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On the order and hyper-order of meromorphic solutions of higher order linear differential equations

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Abstract. In this paper, we investigate the order of growth of solutions of the higher order linear differential equation

$$f^{(k)} + \sum_{j=0}^{k-1} \big(h_j e^{P_j(z)} + d_j\big) f^{(j)} = 0,$$

where $P_j(z)$ $(j = 0, 1, \dots, k-1)$ are nonconstant polynomials such that $\deg P_j = n \geq 1$ and $h_j(z)$, $d_j(z)$ $(j = 0, 1, \dots, k-1)$ with $h_0 \not\equiv 0$ are meromorphic functions of finite order such that $\max\{\rho(h_j), \rho(d_j) : j = 0, 1, \dots, k-1\} < n$. We prove that every meromorphic solution $f \not\equiv 0$ of the above equation is of infinite order. Then, we use the exponent of convergence of zeros or the exponent of convergence of poles of solutions to obtain an estimation of the hyper-order of solutions.

Key words: Linear differential equations, Meromorphic solutions, Order of growth, Hyper-order.

1. Introduction and statement of results

Throughout this paper, f will denote a transcendental meromorphic function in the whole complex plane, we use the standard notations of Nevanlinna's value distribution theory ([11], [16]). Let f be a meromorphic function, we define

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{it})| dt,$$

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

and

$$T(r, f) = m(r, f) + N(r, f)$$

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is the Nevanlinna characteristic function of f, where $\log^+ x = \max(0, \log x)$ for $x \geq 0$, and $n(t, f)$ is the number of poles of $f(z)$ lying in $|z| \leq t$, counted according to their multiplicity. Also, we define

$$N\left(r, \frac{1}{f}\right) = \int_0^r \frac{n\left(t, \frac{1}{f}\right) - n\left(0, \frac{1}{f}\right)}{t} dt + n\left(0, \frac{1}{f}\right) \log r,$$

$$\overline{N}\left(r, \frac{1}{f}\right) = \int_0^r \frac{\overline{n}\left(t, \frac{1}{f}\right) - \overline{n}\left(0, \frac{1}{f}\right)}{t} dt + \overline{n}\left(0, \frac{1}{f}\right) \log r,$$

where $n(t, 1/f)$ is the number of zeros of $f(z)$ lying in $|z| \leq t$, counted according to their multiplicity, and $\overline{n}(t, 1/f)$ indicate the number of distinct zeros of $f(z)$ lying in $|z| \leq t$. In addition, we will use notations $\lambda(f) = \limsup_{r \to +\infty} (\log N(r, 1/f))/(\log r)$, to denote the exponent of convergence of the zero-sequence and $\lambda(1/f) = \limsup_{r \to +\infty} (\log N(r, f))/(\log r)$, to denote the exponent of convergence of the pole-sequence of a meromorphic function $f(z)$. See ([11], [13], [16]) for notations and definitions.

Definition 1.1 ([19]) Let f be a meromorphic function. Then the order $\rho(f)$ and the lower order $\mu(f)$ of $f(z)$ are defined respectively by

$$\rho(f) = \limsup_{r \to +\infty} \frac{\log T(r, f)}{\log r}, \quad \mu(f) = \liminf_{r \to +\infty} \frac{\log T(r, f)}{\log r}.$$

To express the rate of growth of meromorphic solutions of infinite order, we recall the following definition.

Definition 1.2 ([19]) Let f be a meromorphic function. Then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r \to +\infty} \frac{\log \log T(r, f)}{\log r}.$$

To give the precise estimate of fixed points, we define:

Definition 1.3 ([17]) Let f be a meromorphic function and let $z_1, z_2, \ldots$ ($|z_j| = r_j$, $0 < r_1 \leq r_2 \leq \cdots$) be the sequence of the fixed points of f, each point being repeated only once. The exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}(f) = \inf\left\{\tau > 0 : \sum_{j=1}^{+\infty} |z_j|^{-\tau} < +\infty\right\}.$$

Clearly,

$$\overline{\tau}(f) = \limsup_{r\to+\infty} \frac{\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r},$$

where $\overline{N}(r, 1/(f-z))$ is the counting function of distinct fixed points of $f(z)$ in $\{z : |z| < r\}$.

Several authors, such as Kwon [12], Chen [5], Gundersen [10] have investigated the second order linear differential equation

$$f'' + A_1(z)e^{P(z)}f' + A_0(z)e^{Q(z)}f = 0, \tag{1.1}$$

where $P(z), Q(z)$ are nonconstant polynomials, $A_1(z)$, $A_0(z) \not\equiv 0$ are entire functions such that $\rho(A_1) < \deg P(z)$, $\rho(A_0) < \deg Q(z)$. Gundersen showed in [10, p. 419] that if $\deg P(z) \neq \deg Q(z)$, then every nonconstant solution of (1.1) is of infinite order. If $\deg P(z) = \deg Q(z)$, then (1.1) may have nonconstant solutions of finite order. For instanse $f(z) = e^z + 1/2$ satisfies $f'' + 2e^z f' - 2e^z f = 0$. In [6], Chen and Shon investigated the case when $\deg P(z) = \deg Q(z) = 1$ and obtained the following result.

Theorem A ([6]) *Let $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ be meromorphic functions with $\rho(A_j) < 1$, let a, b be complex constants such that $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Then every meromorphic solution $f \not\equiv 0$ of the differential equation*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = 0$$

has an infinite order.

In [18], Xu and Yi generalized Theorem A and study fixed points of solutions and their derivatives. In [1], Belaïdi investigated this case and generalized it for a class of higher-order linear differential equations and obtained the following result.

Theorem B ([1]) *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \ldots, k-1)$ be noncon-*

stant polynomials, where $a_{0,j}, a_{1,j}, \ldots, a_{n,j}$ $(j = 0, 1, \ldots, k-1)$ *are complex numbers such that* $a_{n,j}a_{n,0} \neq 0$ $(j = 1, 2, \ldots, k-1)$, *let* $A_j(z) \not\equiv 0$ $(j = 0, 1, \ldots, k-1)$ *be meromorphic functions. Suppose that* $\arg a_{n,j} \neq \arg a_{n,0}$ *or* $a_{n,j} = ca_{n,0}$ $(0 < c < 1)$ $(j = 1, 2, \ldots, k-1)$, $\rho(A_j) < n$ $(j = 0, 1, \ldots, k-1)$. *Then every meromorphic solution* $f(z) \not\equiv 0$ *of the equation*

$$f^{(k)} + A_{k-1}e^{P_{k-1}(z)}f^{(k-1)} + \cdots + A_1e^{P_1(z)}f' + A_0e^{P_0(z)}f = 0 \qquad (1.2)$$

is of infinite order, where $k \geq 2$.

In [14], Liu and Yuan generalized Theorem A and gave an estimation of the hyper-order of solutions. In [7], Chen and Xu replace the condition "coefficients having finite poles" in [14] by the condition "all poles of the solution f of the equation

$$f^{(k)} + h_{k-1}f^{(k-1)} + \cdots + h_se^{P(z)}f^{(s)} + \cdots + h_1f' + h_0e^{Q(z)}f = 0 \quad (1.3)$$

are of uniformly bounded multiplicity" to give an estimation of the hyper-order and obtained the following theorem.

Theorem C ([7]) *Let* $P(z)$ *and* $Q(z)$ *be nonconstant polynomials such that*

$$P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0,$$
$$Q(z) = b_nz^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0$$

for some complex numbers a_i, b_i $(i = 1, 2, \ldots, n)$ *with* $a_n \neq 0$, $b_n \neq 0$, *and let* h_j $(j = 0, 1, \ldots, k-1)$ *be meromorphic functions with* $h_0 \not\equiv 0$ *and* $\rho = \max\{\rho(h_j) : j = 0, 1, \ldots, k-1\} < n$. *Suppose all poles of* f *are of uniformly bounded multiplicity. Then the following three statements hold*:

(i) *If* $a_n = b_n$ *and* $\deg(P-Q) = m \geq 1$, $\rho < m$, *then every transcendental meromorphic solution* f *of equation* (1.3) *is of infinite order and* $m \leq \rho_2(f) \leq n$.

(ii) *If* $a_n = cb_n$ *with* $c > 1$, *and* $\deg(P-Q) = m \geq 1$, $\rho < m$, *then every meromorphic solution* $f \not\equiv 0$ *of equation* (1.3) *is of infinite order and* $\rho_2(f) = n$.

(iii) *If* $\max\{\rho(h_j) : j = 1, \ldots, k-1\} < \rho(h_0) < 1/2$, $a_n = cb_n$ *with* $c \geq 1$, *and* $P(z) - cQ(z)$ *is a constant, then every meromorphic solution*

$f \not\equiv 0$ of equation (1.3) is of infinite order and $\rho(h_0) \le \rho_2(f) \le n$.

In this paper, we continue the research in this type of problems, we extend the above results, and we obtain the following theorems.

Theorem 1.1 *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \dots, k-1)$ be nonconstant polynomials, where $a_{0,j}, a_{1,j}, \dots, a_{n,j}$ $(j = 0, 1, \dots, k-1)$ are complex numbers such that $a_{n,j}a_{n,0} \neq 0$ $(j = 1, 2, \dots, k-1)$, let $h_j(z), d_j(z)$ $(j = 0, 1, \dots, k-1)$ be meromorphic functions with $h_0 \not\equiv 0$. Suppose that $\arg a_{n,j} \neq \arg a_{n,0}$ or $a_{n,j} = c_j a_{n,0}$ $(c_j > 0,\ c_j \neq 1)$ for all $j = 1, \dots, k-1$ and $\rho = \max\{\rho(h_j), \rho(d_j) : j = 0, \dots, k-1\} < n$. Then the following two statements hold*:

(i) *Every meromorphic solution $f \not\equiv 0$ of the equation*

$$f^{(k)} + \big(h_{k-1}e^{P_{k-1}(z)} + d_{k-1}\big)f^{(k-1)} + \cdots + \big(h_1 e^{P_1(z)} + d_1\big)f' + \big(h_0 e^{P_0(z)} + d_0\big)f = 0 \tag{1.4}$$

is transcendental.

(ii) *For every meromorphic solution f of (1.4) with infinite order $\rho(f) = \infty$, we have if $\lambda(1/f) < n$ and $\lambda(f) < n$, then $\rho_2(f) = n$.*

Theorem 1.2 *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \dots, k-1)$ be nonconstant polynomials, where $a_{0,j}, a_{1,j}, \dots, a_{n,j}$ $(j = 0, 1, \dots, k-1)$ are complex numbers such that $a_{n,j}a_{n,0} \neq 0$ $(j = 1, 2, \dots, k-1)$, let $h_j(z), d_j(z)$ $(j = 0, 1, \dots, k-1)$ be meromorphic functions with $h_0 \not\equiv 0$. Suppose that $\arg a_{n,j} \neq \arg a_{n,0}$ or $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$ $(j = 1, 2, \dots, k-1)$, $\rho = \max\{\rho(h_j),\ \rho(d_j) : j = 0, \dots, k-1\} < n$. Then every meromorphic solution $f(z) \not\equiv 0$ of equation (1.4) is of infinite order and the hyper-order of f satisfies $\rho_2(f) \ge n$. In addition, if $\lambda(1/f) < \infty$, then $\rho_2(f) = n$.*

Theorem 1.3 *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \dots, k-1)$ be nonconstant polynomials, where $a_{0,j}, a_{1,j}, \dots, a_{n,j}$ $(j = 0, 1, \dots, k-1)$ are complex numbers such that $a_{n,j}a_{n,0} \neq 0$ $(j = 1, 2, \dots, k-1)$, let $h_j(z), d_j(z)$ $(j = 0, 1, \dots, k-1)$ be meromorphic functions with $h_0 \not\equiv 0$. Suppose that $a_{n,j} = c_j a_{n,0}$ $(c_j \ge 1)$, $\deg(P_0(z) - (1/c_j)P_j(z)) = 0$ $(j = 1, 2, \dots, k-1)$ and $\rho = \max\{\rho(h_j)\ (j = 1, \dots, k-1),\ \rho(d_j)\ (j = 0, \dots, k-1)\} < \rho(h_0)$. If $\lambda(1/h_0) < \mu(h_0) \le \rho(h_0) < 1/2$, then every transcendental meromorphic solution f of equation (1.4) is of infinite order and the hyper-order of f*

satisfies $\rho_2(f) \geq \rho(h_0)$. In addition, the following two statements hold:

(i) *If $\lambda(1/f) < \infty$, then $\rho(h_0) \leq \rho_2(f) \leq n$.*

(ii) *For $c_j \neq 1$ $(j = 1, 2, \ldots, k-1)$ if $\lambda(f) < n$ and $\lambda(1/f) < n$, then $\rho_2(f) = n$.*

Theorem 1.4 *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \ldots, k-1)$ be nonconstant polynomials, where $a_{0,j}, a_{1,j}, \ldots, a_{n,j}$ $(j = 0, 1, \ldots, k-1)$ are complex numbers such that $a_{n,j} a_{n,0} \neq 0$ $(j = 1, 2, \ldots, k-1)$, let $h_j(z), d_j(z)$ $(j = 0, 1, \ldots, k-1)$ be meromorphic functions with $h_0 \not\equiv 0$. Suppose that* $\arg a_{n,j} \neq \arg a_{n,0}$ $(j = 1, 2, \ldots, k-1)$, $\arg(a_{n,1} + a_{n,j}) \neq \arg a_{n,0}$ $(j = 2, 3)$ *or $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$ $(j = 1, 2, \ldots, k-1)$ and $\rho = \max\{\rho(h_j), \rho(d_j) : j = 0, 1, \ldots, k-1\} < n$. Then for any meromorphic solution $f \not\equiv 0$ of equation (1.4), f, f', f'' all have infinitely many fixed points and satisfy*

$$\overline{\tau}(f) = \overline{\tau}(f') = \overline{\tau}(f'') = \infty.$$

2. Lemmas for the proofs of theorems

First, we recall the following definitions. We define the linear measure of a set $H \subset [0, +\infty)$ by $m(H) = \int_0^{+\infty} \chi_H(t)dt$, where χ_H is the characteristic function of H, and the logarithmic measure of a set $E \subset [1, +\infty)$ by $lm(E) = \int_1^{+\infty} (\chi_E(t)dt/t)$. The upper and the lower densities of H are defined by

$$\overline{dens}\, H = \limsup_{r \to +\infty} \frac{m(H \cap [0, r])}{r}, \quad \underline{dens}\, H = \liminf_{r \to +\infty} \frac{m(H \cap [0, r])}{r}.$$

The upper and the lower logarithmic densities of E are defined by

$$\overline{\log dens}(E) = \limsup_{r \to +\infty} \frac{lm(E \cap [1, r])}{\log r}, \quad \underline{\log dens}(E) = \liminf_{r \to +\infty} \frac{lm(E \cap [1, r])}{\log r}.$$

Lemma 2.1 ([15, pp. 253–255]) *Let $P(z) = \sum_{i=0}^{n} b_i z^i$, where n is a positive integer and $b_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0, 2\pi)$. For any given ε $(0 < \varepsilon < \pi/4n)$, we introduce $2n$ closed angles*

$$S_j : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon \leq \theta \leq -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon \qquad (j = 0, 1, \ldots, 2n-1). \quad (2.1)$$

Then there exists a positive number $R_1 = R_1(\varepsilon)$ *such that for* $|z| = r > R_1$,

$$Re(P(z)) > \alpha_n r^n (1-\varepsilon)\sin(n\varepsilon), \tag{2.2}$$

if $z = re^{i\theta} \in S_j$, *when* j *is even; while*

$$Re(P(z)) < -\alpha_n r^n (1-\varepsilon)\sin(n\varepsilon), \tag{2.3}$$

if $z = re^{i\theta} \in S_j$, *when* j *is odd.*

Lemma 2.2 ([3]) *Let* $g(z)$ *be a transcendental meromorphic function of order* $\rho(g) = \rho < +\infty$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_1 \subset (1, \infty)$ *that has finite linear measure and finite logarithmic measure, such that*

$$|g(z)| \le \exp\{r^{\rho+\varepsilon}\} \tag{2.4}$$

holds for $|z| = r \notin [0,1] \cup E_1$, $r \to +\infty$.

Lemma 2.3 ([9]) *Let* $f(z)$ *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant. Then there exist a set* $E_2 \subset (1, +\infty)$ *of finite logarithmic measure and a constant* $A > 0$ *that depends only on* α *and* (m,n) *(*m, n *positive integers with* $m < n$*) such that for all* z *satisfying* $|z| = r \notin [0,1] \cup E_2$, *we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \le A\left[\frac{T(\alpha r,\, f)}{r}(\log^{\alpha} r)\log T(\alpha r,\, f)\right]^{n-m}. \tag{2.5}$$

Lemma 2.4 ([4]) *Suppose that* $h(z)$ *is a meromorphic function with* $\lambda(1/h) < \mu(h) \le \rho(h) = \rho < 1/2$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_3 \subset (1, +\infty)$ *that has a positive upper logarithmic density such that for all* z *satisfying* $|z| = r \in E_3$, *we have*

$$|h(z)| \ge \exp\left\{(1+o(1))r^{\rho-\varepsilon}\right\}. \tag{2.6}$$

It is well-known that it is very important of the Wiman-Valiron theory [13] to investigate the properties of entire solutions of differential equations. In [4], Z. X. Chen has extend the Wiman-Valiron theory from entire functions to meromorphic functions. Here we give a special form of the result given by J. Wang and H. X. Yi in [17], when meromorphic function has infinite order.

Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. We define by $\mu(r) = \max\{|a_n|r^n;\ n = 0, 1, \dots\}$ the maximum term of g, and define by $\nu_g(r) = \max\{m;\ \mu(r) = |a_m|r^m\}$ the central index of g.

Lemma 2.5 ([17]) *Let $f(z) = g(z)/d(z)$ be an infinite order meromorphic function with $\rho_2(f) = \sigma$, $g(z)$ and $d(z)$ are entire functions, where $\rho(d) < +\infty$. Then there exists a sequence of complex numbers $\{z_k = r_k e^{i\theta_k}\}_{k\in\mathbb{N}}$ satisfying $r_k \to +\infty$, $\theta_k \in [0, 2\pi); k \in \mathbb{N}$, $\lim_{k\to+\infty} \theta_k = \theta_0 \in [0, 2\pi)$, $|g(z_k)| = M(r_k, g)$ and for sufficiently large k, we have*

$$\frac{f^{(n)}(z_k)}{f(z_k)} = \left(\frac{\nu_g(r_k)}{z_k}\right)^n (1 + o(1)) \quad (n \in \mathbb{N}), \tag{2.7}$$

$$\limsup_{r_k\to+\infty} \frac{\log\log \nu_g(r_k)}{\log r_k} = \rho_2(g). \tag{2.8}$$

Lemma 2.6 ([10]) *Let $\varphi : [0, +\infty) \to \mathbb{R}$ and $\psi : [0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $\varphi(r) \le \psi(r)$ for all $r \notin E_4 \cup [0, 1]$, where $E_4 \subset (1, +\infty)$ is a set of finite logarithmic measure. Let $\alpha > 1$ be a given constant. Then there exists an $r_1 = r_1(\alpha) > 0$ such that $\varphi(r) \le \psi(\alpha r)$ for all $r > r_1$.*

Lemma 2.7 *Suppose that $k \ge 2$ and $A_0, A_1, \dots, A_{k-1}$ are meromorphic functions. Let $\rho = \max\{\rho(A_j) : j = 0, \dots, k-1\}$ and let $f(z)$ be a transcendental meromorphic solution with $\lambda(1/f) < \infty$ of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0. \tag{2.9}$$

Then $\rho_2(f) \le \rho$.

Proof. We assume that f is a transcendental meromorphic solution of equation (2.9). If $\rho(f) < \infty$, then $\rho_2(f) = 0 \le \rho$. If $\rho(f) = \infty$. We can rewrite (2.9) as

$$-\frac{f^{(k)}}{f} = A_{k-1}\frac{f^{(k-1)}}{f} + A_{k-2}\frac{f^{(k-2)}}{f} + \cdots + A_1\frac{f'}{f} + A_0. \tag{2.10}$$

By Hadamard factorization theorem, we can write f as $f(z) = g(z)/d(z)$, where $g(z)$ and $d(z)$ are entire functions, with $\lambda(d) = \rho(d)$. Since $\lambda(1/f) <$

∞, we have $\lambda(d) = \rho(d) = \lambda(1/f) < \infty$ and $\rho_2(f) = \rho_2(g)$. By Lemma 2.2 and Lemma 2.5, for any small $\varepsilon > 0$, there exists a sequence $\{z_j = r_j e^{i\theta_j}\}$ satisfying $r_j \notin [0,1] \cup E_1$, $r_j \to +\infty$, $\theta_j \in [0, 2\pi)$, $\lim_{j\to\infty} \theta_j = \theta_0 \in [0, 2\pi)$, $|g(z_j)| = M(r_j, g)$ such that for j sufficiently large, we have

$$\frac{f^{(n)}(z_j)}{f(z_j)} = \left(\frac{\nu_g(r_j)}{z_j}\right)^n (1 + o(1)) \quad (n \in \mathbb{N}), \tag{2.11}$$

$$\limsup_{r_j\to+\infty} \frac{\log\log \nu_g(r_j)}{\log r_j} = \rho_2(g), \tag{2.12}$$

$$|A_s(z_j)| \le \exp\left\{r_j^{\rho+\varepsilon}\right\}, \quad s = 0, 1, \ldots, k-1. \tag{2.13}$$

Substituting (2.11) and (2.13) into (2.10), we obtain

$$\left(\frac{\nu_g(r_j)}{r_j}\right)^k |1 + o(1)| \le e^{r_j^{\rho+\varepsilon}} + \sum_{s=1}^{k-1} e^{r_j^{\rho+\varepsilon}} \left(\frac{\nu_g(r_j)}{r_j}\right)^s |1 + o(1)|.$$

It follows that

$$(\nu_g(r_j))^k |1 + o(1)| \le k e^{r_j^{\rho+\varepsilon}} r_j^k (\nu_g(r_j))^{k-1} |1 + o(1)|.$$

Hence

$$\nu_g(r_j) \le k M r_j^k e^{r_j^{\rho+\varepsilon}}, \tag{2.14}$$

where the sequence $\{z_j = r_j e^{i\theta_j}\}$ satisfies $r_j \notin [0,1] \cup E_1$, $r_j \to +\infty$, $\theta_j \in [0, 2\pi)$, $\lim_{j\to\infty} \theta_j = \theta_0 \in [0, 2\pi)$, $|g(z_j)| = M(r_j, g)$ and $M > 0$ is some constant. Then by (2.12), (2.14), Lemma 2.6 and $\varepsilon > 0$ being arbitrary, we obtain that $\rho_2(g) = \rho_2(f) \le \rho$. $\square$

Lemma 2.8 ([8, p. 30]) *Let $P_1, P_2, \ldots, P_n$ $(n \ge 1)$ be non-constant polynomials with the degree in order $d_1, d_2, \ldots, d_n$, respectively. Suppose that when $i \ne j$, then $\deg(P_i - P_j) = \max\{d_i, d_j\}$. Let $A(z) = \sum_{j=1}^n B_j(z) e^{P_j(z)}$, where $B_j(z) \not\equiv 0$ are meromorphic functions satisfying $\rho(B_j) < d_j$. Then*

$$\rho(A) = \max_{1\le j\le n} \{d_j\}. \tag{2.15}$$

Lemma 2.9 ([2]) *Let $A_0,\ A_1, \ldots, A_{k-1},\ F \not\equiv 0$ be finite order meromor-*

phic functions. If f is a meromorphic solution with $\rho(f) = +\infty$ of the equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F, \tag{2.16}$$

then

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty. \tag{2.17}$$

Lemma 2.10 *Let $P_j(z)$ $(j = 0, 1, \ldots, k)$ be polynomials with $\deg P_0(z) = n$ $(n \geq 1)$ and $\deg P_j(z) \leq n$ $(j = 1, 2, \ldots, k)$. Let $A_j(z)$ $(j = 0, 1, \ldots, k)$ be meromorphic functions with finite order and $\max\{\rho(A_j) : j = 0, 1, \ldots, k\} < n$ such that $A_0(z) \not\equiv 0$. We denote*

$$F(z) = A_k e^{P_k(z)} + A_{k-1}e^{P_{k-1}(z)} + \cdots + A_1 e^{P_1(z)} + A_0 e^{P_0(z)}. \tag{2.18}$$

If $\deg(P_0(z) - P_j(z)) = n$ for all $j = 1, \ldots, k$, then F is a nontrivial meromorphic function with finite order and satisfies $\rho(F) = n$.

Proof. Set $P_j(z) = a_{n,j}z^n + a_{n-1,j}z^{n-1} + \cdots + a_{1,j}z + a_{0,j}$ $(j = 0, 1, \ldots, k)$. Suppose that $\deg(P_0(z) - P_j(z)) = n$ for all $j = 1, \ldots, k$. Then, $a_{n,0} \neq a_{n,j}$, for all $j = 1, \ldots, k$. Let $\{a_{n,j_1}, a_{n,j_2}, \ldots, a_{n,j_m}\} \subset \{a_{n,1}, a_{n,2}, \ldots, a_{n,k}\}$ such that a_{n,j_l} $(l = 1, 2, \ldots, m)$ are different from each other. For each $a_{n,j} \in \{a_{n,1}, a_{n,2}, \ldots, a_{n,k}\}$ we have $a_{n,j} = 0$ or $a_{n,j} \neq 0$. In the case when $a_{n,j} \neq 0$, there exists only one $a_{n,j_l} \in \{a_{n,j_1}, a_{n,j_2}, \ldots, a_{n,j_m}\}$ such that $a_{n,j} = a_{n,j_l}$. We can write

$$\left(A_j(z)e^{P_j(z) - a_{n,j}z^n} + A_{j_l}(z)e^{P_{j_l}(z) - a_{n,j_l}z^n}\right)e^{a_{n,j_l}z^n}$$

instead of $A_j(z)e^{P_j(z)} + A_{j_l}(z)e^{P_{j_l}(z)}$ when $a_{n,j} = a_{n,j_l}$ $(a_{n,j} \in \{a_{n,1}, a_{n,2}, \ldots, a_{n,k}\})$. For $a_{n,j} = 0$, we set $a_{n,j} = a_{n,s_i} \in \{a_{n,s_1}, a_{n,s_2}, \ldots, a_{n,s_t}\}$ where $a_{n,s_i} = 0$ (i.e., $\deg(P_{s_i}) < n$ for $i = 1, 2, \ldots, t$). By $A_0(z) \not\equiv 0$ we can write equation (2.18) in the form

$$A_0(z)e^{P_0(z)} + \sum_{l=1}^{m} B_{j_l}(z)e^{a_{n,j_l}z^n} + \sum_{i=1}^{t} A_{s_i}(z)e^{P_{s_i}(z)} = F(z),$$

it follows that

$$A_0(z)e^{P_0(z)} + \sum_{l=1}^{m} B_{j_l}(z)e^{a_{n,j_l}z^n} = B(z), \tag{2.19}$$

where $B(z) = F(z) - \sum_{i=1}^{t} A_{s_i}(z)e^{P_{s_i}(z)}$ and $A_{s_i}(z)$ $(i = 1, 2, \ldots, t)$, $B_{j_l}(z)$ $(l = 1, 2, \ldots, m)$ are meromorphic functions of finite order which is less than n. Suppose that $\rho(F) \neq n$. Since $\deg P_{s_i}(z) < n$ and $\rho(A_{s_i}) < n$ $(i = 1, 2, \ldots, t)$, then

$$\rho(B) \neq n. \tag{2.20}$$

By $a_{n,0}$ and a_{n,j_l} $(l = 1, 2, \ldots, m)$ are different from each other, then $\deg(P_0(z) - a_{n,j_l}z^n) = n$ $(l = 1, 2, \ldots, m)$ and $\deg(a_{n,j_l}z^n - a_{n,j_i}z^n) = n$ $(1 \leq l \neq i \leq m)$. Since $\rho(B_{j_l}) < n$, $A_0(z) \not\equiv 0$, $\rho(A_0) < n$, by (2.19) and Lemma 2.8, we find that $\rho(B) = n$, this contradicts (2.20). Hence, $\rho(F) = n$. □

3. Proof of Theorem 1.1

(i) We assume that the meromorphic solution $f(z) \not\equiv 0$ of equation (1.4) is not transcendental, then $\rho(f) = 0$ (f is a rational function or is a polynomial). Since we have $h_0 f \not\equiv 0$, we write equation (1.4) in the form

$$h_0 f\ e^{P_0(z)} + \sum_{j=1}^{k-1} h_j f^{(j)} e^{P_j(z)} = B(z), \tag{3.1}$$

where $B(z) = -(f^{(k)} + d_{k-1}f^{(k-1)} + \cdots + d_0 f)$, $h_0 f$ and $h_j f^{(j)}$ are meromorphic functions of finite order with $\rho(B) < n$, $\rho(h_0 f) < n$ and $\rho(h_j f^{(j)}) < n$ $(j = 1, 2, \ldots, k-1)$. Since $\arg a_{n,j} \neq \arg a_{n,0}$ or $a_{n,j} = c_j a_{n,0}$ $(c_j > 0$, $c_j \neq 1)$ for all $j = 1, 2, \ldots, k-1$ then $\deg(P_0(z) - P_j(z)) = n$. From (3.1) and by Lemma 2.10, we have $\rho(B) = n$, this contradicts the fact $\rho(B) < n$. Hence, every meromorphic solution $f \not\equiv 0$ of equation (1.4) is transcendental.

(ii) Set $H_j(z) = A_j(z)e^{P_j(z)} + d_j(z)$ $(j = 0, 1, \ldots, k-1)$. Suppose that $f \not\equiv 0$ is a meromorphic solution of (1.4) with $\rho(f) = \infty$, $\lambda(1/f) < n$ and $\lambda(f) < n$. Then, by Hadamard factorization theorem, we can write f as $f(z) = (\pi(z)/d(z))e^{h(z)}$, where $\pi(z), d(z)$ are entire functions with $\lambda(\pi) = \rho(\pi) = \lambda(f) < n$, $\lambda(d) = \rho(d) = \lambda(1/f) < n$ and h is a transcendental entire function with $\rho_2(f) = \rho(h)$. Put $g = f'/f$, then (see [13, Lemma 2.3.7])

$$\frac{f^{(j)}}{f} = g^j + \frac{1}{2}j(j-1)g^{j-2}g' + G_{j-2}(g) \quad (j = 2, 3, \ldots, k), \tag{3.2}$$

where $G_{j-2}(g)$ is a differential polynomial of the meromorphic function g with constant coefficients and the degree no more than $j-2$. Substituting (3.2) into (1.4), we obtain

$$g^k = T_{k-1}(g), \tag{3.3}$$

where $T_{k-1}(g)$ is a differential polynomial of the meromorphic function g with the coefficients $H_0, H_1, \ldots, H_{k-1}$ and the degree no more than $k-1$. Applying Clunie Lemma [13] to (3.3), we have

$$m(r, g) \leq O\left(\sum_{j=0}^{k-1} m(r, H_j)\right) + S(r, g).$$

We know that

$$N(r, g) = N\left(r, \frac{f'}{f}\right) = \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right),$$

hence

$$T(r, g) \leq O\left(\sum_{j=0}^{k-1} T(r, H_j)\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, g). \tag{3.4}$$

It follows from (3.4) and the fact $\lambda(f) < n$, $\lambda(1/f) < n$, $\rho(H_j) = n$ $(j = 0, 1, \ldots, k-1)$ that $\rho(g) \leq n$. We assert that $\rho(g) = n$. If $\rho(g) < n$, then by (3.2) we have

$$\rho\left(\frac{f^{(j)}}{f}\right) < n \quad (j = 1, 2, \ldots, k).$$

Since we have $h_0 f \not\equiv 0$, we write equation (1.4) in the form

$$h_0 e^{P_0(z)} f + \sum_{j=1}^{k-1} h_j e^{P_j(z)} f^{(j)} + \left(f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_0 f' + d_0 f\right) = 0. \tag{3.5}$$

It follows that

$$h_0e^{P_0(z)}+\sum_{j=1}^{k-1}\left(h_j\frac{f^{(j)}}{f}\right)e^{P_j(z)}=-\left(\frac{f^{(k)}}{f}+d_{k-1}\frac{f^{(k-1)}}{f}+\cdots+d_1\frac{f'}{f}+d_0\right). \tag{3.6}$$

Hence

$$h_0(z)e^{P_0(z)}+\sum_{j=1}^{k-1}B_j(z)e^{P_j(z)}=F(z), \tag{3.7}$$

where $B_j=h_j(f^{(j)}/f)$ $(j=1,2,\ldots,k-1)$, $h_0\not\equiv 0$ and $F(z)=-(f^{(k)}/f+d_{k-1}(f^{(k-1)}/f)+\cdots+d_1(f'/f)+d_0)$ are meromorphic functions of finite order with $\rho(B_j)<n$ $(j=1,2,\ldots,k-1)$, $\rho(h_0)<n$ and $\rho(F)<n$. Since $\arg a_{n,j}\neq\arg a_{n,0}$ or $a_{n,j}=c_ja_{n,0}$ $(c_j>0,\ c_j\neq 1)$ for all $j=1,2,\ldots,k-1$ then $\deg(P_0(z)-P_j(z))=n$ for all $j=1,2,\ldots,k-1$. From (3.7) and Lemma 2.10, we have $\rho(F)=n$, this contradicts the fact that $\rho(F)<n$. Hence $\rho(g)=n$. Since $\rho_2(f)=\rho(h)$ and $\lambda(d)=\lambda(1/f)<n$, then by Lemma 2.7 we have $\rho_2(f)=\rho(h)\le\max\{\rho(H_j):j=0,1,\ldots,k-1\}=n$. Suppose that $\rho(h)<n$. Then, it follows from $f'/f=\pi'/\pi-d'/d+h'$ that

$$\begin{aligned}T\left(r,\frac{f'}{f}\right)&\le T\left(r,\frac{\pi'}{\pi}\right)+T\left(r,\frac{d'}{d}\right)+T(r,h')+O(1)\\&=m\left(r,\frac{\pi'}{\pi}\right)+\overline{N}\left(r,\frac{1}{\pi}\right)+m\left(r,\frac{d'}{d}\right)+\overline{N}\left(r,\frac{1}{d}\right)+T(r,h')+O(1)\\&=O(\log r)+\overline{N}\left(r,\frac{1}{\pi}\right)+O(\log r)+\overline{N}\left(r,\frac{1}{d}\right)+T(r,h')\\&=O(\log r)+\overline{N}\left(r,\frac{1}{\pi}\right)+\overline{N}\left(r,\frac{1}{d}\right)+T(r,h').\end{aligned} \tag{3.8}$$

By (3.8) and the fact $\lambda(\pi)<n$, $\lambda(d)<n$, we get $\rho(f'/f)=\rho(g)<n$, a contradiction to $\rho(g)=n$, hence $\rho(h)=n$, then $\rho_2(f)=n$. □

4. Proof of Theorem 1.2

Let $f\not\equiv 0$ be a meromorphic solution of equation (1.4). Then, by Theorem 1.1, f is transcendental.

Case 1: When $\arg a_{n,j} \neq \arg a_{n,0}$ $(j = 1, 2, \ldots, k-1)$. From (1.4), we have

$$|e^{P_0(z)}| \leq \left|\frac{1}{h_0(z)}\right|\left|\frac{f^{(k)}(z)}{f(z)}\right| + \left(\left|\frac{h_{k-1}(z)}{h_0(z)}\right||e^{P_{k-1}(z)}| + \left|\frac{d_{k-1}(z)}{h_0(z)}\right|\right)\left|\frac{f^{(k-1)}(z)}{f(z)}\right|$$
$$+ \cdots + \left(\left|\frac{h_1(z)}{h_0(z)}\right||e^{P_1(z)}| + \left|\frac{d_1(z)}{h_0(z)}\right|\right)\left|\frac{f'(z)}{f(z)}\right| + \left|\frac{d_0(z)}{h_0(z)}\right|. \tag{4.1}$$

By Lemma 2.1, there exist constants $R_0 > 0$, $L > 0$ and $\theta_1 < \theta_2$, such that for all $z = re^{i\theta}$, $|z| = r > R_0$, $\theta \in (\theta_1, \theta_2)$, we have

$$Re(P_j(z)) \leq 0 \ (j = 1, 2, \ldots, k-1) \ \text{ and } \ Re(P_0(z)) > Lr^n. \tag{4.2}$$

Set $\rho = \max\{\rho(h_j), \rho(d_j) : j = 0, 1, \ldots, k-1\} < n$, by Lemma 2.2, for any given ε $(0 < \varepsilon < n - \rho)$, there exists a set $E_1 \subset (1, +\infty)$ that has finite logarithmic measure such that when $|z| = r \notin [0,1] \cup E_1$, $r \longrightarrow +\infty$, we have

$$\left|\frac{1}{h_0(z)}\right| \leq e^{r^{\rho+\varepsilon}}, \ \left|\frac{d_j(z)}{h_0(z)}\right| \leq e^{r^{\rho+\varepsilon}} \quad (j = 0, 1, \ldots, k-1) \tag{4.3}$$

and

$$\left|\frac{h_j(z)}{h_0(z)}\right| \leq e^{r^{\rho+\varepsilon}} \quad (j = 1, 2, \ldots, k-1). \tag{4.4}$$

By Lemma 2.3, there exist a constant $A > 1$ and a set $E_2 \subset (1, +\infty)$ which has finite logarithmic measure, such that for all $|z| = r \notin [0,1] \cup E_2$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq A[T(2r, f)]^{2k}, \quad j = 1, 2, \ldots, k. \tag{4.5}$$

In accordance with (4.2), (4.3), (4.4) and (4.5), for $z = re^{i\theta}$, $\theta \in (\theta_1, \theta_2)$, $r \notin [0,1] \cup E_1 \cup E_2$, $r \longrightarrow +\infty$ the inequality (4.1) gives

$$e^{Lr^n} \leq e^{r^{\rho+\varepsilon}} A[T(2r, f)]^{2k} + 2(k-1)e^{r^{\rho+\varepsilon}} A[T(2r, f)]^{2k} + e^{r^{\rho+\varepsilon}}$$
$$\leq 2kAe^{r^{\rho+\varepsilon}}[T(2r, f)]^{2k}. \tag{4.6}$$

Since $\rho+\varepsilon<n$, by Lemma 2.6 and (4.6) we obtain

$$\rho(f)=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r}=+\infty$$

and

$$\rho_2(f)=\limsup_{r\to+\infty}\frac{\log\log T(r,f)}{\log r}\geq n.$$

In addition, if $\lambda(1/f)<+\infty$ then by Lemma 2.7 and from equation (1.4), we have $\rho_2(f)\leq n$, so $\rho_2(f)=n$.

Case 2: When $a_{n,j}=c_j a_{n,0}$ $(0<c_j<1)$ $(j=1,2,\ldots,k-1)$. We put $\deg(P_j(z)-c_jP_0(z))=m_j$ (m_j is a positive integer and $0\leq m_j<n$). By Lemma 2.1, there exist constants $R_1>0$, $L_1>0$, $\lambda>0$ and $\theta_1<\theta_2$, such that for all$z=re^{i\theta}$, $|z|=r>R_1$, $\theta\in(\theta_1,\theta_2)$, we have

$$Re(P_j(z)-c_jP_0(z))<\lambda\ (j=1,2,\ldots,k-1)$$

and

$$Re(P_0(z))>L_1r^n.$$

Set $c=\max\{c_j: j=1,\ldots,k-1\}$, then we have $0<c<1$ and

$$e^{(1-c)L_1r^n}\leq\left|e^{(1-c)P_0(z)}\right|,\quad \left|e^{-cP_0(z)}\right|\leq e^{-cL_1r^n}<1. \tag{4.7}$$

Since $(c_j-c)\leq 0$ for all $j=1,2,\ldots,k-1$, we obtain

$$\begin{aligned}\left|e^{P_j(z)-cP_0(z)}\right|&=\left|e^{P_j(z)-c_jP_0(z)+(c_j-c)P_0(z)}\right|\\&=\left|e^{P_j(z)-c_jP_0(z)}\right|\left|e^{(c_j-c)P_0(z)}\right|<e^{\lambda}.\end{aligned} \tag{4.8}$$

From (1.4) we have

$$\left|e^{(1-c)P_0(z)}\right|\leq\left|\frac{1}{h_0(z)}\right|\left|e^{-cP_0(z)}\right|\left|\frac{f^{(k)}(z)}{f(z)}\right|$$

$$+\left(\left|\frac{h_{k-1}(z)}{h_0(z)}\right|\left|e^{P_{k-1}(z)-cP_0(z)}\right|+\left|\frac{d_{k-1}(z)}{h_0(z)}\right|\left|e^{-cP_0(z)}\right|\right)\left|\frac{f^{(k-1)}(z)}{f(z)}\right|$$
$$+\cdots+\left(\left|\frac{h_1(z)}{h_0(z)}\right|\left|e^{P_1(z)-cP_0(z)}\right|+\left|\frac{d_1(z)}{h_0(z)}\right|\left|e^{-cP_0(z)}\right|\right)\left|\frac{f'(z)}{f(z)}\right|$$
$$+\left|\frac{d_0(z)}{h_0(z)}\right|\left|e^{-cP_0(z)}\right|. \tag{4.9}$$

In accordance with (4.3), (4.4), (4.5), (4.7) and (4.8), for $z = re^{i\theta}$, $\theta \in (\theta_1, \theta_2)$, $r \notin [0,1] \cup E_1 \cup E_2$, $r \longrightarrow +\infty$ the inequality (4.9) gives

$$e^{(1-c)L_1 r^n} \le e^{r^{\rho+\varepsilon}} A[T(2r,f)]^{2k} + (k-1)e^{r^{\rho+\varepsilon}}(e^{\lambda}+1)A[T(2r,f)]^{2k} + e^{r^{\rho+\varepsilon}}$$
$$\le A(k+1)(e^{\lambda}+1)e^{r^{\rho+\varepsilon}}[T(2r,f)]^{2k}. \tag{4.10}$$

Since $\rho+\varepsilon < n$ and $1-c > 0$, by Lemma 2.6 and (4.10) we obtain

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r} = +\infty$$

and

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} \ge n.$$

In addition, if $\lambda(1/f) < \infty$ then by Lemma 2.7 and from equation (1.4), we have $\rho_2(f) \le n$, so $\rho_2(f) = n$. □

5. Proof of Theorem 1.3

Let f be a transcendental meromorphic solution of (1.4). We assume that $P_0(z) - (1/c_j)P_j(z) = A_j$ (A_j is a constant) for all $j = 1, \ldots, k$, then $P_j(z) = c_j P_0(z) - c_j A_j$. Let $j_0 \in \{1, 2, \ldots, k-1\}$, we have

$$P_j(z) - \frac{1}{c_{j_0}} P_{j_0}(z) = (c_j - 1)P_0(z) - c_j A_j + A_{j_0}.$$

According to Lemma 2.1, there exists a continuous curve Γ (see also [12]), such that for all $z = re^{i\theta}$ with $z \in \Gamma$ and $|z| = r > R_2$, we have

$$Re(P_0(z)) = 0.$$

Consequently, there exists $\lambda > 0$ such that for all $j = 1, 2, \ldots, k-1$, we have

$$\left|e^{P_j(z)-(1/c_{j_o})P_{j_0}(z)}\right| = \left|e^{(c_j-1)P_0(z)-c_jA_j+A_{j_0}}\right| = e^{Re(-c_jA_j+A_{j_0})} < e^{\lambda} \quad (5.1)$$

and

$$\left|e^{-(1/c_{j_o})P_{j_0}(z)}\right| = \left|e^{-P_0(z)+A_{j_0}}\right| = e^{Re(A_{j_0})} < e^{\lambda}. \quad (5.2)$$

Since $\rho = \max\{\rho(h_j)\ (j = 1, \ldots, k-1),\ \rho(d_j)\ (j = 0, 1, \ldots, k-1)\} < \rho(h_0)$, let α, β be two real numbers that satisfy $\rho < \beta < \alpha < \rho(h_0)$. Then, by Lemma 2.2, there exists a set $E_1 \subset (1, +\infty)$ that has finite logarithmic measure such that when $|z| = r \notin [0, 1] \cup E_1$, $r \longrightarrow +\infty$, we have

$$|h_j(z)| \leq e^{r^{\beta}},\ j = 1, 2, \ldots, k-1 \text{ and } |d_j(z)| \leq e^{r^{\beta}},\ j = 0, 1, \ldots, k-1. \quad (5.3)$$

Since h_0 is a meromorphic function and $\lambda(1/h_0) < \mu(h_0) \leq \rho(h_0) < 1/2$, by Lemma 2.4, there exists a set $E_3 \subset (1, +\infty)$ that has a positive upper logarithmic density such that for all z satisfying $|z| = r \in E_3$, we have

$$|h_0(z)| \geq e^{r^{\alpha}}. \quad (5.4)$$

From equation (1.4) it follows that

$$\begin{aligned}
&\left|h_0(z)e^{P_0(z)-(1/c_{j_o})P_{j_0}(z)}\right| \\
&\leq \left|e^{-(1/c_{j_o})P_{j_0}(z)}\right|\left|\frac{f^{(k)}(z)}{f(z)}\right| \\
&\quad + \left(|h_{k-1}(z)|\left|e^{P_{k-1}(z)-(1/c_{j_o})P_{j_0}(z)}\right| + |d_{k-1}(z)|\left|e^{-(1/c_{j_o})P_{j_0}(z)}\right|\right)\left|\frac{f^{(k-1)}(z)}{f(z)}\right| \\
&\quad + \cdots + \left(|h_1(z)|\left|e^{P_1(z)-(1/c_{j_o})P_{j_0}(z)}\right| + |d_1(z)|\left|e^{-(1/c_{j_o})P_{j_0}(z)}\right|\right)\left|\frac{f'(z)}{f(z)}\right| \\
&\quad + |d_0(z)|\left|e^{-(1/c_{j_o})P_{j_0}(z)}\right|. \qquad (5.5)
\end{aligned}$$

In accordance with (4.5), (5.1), (5.2), (5.3) and (5.4), for all $z = re^{i\theta}$ with $z \in \Gamma$ and $r \in E_3 - [0, 1] \cup E_1 \cup E_2$, $r \longrightarrow +\infty$, the inequality (5.5) gives

$$\begin{aligned} e^{r^\alpha} e^{Re(A_{j_0})} &\le e^\lambda A[T(2r,f)]^{2k} + 2(k-1)e^\lambda e^{r^\beta} A[T(2r,f)]^{2k} + e^\lambda e^{r^\beta} \\ &\le 2Ake^\lambda e^{r^\beta}[T(2r,f)]^{2k}. \end{aligned} \tag{5.6}$$

Since $\beta < \alpha$, by Lemma 2.6 and (5.6) we find

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r} = +\infty$$

and

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} \ge \alpha.$$

Since that α is an arbitrary number in the interval $]\beta, \rho(h_0)[$, we obtain that $\rho_2(f) \ge \rho(h_0)$. In addition:

(i) If $\lambda(1/f) < \infty$, then by Lemma 2.7 and from equation (1.4), we have $\rho_2(f) \le n$, so $\rho(h_0) \le \rho_2(f) \le n$.

(ii) For $c_j \ne 1$ $(j = 1, 2, \ldots, k-1)$, if $\lambda(f) < n$ and $\lambda(1/f) < n$, then by Theorem 1.1, we have $\rho_2(f) = n$. □

6. Proof of Theorem 1.4

Let f be a nontrivial meromorphic solution of equation (1.4). Then, by Theorem 1.2, we have $\rho(f) = \infty$.

Step 1. We consider the fixed points of $f(z)$. Let $g_0(z) = f(z) - z$, then z is a fixed point of $f(z)$ if and only if $g_0(z) = 0$. We have $g_0(z)$ is a meromorphic function and $\rho(g_0(z)) = \rho(f(z)) = \infty$. Substituting $f(z) = g_0(z) + z$ into equation (1.4), we obtain

$$\begin{aligned} &g_0^{(k)} + \big(h_{k-1}e^{P_{k-1}(z)} + d_{k-1}\big)g_0^{(k-1)} \\ &\quad + \cdots + \big(h_1 e^{P_1(z)} + d_1\big)g_0' + \big(h_0 e^{P_0(z)} + d_0\big)g_0 \\ &= -\big(h_1 e^{P_1(z)} + d_1\big) - z\big(h_0 e^{P_0(z)} + d_0\big). \end{aligned} \tag{6.1}$$

We rewrite (6.1) in the form

$$g_0^{(k)} + A_{0,k-1}g_0^{(k-1)} + \cdots + A_{0,1}g_0' + A_{0,0}g_0 = -A_{0,1} - z\, A_{0,0} = A_0. \quad (6.2)$$

For equation (6.2), we consider just meromorphic solutions of infinite order satisfying $g_0(z) = f(z) - z$. We have

$$\begin{aligned} A_0 = -A_{0,1} - z\, A_{0,0} &= -\big(h_1 e^{P_1(z)} + d_1\big) - z\big(h_0 e^{P_0(z)} + d_0\big) \\ &= -zh_0 e^{P_0(z)} - h_1 e^{P_1(z)} - d_1 - zd_0 = B_0 e^{P_0(z)} + B_1 e^{P_1(z)} + B_2, \end{aligned}$$

where $B_0 = -zh_0 \not\equiv 0$, $B_1 = -h_1$, $B_2 = -d_1 - zd_0$. Since $\deg(P_0(z) - P_1(z)) = n$, $\max\{\rho(B_0), \rho(B_1), \rho(B_2)\} < n$ and $B_0 \not\equiv 0$, according to Lemma 2.10, we have $A_0 \not\equiv 0$. By using Lemma 2.9 to equation (6.2) above, we obtain

$$\overline{\lambda}(g_0(z)) = \overline{\tau}(f) = \rho(g_0(z)) = \infty.$$

Step 2. We consider the fixed points of $f'(z)$. Let $g_1(z) = f'(z) - z$, then z is a fixed point of $f'(z)$ if and only if $g_1(z) = 0$. We have $g_1(z)$ is a meromorphic function and $\rho(g_1(z)) = \rho(f'(z)) = \rho(f(z)) = \infty$. By differentiating the both sides of equation (1.4), we obtain

$$\begin{aligned} & f^{(k+1)} + \big(h_{k-1}e^{P_{k-1}(z)} + d_{k-1}\big) f^{(k)} \\ &\quad + \big[(h_{k-1}e^{P_{k-1}(z)} + d_{k-1})' + (h_{k-2}e^{P_{k-2}(z)} + d_{k-2})\big] f^{(k-1)} \\ &\quad + \cdots + \big[(h_2 e^{P_2(z)} + d_2)' + (h_1 e^{P_1(z)} + d_1)\big] f'' \\ &\quad + \big[(h_1 e^{P_1(z)} + d_1)' + (h_0 e^{P_0(z)} + d_0)\big] f' + \big(h_0 e^{P_0(z)} + d_0\big)' f = 0. \quad (6.3) \end{aligned}$$

By equation (1.4) we have

$$\begin{aligned} f = -\frac{1}{(h_0 e^{P_0(z)} + d_0)} \big[& f^{(k)} + (h_{k-1}e^{P_{k-1}(z)} + d_{k-1}) f^{(k-1)} \\ & + \cdots + (h_1 e^{P_1(z)} + d_1) f' \big]. \quad (6.4) \end{aligned}$$

Substituting (6.4) into (6.3), we obtain

$$\begin{aligned}
f^{(k+1)} &+ \left[\left(h_{k-1}e^{P_{k-1}(z)} + d_{k-1}\right) - \frac{(h_0e^{P_0(z)} + d_0)'}{(h_0e^{P_0(z)} + d_0)}\right] f^{(k)} \\
&+ \left[\left(h_{k-1}e^{P_{k-1}(z)} + d_{k-1}\right)' + \left(h_{k-2}e^{P_{k-2}(z)} + d_{k-2}\right)\right. \\
&\qquad\qquad \left. - \frac{(h_0e^{P_0(z)} + d_0)'}{(h_0e^{P_0(z)} + d_0)}\left(h_{k-1}e^{P_{k-1}(z)} + d_{k-1}\right)\right] f^{(k-1)} \\
&+ \cdots + \left[\left(h_2e^{P_2(z)} + d_2\right)' + \left(h_1e^{P_1(z)} + d_1\right)\right. \\
&\qquad\qquad\qquad \left. - \frac{(h_0e^{P_0(z)} + d_0)'}{(h_0e^{P_0(z)} + d_0)}\left(h_2e^{P_2(z)} + d_2\right)\right] f'' \\
&+ \left[\left(h_1e^{P_1(z)} + d_1\right)' + \left(h_0e^{P_0(z)} + d_0\right)\right. \\
&\qquad\qquad\qquad \left. - \frac{(h_0e^{P_0(z)} + d_0)'}{(h_0e^{P_0(z)} + d_0)}\left(h_1e^{P_1(z)} + d_1\right)\right] f' = 0. \qquad (6.5)
\end{aligned}$$

We can denote equation (6.5) by the following shape

$$f^{(k+1)} + A_{1,k-1}f^{(k)} + A_{1,k-2}f^{(k-1)} + \cdots + A_{1,1}f'' + A_{1,0}f' = 0, \qquad (6.6)$$

where $A_{1,j}$ $(j = 0, 1, \ldots, k-1)$ are meromorphic functions defined by equation (6.5). Substituting $f'(z) = g_1(z) + z$, $f''(z) = g_1'(z) + 1$, $f^{(j+1)} = g_1^{(j)}$ $(j = 2, 3, \ldots, k)$ into equation (6.6), we obtain

$$\begin{aligned}
g_1^{(k)} &+ A_{1,k-1}g_1^{(k-1)} + A_{1,k-2}g_1^{(k-2)} + \cdots + A_{1,1}g_1' + A_{1,0}g_1 \\
&= -A_{1,1} - z\, A_{1,0} = A_1, \qquad (6.7)
\end{aligned}$$

where

$$\begin{aligned}
A_1 = -\Bigg[&\left(h_2e^{P_2(z)} + d_2\right)' + \left(h_1e^{P_1(z)} + d_1\right) \\
&- \frac{(h_0e^{P_0(z)} + d_0)'}{(h_0e^{P_0(z)} + d_0)}\left(h_2e^{P_2(z)} + d_2\right)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&- z\Big[\big(h_1 e^{P_1(z)} + d_1\big)' + \big(h_0 e^{P_0(z)} + d_0\big) \\
&\qquad\qquad - \frac{(h_0 e^{P_0(z)} + d_0)'}{(h_0 e^{P_0(z)} + d_0)}\big(h_1 e^{P_1(z)} + d_1\big)\Big] \\
&= -\frac{1}{(h_0 e^{P_0(z)} + d_0)}\big[z h_0^2 e^{2P_0(z)} + B_1 e^{P_0} + B_2 e^{P_0+P_1} \\
&\qquad\qquad + B_3 e^{P_0+P_2} + B_4 e^{P_2} + B_5 e^{P_1} + B_6\big] \\
&= -\frac{1}{(h_0 e^{P_0(z)} + d_0)}\Big[\sum_{j=0}^{6} B_j e^{G_j}\Big],
\end{aligned}
$$

G_j are polynomials defined as above, where $G_0 = 2P_0(z)$ and $B_0 = zh_0^2 \not\equiv 0$, B_j $(j = 1,2,3,4,5,6)$ are meromorphic functions of finite order which is less than n, written on the form of a sum of terms of kinds of multiplications of the functions z, h_i, h_i', P_i', d_i, d_i' $(i = 0,1,2)$. We have if $G_j = P_0$; $P_0 + P_1$; $P_0 + P_2$; P_2; P_1 then $G_j - 2P_0(z) = -P_0$; $P_1 - P_0$; $P_2 - P_0$; $P_2 - 2P_0$; $P_1 - 2P_0$. Since $a_{n,0} \neq 0$ and $\arg a_{n,j} \neq \arg a_{n,0}$ or $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$ $(j = 1,2,\ldots,k-1)$, then

$$a_{n,j} - a_{n,0} \neq 0, \quad a_{n,j} - 2a_{n,0} \neq 0 \ (j = 1,2).$$

Hence $\deg(G_j - 2P_0(z)) = \deg(2P_0(z)) = n$ $(j = 1,2,3,4,5,6)$. Since $B_0 = zh_0^2 \not\equiv 0$, then according to Lemma 2.10, we have $A_1 \not\equiv 0$. By using Lemma 2.9 to equation (6.7) above, we obtain

$$\overline{\lambda}(g_1) = \overline{\lambda}(f' - z) = \overline{\tau}(f') = \rho(g_1) = \rho(f) = \infty.$$

Step 3. We prove that $\overline{\tau}(f'') = \overline{\lambda}(f'' - z) = \infty$. Let $g_2(z) = f''(z) - z$, then z is a fixed point of $f''(z)$ if and only if $g_2(z) = 0$. We have $g_2(z)$ is a meromorphic function and $\rho(g_2(z)) = \rho(f''(z)) = \rho(f(z)) = \infty$. We just prove that $\overline{\lambda}(g_2) = \infty$. By differentiating the both sides of equation (6.6), we obtain

$$
\begin{aligned}
&f^{(k+2)} + A_{1,k-1} f^{(k+1)} + (A_{1,k-1}' + A_{1,k-2}) f^{(k)} \\
&\qquad + \cdots + (A_{1,1}' + A_{1,0}) f'' + A_{1,0}' f' = 0. \qquad (6.8)
\end{aligned}
$$

By equation (6.6) we have

$$f' = -\frac{1}{A_{1,0}}\big[f^{(k+1)} + A_{1,k-1}f^{(k)} + A_{1,k-2}f^{(k-1)} + \cdots + A_{1,1}f''\big]. \quad (6.9)$$

We remark that $A_{1,0} \not\equiv 0$, because $h_0 \not\equiv 0$ (for the proof, we can apply Lemma 2.10). Substituting (6.9) into (6.8) we obtain

$$\begin{aligned} f^{(k+2)} &+ \left[A_{1,k-1} - \frac{A'_{1,0}}{A_{1,0}}\right] f^{(k+1)} + \left[A'_{1,k-1} + A_{1,k-2} - \frac{A'_{1,0}}{A_{1,0}} A_{1,k-1}\right] f^{(k)} \\ &+ \cdots + \left[A'_{1,2} + A_{1,1} - \frac{A'_{1,0}}{A_{1,0}} A_{1,2}\right] f^{(3)} + \left[A'_{1,1} + A_{1,0} - \frac{A'_{1,0}}{A_{1,0}} A_{1,1}\right] f'' \\ &= 0. \end{aligned} \quad (6.10)$$

We can denote equation (6.10) by the following shape

$$f^{(k+2)} + A_{2,k-1}f^{(k+1)} + A_{2,k-2}f^{(k)} + \cdots + A_{2,1}f^{(3)} + A_{2,0}f'' = 0, \quad (6.11)$$

where $A_{2,j}$ ($j = 0, 1, \ldots, k-1$) are meromorphic functions defined by equation (6.10) above, and we have

$$A_{2,0} = A'_{1,1} + A_{1,0} - \frac{A'_{1,0}}{A_{1,0}} A_{1,1}, \quad A_{2,1} = A'_{1,2} + A_{1,1} - \frac{A'_{1,0}}{A_{1,0}} A_{1,2}.$$

Substituting $f''(z) = g_2(z) + z$, $f^{(3)}(z) = g'_2(z) + 1$, $f^{(j+2)} = g_1^{(j)}$ ($j = 2, 3, \ldots, k$) into equation (6.11), we obtain

$$g_2^{(k)} + A_{2,k-1}g_2^{(k-1)} + A_{2,k-2}g_2^{(k-2)} + \cdots + A_{2,1}g'_2 + A_{2,0}g_2 = A_2, \quad (6.12)$$

where

$$\begin{aligned} A_2 &= -A_{2,1} - zA_{2,0} \\ &= -\left[A'_{1,2} + A_{1,1} - \frac{A'_{1,0}}{A_{1,0}} A_{1,2}\right] - z\left[A'_{1,1} + A_{1,0} - \frac{A'_{1,0}}{A_{1,0}} A_{1,1}\right] \\ &= -\frac{1}{A_{1,0}}\big[A'_{1,2}A_{1,0} + A_{1,1}A_{1,0} - A'_{1,0}A_{1,2} \\ &\qquad + zA'_{1,1}A_{1,0} + zA_{1,0}^2 - zA'_{1,0}A_{1,1}\big]. \end{aligned} \quad (6.13)$$

We have

$$A_{1,0} = (h_1 e^{P_1(z)} + d_1)' + (h_0 e^{P_0(z)} + d_0) - \frac{(h_0 e^{P_0(z)} + d_0)'}{(h_0 e^{P_0(z)} + d_0)}(h_1 e^{P_1(z)} + d_1),$$

$$A_{1,1} = (h_2 e^{P_2(z)} + d_2)' + (h_1 e^{P_1(z)} + d_1) - \frac{(h_0 e^{P_0(z)} + d_0)'}{(h_0 e^{P_0(z)} + d_0)}(h_2 e^{P_2(z)} + d_2),$$

$$A_{1,2} = (h_3 e^{P_3(z)} + d_3)' + (h_2 e^{P_2(z)} + d_2) - \frac{(h_0 e^{P_0(z)} + d_0)'}{(h_0 e^{P_0(z)} + d_0)}(h_3 e^{P_3(z)} + d_3).$$

Therefore

$$\begin{aligned}
A_{1,0} &= \frac{1}{(h_0 e^{P_0(z)} + d_0)}\big(h_0^2 e^{2P_0} + \alpha_{1,0}^{(1)} e^{P_0} + \alpha_{1,0}^{(2)} e^{P_0+P_1} + \alpha_{1,0}^{(3)} e^{P_1} + \alpha_{1,0}^{(4)}\big),\\
A_{1,1} &= \frac{1}{(h_0 e^{P_0(z)} + d_0)}\big(\alpha_{1,1}^{(0)} e^{P_0} + \alpha_{1,1}^{(1)} e^{P_0+P_2} + \alpha_{1,1}^{(2)} e^{P_0+P_1}\\
&\qquad + \alpha_{1,1}^{(3)} e^{P_2} + \alpha_{1,1}^{(4)} e^{P_1} + \alpha_{1,1}^{(5)}\big),\\
A_{1,2} &= \frac{1}{(h_0 e^{P_0(z)} + d_0)}\big(\alpha_{1,2}^{(0)} e^{P_0} + \alpha_{1,2}^{(1)} e^{P_0+P_2} + \alpha_{1,2}^{(2)} e^{P_0+P_3}\\
&\qquad + \alpha_{1,2}^{(3)} e^{P_2} + \alpha_{1,2}^{(4)} e^{P_3} + \alpha_{1,2}^{(5)}\big)
\end{aligned}$$

and

$$\begin{aligned}
A'_{1,0} &= \frac{1}{(h_0 e^{P_0(z)} + d_0)^2}\big(\beta_{1,0}^{(0)} e^{3P_0} + \beta_{1,0}^{(1)} e^{2P_0} + \beta_{1,0}^{(2)} e^{2P_0+P_1} + \beta_{1,0}^{(3)} e^{P_0+P_1}\\
&\qquad + \beta_{1,0}^{(4)} e^{P_0} + \beta_{1,0}^{(5)} e^{P_1} + \beta_{1,0}^{(6)}\big),\\
A'_{1,1} &= \frac{1}{(h_0 e^{P_0(z)} + d_0)^2}\big(\beta_{1,1}^{(0)} e^{2P_0} + \beta_{1,1}^{(1)} e^{2P_0+P_2} + \beta_{1,1}^{(2)} e^{2P_0+P_1} + \beta_{1,1}^{(4)} e^{P_0+P_2}\\
&\qquad + \beta_{1,1}^{(5)} e^{P_0+P_1} + \beta_{1,1}^{(6)} e^{P_0} + \beta_{1,1}^{(7)} e^{P_2} + \beta_{1,1}^{(8)} e^{P_1} + \beta_{1,1}^{(9)}\big),\\
A'_{1,2} &= \frac{1}{(h_0 e^{P_0(z)} + d_0)^2}\big(\beta_{1,2}^{(0)} e^{2P_0} + \beta_{1,2}^{(1)} e^{2P_0+P_2} + \beta_{1,2}^{(2)} e^{2P_0+P_3} + \beta_{1,2}^{(4)} e^{P_0+P_2}\\
&\qquad + \beta_{1,2}^{(5)} e^{P_0+P_3} + \beta_{1,2}^{(6)} e^{P_0} + \beta_{1,2}^{(7)} e^{P_2} + \beta_{1,2}^{(8)} e^{P_3} + \beta_{1,2}^{(9)}\big),
\end{aligned}$$

where $\alpha_{i,j}^{(l)}$, $\beta_{i,j}^{(l)}$ are meromorphic functions of finite order which is less than n, written on the form of a sum of terms of kinds of multiplications of the functions h_i, h'_i, P'_i, d_i, d'_i $(i = 0, 1, 2, 3)$. From (6.13) we have

$$
\begin{aligned}
A_2 = &-\frac{1}{A_{1,0}(h_0e^{P_0(z)}+d_0)^3} \\
&\times \big[zh_0^5e^{5P_0} + B_1e^{4P_0} + B_2e^{3P_0} + B_3e^{2P_0} + B_4e^{P_0} + B_5e^{4P_0+P_1} \\
&\quad + B_6e^{4P_0+P_2} + B_7e^{4P_0+P_3} + B_8e^{3P_0+P_1} + B_9e^{3P_0+P_2} + B_{10}e^{3P_0+P_3} \\
&\quad + B_{11}e^{3P_0+P_1+P_2} + B_{12}e^{3P_0+P_1+P_3} + B_{13}e^{3P_0+2P_1} + B_{14}e^{2P_0+P_1} \\
&\quad + B_{15}e^{2P_0+P_2} + B_{16}e^{2P_0+P_3} + B_{17}e^{2P_0+2P_1} + B_{18}e^{2P_0+P_1+P_2} \\
&\quad + B_{19}e^{2P_0+P_1+P_3} + B_{20}e^{P_0+P_1} + B_{21}e^{P_0+P_2} + B_{22}e^{P_0+P_3} \\
&\quad + B_{23}e^{P_0+2P_1} + B_{24}e^{P_0+P_1+P_2} + B_{25}e^{P_0+P_1+P_3} + B_{26}e^{2P_1} + B_{27}e^{P_1} \\
&\quad + B_{28}e^{P_1+P_2} + B_{29}e^{P_1+P_3} + B_{30}e^{P_2} + B_{31}e^{P_3} + B_{32}\big] \\
= &-\frac{1}{A_{1,0}(h_0e^{P_0(z)}+d_0)^3}\left[\sum_{j=0}^{32} B_j e^{G_j}\right],
\end{aligned}
$$

G_j are polynomials defined as above, where $G_0 = 5P_0(z)$ and $B_0 = zh_0^5 \not\equiv 0$, B_j $(j = 1, 2, \ldots, 32)$ are meromorphic functions of finite order which is less than n, written on the form of a sum of terms of kinds of multiplications of the functions z, h_i, h_i', P_i', d_i, d_i' $(i = 0, 1, 2, 3)$. We certify that $A_2 \not\equiv 0$. We have

(i) if

$$G_j = 4P_0;\ 3P_0;\ 2P_0;\ P_0;\ 4P_0 + P_1;\ 4P_0 + P_2;\ 4P_0 + P_3$$

then

$$G_j - 5P_0 = -P_0;\ -2P_0;\ -3P_0;\ -4P_0;\ P_1 - P_0;\ P_2 - P_0;\ P_3 - P_0.$$

(ii) If

$$
\begin{aligned}
G_j = &\ 3P_0 + P_1;\ 3P_0 + P_2;\ 3P_0 + P_3;\ 3P_0 + P_1 + P_2;\ 3P_0 + P_1 + P_3; \\
&\ 3P_0 + 2P_1
\end{aligned}
$$

then

$$G_j - 5P_0 = P_1 - 2P_0;\ P_2 - 2P_0;\ P_3 - 2P_0;\ P_1 + P_2 - 2P_0;\\ P_1 + P_3 - 2P_0;\ 2P_1 - 2P_0.$$

(iii) If

$$G_j = 2P_0 + P_1;\ 2P_0 + P_2;\ 2P_0 + P_3;\ 2P_0 + 2P_1;\ 2P_0 + P_1 + P_2;\\ 2P_0 + P_1 + P_3$$

then

$$G_j - 5P_0 = P_1 - 3P_0;\ P_2 - 3P_0;\ P_3 - 3P_0;\ 2P_1 - 3P_0;\\ P_1 + P_2 - 3P_0;\ P_1 + P_3 - 3P_0.$$

(iv) If

$$G_j = P_0 + P_1;\ P_0 + P_2;\ P_0 + P_3;\ P_0 + 2P_1;\ P_0 + P_1 + P_2;\\ P_0 + P_1 + P_3$$

then

$$G_j - 5P_0 = P_1 - 4P_0;\ P_2 - 4P_0;\ P_3 - 4P_0;\ 2P_1 - 4P_0;\\ P_1 + P_2 - 4P_0;\ P_1 + P_3 - 4P_0.$$

(v) If

$$G_j = 2P_1;\ P_1;\ P_1 + P_2;\ P_1 + P_3;\ P_2;\ P_3$$

then

$$G_j - 5P_0 = 2P_1 - 5P_0;\ P_1 - 5P_0;\ P_1 + P_2 - 5P_0;\ P_1 + P_3 - 5P_0;\\ P_2 - 5P_0;\ P_3 - 5P_0.$$

Since $a_{n,0} \neq 0$ and $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$ $(j = 1, 2, \ldots, k-1)$, then

$$a_{n,j} - \lambda a_{n,0} = (c_j - \lambda) a_{n,0} \neq 0 \text{ for } \lambda = 1, 2, 3, 4, 5;\ j = 1, 2, 3,$$
$$2a_{n,1} - \lambda a_{n,0} = (2c_1 - \lambda) a_{n,0} \neq 0 \text{ for } \lambda = 3, 5,$$
$$a_{n,1} + a_{n,j} - \lambda a_{n,0} = ((c_1 + c_j) - \lambda) a_{n,0} \neq 0 \text{ for } \lambda = 2, 3, 4, 5;\ j = 2, 3$$

or $\arg a_{n,j} \neq \arg a_{n,0}$ ($j = 1, 2, \ldots, k-1$) and $\arg(a_{n,1} + a_{n,j}) \neq \arg a_{n,0}$ ($j = 2, 3$). Hence $\deg(G_j - 5P_0(z)) = \deg(5P_0(z)) = n$ ($j = 1, 2, \ldots, 32$). By Lemma 2.10 and the fact $B_0 = zh_0^5 \not\equiv 0$, we obtain $A_2 \not\equiv 0$. By using Lemma 2.9 to equation (6.12) above, we have

$$\overline{\lambda}(g_2) = \overline{\lambda}(f'' - z) = \overline{\tau}(f'') = \rho(g_2) = \rho(f) = \infty. \qquad \square$$

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