^4x_2x_3 + x_1x_2x_3^4)\bigr\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[u_2, u_3] &= \left[\frac{t}{2}(x_1^2x_2 + x_1x_2^2) + (1-t)x_1x_2x_3, u_3\right] \\
&= -2(1-t)x_1^3x_2^3 + \frac{5}{2}t(1-t)\big(x_1^4x_2 + x_1^2x_2^4\big) \\
&\quad + (t^2+t-3)\big(x_1^5x_2 + x_1x_2^5\big) \\
&\quad + (1-t)\big\{x_1x_2^4x_3 + (x_1^2x_2x_3^3 + x_1^3x_2x_3^2) + (x_1^2x_2^3x_3 + x_1x_2^3x_3^2) \\
&\qquad\qquad + (x_1^3x_2^2x_3 + x_1x_2^2x_3^3) + (x_1^4x_2x_3 + x_1x_2x_3^4)\big\}.
\end{aligned}$$

Thus we have $u_5 \neq [u_2, u_3]$, which contradicts s is a homomorphism of Lie algebras. This completes the proof. □

Acknowledgments The authors wish to express their gratitude to Yusuke Kuno for valuable discussions, and to Jean-Louis Loday for careful comments to the first version of this paper. The second-named author also thanks Ralph Kaufmann and Robert Penner for helpful advices. The first-named author is supported by JSPS Research Fellowships for Young Scientists (23·1352). The second-named author is partially supported by the Grant-in-Aid for Scientific Research (A) (No. 20244003) from the Japan Society for Promotion of Sciences.

References

[1] Feigin B. L. and Fuks D. B., *Homology of the Lie algebra of vector fields on the line*. Functional Anal. Appl. **14**(2) (1980), 201–212.

[2] Goncharova L. V., *Cohomology of Lie algebra of formal vector fields on the line*. Functional Anal. Appl. **7**(2) (1973), 6–14.

[3] Ishida T., *Second cohomology classes of the group of C^1-flat diffeomorphisms of the line*. Ann. Inst. Fourier **62**(1) (2012), 77–85.

[4] Kapranov M. and Manin Yu., *Modules and Morita theorem for operads*. Amer. J. Math. **123** (2001), 811–838.

[5] Kawazumi N., *On the complex analytic Gel'fand-Fuks cohomology of open Riemann surfaces*. Ann. Inst. Fourier **43**(2) (1993), 655–712.

[6] Kawazumi N., *An application of the second Riemann continuation theorem to cohomology of the Lie algebra of vector fields on the complex line*. preprint, arXiv:math/0508390 (2005).

[7] Kawazumi N. and Kuno Y., *The center of the Goldman Lie algebra of a surface of infinite genus*, to appear in: Q. J. Math.

[8] Kluge L. and Paal E., *On derivation deviation in an abstract pre-operad.* Comm. Algebra **29** (2001), 1609–1626.

[9] Loday J.-L. and Ronco M., *Hopf algebra of the planar binary trees.* Adv. Math. **139** (1998), 293–309.

[10] Loday J.-L. and Vallette B., Algebraic operads, available at: http://www-irma.u-strasbg.fr/ loday/PAPERS/LodayVallette.pdf

[11] Markl M., Shnider S. and Stasheff J., Operads in algebra, topology and physics, American Mathematical Society, Providence, Rhode Island (2002).

[12] Millionschikov D., *Algebra of formal vector fields on the line and Buchstaber's conjecture.* Funct. Anal. Appl. **43** (2009), 264–278.

[13] Stasheff J., *Homotopy associativity of H spaces, I.* Trans. Amer. Math. Soc. **108** (1963), 275–292.

[14] Vainshtein F. V., *Filtering bases and cohomology of infinite-dimensional Lie algebras and Laplace operators.* Functional Anal. Appl. **19**(4) (1985), 259–269.

Tomohiko Ishida
Graduate School of Mathematical Sciences
University of Tokyo, 3-8-1 Komaba
Meguro-ku, Tokyo 153-8914, Japan
E-mail: ishidat@ms.u-tokyo.ac.jp

Nariya Kawazumi
Department of Mathematical Sciences
University of Tokyo, 3-8-1 Komaba
Meguro-ku, Tokyo 153-8914, Japan
E-mail: kawazumi@ms.u-tokyo.ac.jp