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Continuity of Julia sets and its Hausdorff dimension of $P_c(z) = z^d + c$

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Abstract. Given $d \geq 2$ consider the family of monic polynomials $P_c(z) = z^d + c$, for $c \in \mathbb{C}$. Denote by J_c and $HD(J_c)$ the Julia set of P_c and the Hausdorff dimension of J_c respectively, and let $\mathcal{M}_d = \{c | J_c \text{ is connected}\}$ be the connectedness locus; for $d = 2$ it is called the Mandelbrot set. We study semihyperbolic parameters $c_0 \in \partial\mathcal{M}_d$: those for which the critical point is not recurrent by P_{c_0}, $0 \in J_{c_0}$, and without parabolic cycles. We prove that if $P_{c_n} \to P_{c_0}$ algebraically, then for some $C > 0$,

$$d_H(J_{c_n}, J_{c_0}) \leq C|c_n - c_0|^{1/d},$$

where d_H denotes the Hausdorff distance. If, in addition, $P_{c_n} \to P_{c_0}$ preserving critical relations, then P_{c_n} is semihyperbolic for all $n \gg 0$, and

$$HD(J_{c_n}) \to HD(J_{c_0}).$$

Key words: Julia set, Hausdorff dimension, net, conformal measure.

1. Introduction and main results

Let $R(z)$ be a rational map of degree $d = \deg R \geq 2$ on the complex sphere $\overline{\mathbb{C}}$. The Julia set $J(R)$ of a rational function R is defined to be the closure of all repelling periodic points of R, its complement set is called Fatou set $F(R)$. It is known that $J(R)$ is a perfect set (so $J(R)$ is uncountable, and no point of $J(R)$ is isolated), and also that if $J(R)$ is disconnected, then it has infinitely many components.

Let $\mathcal{C}$ be the set of critical points of a rational map R. Then the set of critical values of R^n is

$$Ctv_n(R) = R(\mathcal{C}) \cup R^2(\mathcal{C}) \cup \cdots R^n(\mathcal{C}).$$

The ω-limit set of the set $Ctv_n(R)$ of critical values of $R : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$ is defined by

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$$\Omega(R) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{k=n}^{\infty} R^k(Ctv_n(R))}.$$

In other words $z \in \Omega(R)$ if and only if there exist $c \in Ctv_n(R)$ and a sequence $n_k \to \infty$ $(k \geq 1)$ of positive integers such that $z = \lim_{k\to\infty} R^{n_k}(c)$.

We call a critical point c of R recurrent if $c \in \Omega(R)$; otherwise c is called non-recurrent, denoted by NCP maps.

In this paper we consider the NCP maps $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ called semihyperbolic maps: those for which the critical points are not recurrent by R and without parabolic cycles.

We say rational maps R_n converge to R *algebraically* if $\deg R_n = \deg R$ and, when R_n is expressed as the quotient of two polynomials, the coefficients can be chosen to converge to those of R. Equivalently, $R_n \to R$ uniformly in the spherical metric.

Given that $R_n \to R$ algebraically. Let $b \in J(R)$ be a preperiodic critical point, satisfying $R^i(b) = R^j(b)$ for some $i > j > 0$. Suppose for all such b and for all $n \gg 0$, the maps R_n have critical points $b_n \in J(R_n)$ with the same muliplicity as b, $b_n \to b$ and $R_n^i(b_n) = R_n^j(b_n)$. Then we say $R_n \to R$ *preserving critical relations.*

In this paper we study dynamics of polynomials $P_c = z^d + c$, $d \geq 2$, such that the critical point 0 is not recurrent and $0 \in J_c$. These polynomials are *semihyperbolic* in the sense of [1].

HD denotes the Hausdorff dimension; $n \gg 0$ means for all n sufficiently large. We have the following main theorem:

Main Theorem *Let $c_0 \in \partial\mathcal{M}_d$ be such that P_{c_0} is semihyperbolic. If $P_{c_n} \to P_{c_0}$ algebraically, then for some $C > 0$,*

$$d_H(J_{c_n}, J_{c_0}) \leq C|c_n - c_0|^{1/d},$$

where d_H denotes the Hausdorff distance.

If, in addition, $P_{c_n} \to P_{c_0}$ preserving critical relations, then P_{c_n} is semihyperbolic for all $n \gg 0$, and

$$HD(J_{c_n}) \to HD(J_{c_0}).$$

2. Preliminaries and the construction of a net

Let X be a connected complex manifold. A *holomorphic family of rational maps*, parameterized by X, is a holomorphic map $R : X \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. We denote this map by $R_\lambda(z)$, where $\lambda \in X$ and $z \in \overline{\mathbb{C}}$; then $R_\lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map.

Let x be a basepoint in X. A *holomorphic motion* of a set $E \subset \overline{\mathbb{C}}$ parameterized by (X, x) is a family of injections

$$\phi_\lambda : E \to \overline{\mathbb{C}},$$

one for each λ in X, such that $\phi_\lambda(e)$ is a holomorphic function of λ for each fixed e, and $\phi_x = id$.

Given a holomorphic family of rational maps R_λ, we say the corresponding Julia sets $J(R_\lambda) \subset \overline{\mathbb{C}}$ *move holomorphically* if there is a holomorphic motion

$$\phi_\lambda : J(R_x) \to \overline{\mathbb{C}}$$

such that $\phi_\lambda(J(R_x)) = J(R_\lambda)$ and

$$\phi_\lambda \circ R_x(z) = R_\lambda \circ \phi_\lambda(z)$$

for all z in $J(R_x)$. Thus ϕ_λ provides a conjugacy between R_x and R_λ on their respective Julia sets. The motion ϕ_λ is unique if it exists, by density of periodic cycles in $J(R_x)$.

The Julia sets move holomorphically *at x* if they move holomorphically on some neighborhood U of x in X.

A periodic point z of R_x of period n is *persistently indifferent* if there is a neighborhood U of x and a holomorphic map $\mathcal{W} : U \to \overline{\mathbb{C}}$ such that $\mathcal{W}(x) = z$, $R^n_\lambda(\mathcal{W}(\lambda)) = \mathcal{W}(\lambda)$, and $|(R^n_\lambda)'(\mathcal{W}(\lambda))| = 1$ for all λ in U. (Here $(R^n_\lambda)'(z) = dR^n_\lambda/dz$.)

Lemma 2.1 ([2], Characterizations of stability) *Let R_λ be a holomorphic family of rational maps parameterized by X, and let x be a point in X. Then the following conditions are equivalent*:

1. *The number of attracting cycles of R_λ is locally constant at x.*
2. *The maximum period of an attracting cycle of R_λ is locally bounded at x.*

3. *The Julia set moves holomorphically at x.*
4. *For all y sufficiently close to x, every periodic point of R_y is attracting, repelling or persistently indifferent.*
5. *The Julia set J_λ depends continuously on λ (in the Hausdorff topology) on a neighborhood of x.*
   Suppose in addition that $c_i : X \to \overline{\mathbb{C}}$, are holomorphic maps parameterizing the critical points of R_λ. Then the following conditions are also equivalent to those above:
6. *For each i, the function $\lambda \mapsto R^n_\lambda(c_i(\lambda))$, $n = 0, 1, 2, \ldots$ form a normal family at x.*
7. *There is a neighborhood U of x such that for all λ in U, $c_i(\lambda) \in J_\lambda$ if and only if $c_i(x) \in J_x$.*

The definition of *conformal measures* for rational maps was first given by Sullivan as a modification of the Patterson measures for limit sets of Fuchsian groups. A more general definition, showing the connection to ergodic theory, has been given by M. Denker and M. Urbański earlier. Let $t \geq 0$, a probability measure m on $J(R)$ is called t-conformal for $R : J(R) \to J(R)$ if $m(J(R)) = 1$ and

$$m(R(A)) = \int_A |R'|^t dm$$

for every Borel set $A \subset J(R)$ such that $R|_A$ is injective.

Let R be an NCP map. Denote by $\Lambda(R)$ the set of all parabolic periodic points of R (these points belong to the Julia set and have an essential influence on its fractal structure), and $Crit(R)$ of all critical points of R. We put

$$Crit(J(R)) = Ctit(R) \cap J(R).$$

Set

$$Sing(R) = \bigcup_{n \geq 0} R^{-n}(\Lambda(R) \cup Crit(J(R))).$$

Definition 2.1 We define the *conical set* $Con(R)$ of R as follow. First, say x belongs to $Con(R, r)$ if for any $\epsilon > 0$, there is a neighborhood U of x and $n > 0$ such that $diam(U) < \varepsilon$ and

$$R^n : U \to B(R^n(x), r)$$

is a homeomorphism. Then set

$$Con(R) = \bigcup_{r>0} Con(R, r).$$

We have $x \in Con(R)$ if and only if arbitrary small neighborhood of x can be blow up univalently by the dynamics to balls of definite size centered at $R^n(x)$.

Lemma 2.2 ([3]) *If* $R : J(R) \to J(R)$ *is an NCP map, then*

$$Con(R) = J(R) \setminus Sing(R).$$

Note that Curtis T. McMullen used the term *radial Julia set* $J_{rad}(R)$ instead of *conical set* $Con(R)$ in analogy with Kleinian groups; see ref. [4].

By paper [4], we have the set $Sing(R)$ is countable.

Let $0 < \lambda < 1$. Then there exist an integer $m \geq 1$, $C > 0$, an open topological disk U containing no critical values of R up to order m and analytic inverse branches $R_i^{-mn} : U \to \overline{\mathbb{C}}$ of $R^{mn}(i = 1, \ldots, k_n \leq d^{nm}, n \geq 0)$, satisfying:

(1) $\forall n \geq 0$, $\forall 1 \leq i \leq k_{n+1}$, $\exists 1 \leq j \leq k_n$, $R^m \circ R_i^{-m(n+1)} = R_j^{-mn}$,
(2) $diam(R_i^{-mn}(U)) \leq c\lambda^n$ for $n = 0, 1, \ldots$ and $i = 1, \ldots, k_n$,
(3) for each fixed $n \geq 1$, for all $i = 1, \ldots, k_n$ the sets $\overline{R_i^{-mn}(U)}$ are pairwise disjoint and $\overline{R_i^{-mn}(U)} \subset U$.

By Definition 2.1 and Lemma 2.2, the conical set $J_c(R)$ is a hyperbolic set. Now we state as a lemma the following consequence of (1)–(3).

Lemma 2.3 *Let* $R(z)$ *be a semihyperbolic map. For each* n, *let* $\mathcal{N}_n = \bigcup\{R_j^{-n}(U) : j = 1, \ldots, k_n\}$ *and let* $\mathcal{N} = \bigcup \mathcal{N}_n$. *Then* $\mathcal{N}$ *is a net of* $Con(R)$, *i.e. any two sets in* $\mathcal{N}$ *are either disjoint or one is a subset of the other.*

Consider the net $\mathcal{N}$, given by Lemma 2.3. For $n \geq 0$, the preimages of the sets $\mathcal{N}_i$ under R^n that intersect $J(R)$ are called the *nth step pieces* of the net. Note that for $n \geq 1$ the collection of all the nth step pieces also is a net; we call it a *refinement* of the net $\mathcal{N}$.

Lemma 2.4 *Let W be an nth step piece of the net $\mathcal{N}_i$, then the inverse of*

$$P_{c_0}^n : W \to \mathcal{N}_i = P_{c_0}^n(W)$$

extends in a injective way to a neighborhood of $\overline{\mathcal{N}_i}$, only depending on i.

Proof. Refining the net if necessary, we will prove that for some $m \geq 1$ all the mth step pieces (or some of the mth step pieces) of the net are compactly contained in some $\mathcal{N}_i$. Then the net formed by the mth step pieces will be the desired net. Thus it is enough to prove that the diameters of the mth step pieces of the net converge uniformly to zero as $m \to \infty$.

Let $\varepsilon > 0$, and $N \geq 1$ be such that we can partition each $\mathcal{N}_i$ in at most N connected sets of diameter less than $\varepsilon > 0$. If necessary we can refine the disks $\mathcal{N}_i$ small enough, then P_{c_0} is injective in each cover of the net. Let W be an mth step piece of the net, so that $P_{c_0}^m$ is injective in W. Then by the property (2) of net we have $diam(W) \to 0$ as $m \to \infty$. The proof of this lemma is complete. □

As in immediate consequence, together with the Koebe Distortion Theorem, we obtain the Bounded Distortion Property.

Lemma 2.5 (Bounded Distortion Property) *For any $k \geq 0$ the distortion of $P_{c_0}^k$ in each of the kth step pieces of the net is bounded by some constant $K > 1$, independent of k.*

3. Proof of the main Result

Proof of the Main Theorem.

Step 1: Since P_{c_0} is a semihyperbolic map, it has no Siegel disks and Herman rings. For each $x \in F(P_{c_0}) = \overline{\mathbf{C}} - J_{c_0}$ (the Fatou set of P_{c_0}), under iteration $P_{c_0}^i(x)$ converges to an attracting or super-attracting fixed-point c of P_{c_0}. Then this behavior persists under algebraic perturbation of P_{c_0}. In fact there is a small neighborhood U of c such that $P_{c_n}(U) \subset U$ for all $n \gg 1$. Thus $U \subset F(P_{c_n})$, and we have shown a neighborhood of c persists in the Fatou set for large n. Therefore the multiplier of an attracting cycle of a semihyperbolic map P_λ is constant as λ varies small, and hence the number of repelling cycles of P_λ is constant in the neighborhood of λ. Thus the repelling periodic points of sufficiently high period move holomorphically and without collision as λ varies small. Since the repelling points are dense in the Julia set, the Julia set moves holomorphically by the λ-lemma ([2, Theorem 4.1]). It follows by Lemma 2.1 (Characterizations of stability) that the Julia set moves holomorphically at c_0, and there is a unique holomorphic motion

$$\phi_{c_n} : J_{c_0} \to \overline{\mathbb{C}}$$

such that $\phi_{c_n}(J_{c_0}) = J_{c_n}$ and

$$\phi_{c_n} \circ P_{c_0} = P_{c_n} \circ \phi_{c_n}(z) \tag{3.1}$$

for all z in J_{c_0}.

Since the holomorphic motion ϕ_{c_n} is a holomorphic function of c_n in a neighborhood of c_0, and $\phi_{c_0} = id$. We have

$$|\phi_{c_n}(z) - z| = |\phi_{c_n}(z) - \phi_{c_0}(z)|$$

for all z in J_{c_0}. By item 5 in Lemma 2.1, the Julia set J_{c_n} depends continuously on c_n (in the Hausdorff topology) on a neighborhood of c_0. So we have

$$|\phi_{c_n}(z) - z| = |\phi_{c_n}(z) - \phi_{c_0}(z)| \sim |c_n - c_0|, \tag{3.2}$$

where $A \sim B$ means $C^{-1}B < A < CB$ for two numbers A and B and some implicit constant C.

Let $w = \phi_{c_n}(z) \in J_{c_n}$, where $\forall z \in J_{c_0}$. Then it follows by (3.1) and (3.2) that

$$\begin{aligned} |w - z| &\sim \left|P_{c_n}^{-1}(\phi_{c_n}(z)) - P_{c_0}^{-1}(z)\right| \sim \left|P_{c_0}^{-1}(\phi_{c_n}(z)) - P_{c_0}^{-1}(z)\right| \\ &\sim \left|P_{c_0}^{-1}(\phi_{c_n}(z) - z)\right| \sim |\phi_{c_n}(z) - z|^{1/d} \sim |c_n - c_0|^{1/d}. \end{aligned}$$

It follows that $\forall z \in J_{c_0}$, $w = \phi_{c_n}(z) \in J_{c_n}$,

$$dist(w, z) \sim |c_n - c_0|^{1/d}.$$

Thus we get that for any small $\epsilon > 0$ the Julia sets J_{c_n} are contained in the ϵ-neighborhood of J_{c_0} for all $n \gg 0$.

Therefore

$$d_H(J_{c_n}, J_{c_0}) \sim |c_n - c_0|^{1/d}.$$

So we obtain

$$d_H(J_{c_n}, J_{c_0}) \leq C|c_n - c_0|^{1/d}$$

for some constant $C > 0$ only depending on P_{c_0}, where d_H denotes the Hausdorff distance.

Step 2: Since P_{c_n} and P_{c_0} have the same critical point 0, we have if $0 \in J_{c_0}$, then $0 \in J_{c_0}$ is preperiodic, and so is $0 \in J_{c_n}$ and P_{c_n} has no parabolic cycles for all $n \gg 0$ by our assumption that critical point relations are preserved. Hence P_{c_n} is semihyperbolic.

Now we only prove that

$$HD(J_{c_n}) \to HD(J_{c_0}).$$

Let $h = HD(J_{c_0})$ be the Hausdorff dimension of the Julia set J_{c_0} of the semihyperbolic map P_{c_0}. It follows by [6] that there exists exactly one h-conformal measure μ and this measure is atomless (the μ measure of a point is zero). The unique h-conformal measure for $P_{c_0} : J_{c_0} \to J_{c_0}$ supported on J_{c_0} has exponent $h = HD(J_{c_0})$. For all $n \gg 0$, P_{c_n} is a semihyperbolic map. The unique h_n-conformal probability measure μ_n for $P_{c_n} : J_{c_n} \to J_{c_n}$ supported on J_{c_n} has exponent $h_n = HD(J_{c_n})$ and it is atomless; see ref. [6]. Thus to prove that

$$\lim_{n\to\infty} HD(J_{c_n}) = HD(J_{c_0})$$

it is enough to prove that there is a neighborhood $B_r(0)$ of the critical point $0 \in J_{c_0}$ such that

$$\lim_{r\to 0} \lim_{n\to\infty} \mu_n(B_r(0)) = 0.$$

Since P_{c_0} is semihyperbolic, there exists $l > 1$ such that $P_{c_0}^l(0) = w \in \omega(0)$, where the set $\omega(0)$ of accumulation points of the orbit of 0 is a hyperbolic set of P_{c_0}. By the completely invariant property of the Julia set, it is enough that we only prove the following

$$\lim_{r\to 0} \lim_{n\to\infty} \mu_n(B_r(w)) = 0.$$

In fact any weak accumulation point ν of μ_n gives a P_{c_0}-invariant measure for $P_{c_0} : J_{c_0} \to J_{c_0}$. The previous limit implies that $\mu_n \to \mu = \nu$, and it follows that $h_n \to h$. Hence, we obtain that

$$HD(J_{c_n}) \to HD(J_{c_0}).$$

Since P_{c_0} is semihyperbolic, we consider the net $\mathcal{N}$ as in Lemma 2.3 and consider constants $C_0 > 0$ and $\theta_0 \in (0,1)$. Let $w \in \omega(0)$ be any point, then we have

$$\left|(P_{c_n}^m)'(w)\right|^{-1} \le C_0 \theta_0^m, \tag{3.3}$$

for all $m \ge 1$ and $n \gg 0$. Moreover we may suppose that there is a uniform Bounded Distortion property: *There is a constant $K > 1$ so that for every $k \ge 1$ and every kth step piece W of the net $\mathcal{N}_i$, the distortion of $P_{c_n}^k$ in W is bounded by K for all $n \gg 0$*; see Lemma 2.5.

Let $w \in \omega(0)$ be any point and B_q be the *qth* step piece containing $u_w = P_{c_0}^l(w)$ and V_q be the pull-back of B_q by $P_{c_0}^l$ containing w. Since $P_{c_n} \to P_{c_0}$ algebraically and $d_H(J_{c_n}, J_{c_0}) \le C|c_n - c_0|^{1/d}$, we let $\widetilde{V}_q$ be the pull-back of B_q by $P_{c_n}^l$ containing w, $n \gg 0$. It follows that for $r > 0$ small there is $q = q(r) \to \infty$, as $r \to 0$ so that $B_r(w) \subset \widetilde{V}_q$ for all $n \gg 0$. So we only need to prove that

$$\lim_{n\to\infty} \lim_{q\to\infty} \mu_n(\widetilde{V}_q) = 0.$$

Let D be a disc containing w, small enough so that $P_{c_n}^l|_D$ is at most of degree d. Refining the net if necessary, suppose that $B_1 \subset P_{c_n}^l(D)$. Since the probability measure μ_n is atomless for all $n \gg 0$, we have

$$\mu_n(\widetilde{V}_q) = \sum_{m\ge q} \mu_n(\widetilde{V}_m - \widetilde{V}_{m+1}).$$

Note that for $m \ge 1$ we have

$$\mu_n(\widetilde{V}_m - \widetilde{V}_{m+1}) \le d\mu_n(B_m - B_{m+1}) \inf_{(\widetilde{V}_m - \widetilde{V}_{m+1})\cap J_{c_n}} \left|(P_{c_n}^l)'(z)\right|^{-h_n}.$$

By formula (3.3), we have

$$\inf_{(\widetilde{V}_m-\widetilde{V}_{m+1})\cap J_{c_n}} \left|(P^l_{c_n})'(z)\right|^{-h_n} < C_1$$

for all $n \gg 0$ and some constant C_1. By the uniform Bounded Distortion Property and considering that μ_n is a probability measure, for some constant C_2 we have

$$\mu_n(B_m - B_{m+1}) \leq K^{h_n}\left|(P^m_{c_n})'(w)\right|^{-h_n} \leq C_2\theta_0^{mh_n},$$

for all $w \in B_m$. So

$$\mu_n(\widetilde{V}_q) \leq \sum C_1C_2\theta_0^{mh_n} \leq \sum C_3\theta_0^{mh_n}.$$

Since

$$\sum_{m\geq q} \theta_0^{mh_n} = \frac{(\theta_0^{h_n})^q}{1-\theta_0^{h_n}},$$

we conclude that

$$\lim_{n\to\infty}\lim_{q\to\infty} \mu_n(\widetilde{V}_q) = 0.$$

Therefor, we get

$$HD(J_{c_n}) \to HD(J_{c_0}).$$

The proof of the Main Theorem is finished. □

We remark that this theorem is sharp, that is, $O(|c_n - c_0|^{1/d})$ cannot be replaced by $o(|c_n - c_0|^{1/d})$. Assume that $d = 2$ and let $c_0 = -2$. It is well known, and can be easily checked, that the critical point 0 is pre-periodic. It eventually lands on 2, which is a repelling fixed point. Moreover we have $J_{-2} = [-2, 2]$. For any $\varepsilon > 0$ let $c_\varepsilon = -2 - \varepsilon$. The Julia set J_{c_ε} is a Cantor set that lies on the real line and is symmetric with respect to 0. Its extreme points are $Z_\varepsilon = (1 + \sqrt{1 - 4c_\varepsilon})/2$, the positive fixed point, and $-Z_\varepsilon$. Let $z_\varepsilon = \sqrt{-Z_\varepsilon - c_\varepsilon}$. One easily compute that $z_\varepsilon \sim \sqrt{(2/3)\varepsilon}$ and that $(-z_\varepsilon, z_\varepsilon) \not\subseteq J_{c_\varepsilon}$. We can thus conclude that for ε small enough,

$$d_H(J_{c_\varepsilon}, J_{-2}) \sim \sqrt{\varepsilon}.$$

Since $c_0 - c_\varepsilon = \varepsilon$, there is no hope to find a constant C such that

$$d_H(J_{c_\varepsilon}, J_{-2}) \sim o(|c_n - c_0|^{1/d}).$$

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