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Laurent decomposition for harmonic and biharmonic functions in an infinite network

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Abstract. In this article we give a decomposition for harmonic functions in an infinite network X which is similar to the Laurent decomposition of harmonic functions defined on an annulus in $\mathbb{R}^n$, $n \geq 2$. Also we give a decomposition for biharmonic functions on bihyperbolic infinite networks.

Key words: Laurent decomposition in networks, circled sets, bihyperbolic networks.

1. Introduction

The Laurent decomposition for harmonic functions in $\mathbb{R}^n$, $n \geq 2$ is well-known and is given, for example, in Axler et al. [4]: Let k be a compact set contained in an open set ω in $\mathbb{R}^n$, $n \geq 2$. Let h be a harmonic function on $\omega \backslash k$. Then, h can be written in the form $h = s + t$ on $\omega \backslash k$, where s is harmonic on $\mathbb{R}^n \backslash k$ and t is harmonic on ω. Moreover, this representation is unique if we impose the following restrictions,

(1) if $n \geq 3$, $\lim_{x \to \infty} s(x) = 0$; and
(2) if $n = 2$, $\lim_{x \to \infty}[s(x) - \alpha \log |x|] = 0$ for some constant α.

In this article we prove a similar Laurent decomposition for harmonic and biharmonic functions in an infinite network X.

2. Preliminaries

By a network X, we mean an infinite graph which is connected, locally finite and has no self loops. There is a collection of numbers $p(x, y) \geq 0$, called conductance, such that $p(x, y) > 0$ if and only if $x \sim y$ (the symbol $x \sim y$ denotes that x and y are neighbours in X). For any vertex $x \in X$, we write $p(x) = \sum_{y \in X} p(x, y)$. Since X is locally finite, $p(x)$ is finite; since X is connected, $p(x) > 0$. Note that we have not placed the restriction

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$p(x,y)=p(y,x)$ for every pair $x,y\in X$. For any subset E of X, we write $E^0=\{x : x$ and all its neighbours are in $E\}$ and $\partial E = E\setminus E^0$. E^0 is referred to as the interior of E and ∂E is referred to as the boundary of E. Potential Theory on a network is extensively studied by Yamasaki [9], [10] and Soardi [8]. It is generally assumed that the conductance is symmetric (that is $p(x,y)=p(y,x)$ for every pair of vertices x and y in X). But in this note, we do not place the restriction that $p(x,y)=p(y,x)$.

An arbitrary subset E in X is said to be *circled* if any $z\in\partial E$ has at least one neighbour in $\overset{\circ}{E}$. Let F be a finite set of vertices. Let $E_1=V(F)$ be the set consisting of F and also all vertices x in X such that x has a neighbour in F. Then E_1 is a finite set, $F\subset\overset{\circ}{E}_1$ and E_1 is circled. Define by recurrence $E_{i+1}=V(E_i)$ for $i\geq 1$. Since X is connected, any x should be in some E_i. Thus $\{E_i\}$ an increasing sequence of finite circled sets such that $F\subset\overset{\circ}{E}_i\subset_i\subset\overset{\circ}{E}_{i+1}$ for $i\geq 1$ and $X=\cup E_i$. We shall refer to $\{E_i\}$ as an exhaustion of X by finite circled sets. Example for a circled set: Let e be a fixed vertex. For any vertex x, Let $|x|$ denote the distance between e and x. Then $B_m=\{x : |x|\leq m\}$ is a circled set.

If f is a real-valued function on E, the *Laplacian* of f at any $x\in\overset{\circ}{E}$ is defined as $\Delta f(x)=\sum_{y\sim x}p(x,y)[f(y)-f(x)]$. f is said to be *superharmonic* (respectively *harmonic*, *subharmonic*) on E if $\Delta f(x)\leq 0$ (respectively $\Delta f(x)=0$, $\Delta f(x)\geq 0$) for every $x\in\overset{\circ}{E}$. A superharmonic function $f\geq 0$ on E is said to be a *potential* on E if for any subharmonic function g on E such that $g\leq f$ we have $g\leq 0$.

We say that a superharmonic function f on X is said to have the harmonic support in E if $\Delta f(x)=0$ for every $x\in X\setminus E$. If E is a finite set and if $\Delta f(x)=0$ for every $x\in X\setminus E$, then we say that f has finite harmonic support. A superharmonic function f is said to be admissible if and only if it has a harmonic minorant outside a finite set.

If there exists a nonconstant positive superharmonic function on X, then X is said to be a *hyperbolic* network. If it is not hyperbolic, then X is referred to as *parabolic*. This division corresponds to $\mathbb{R}^2$ being parabolic and $\mathbb{R}^n$, $n\geq 3$, being hyperbolic in the classical potential theory.

In a hyperbolic network, for any z, there exists a unique potential $g_z(x)$ (called the Green function of X with pole at z) such that $\Delta g_z(x)=-\delta_z(x)$. Here δ_z denotes the characteristic function of the set $\{z\}$. Similarly, in a parabolic network, for any z, there exists a unique superharmonic function $q_z(x)$ with the following properties,

(1) $q_z(x) \leq 0$ on X,
(2) $\Delta q_z(x) = -\delta_z(x)$ for every $x \in X$,
(3) $q_z(z) = 0$, and
(4) for a constant $\alpha > 0$, $q_z(x) + \alpha H_e(x)$ is bounded on X,

where e is a fixed vertex and H_e is a fixed function such that $H_e(x) \geq 0$ on X, $H_e(e) = 0$ and $\Delta H_e(x) = \delta_e(x)$ for any $x \in X$. Notice that H_e is unbounded ([1]). Here the constant α is uniquely determined if we fix z. We write $\alpha = \phi(z)$. In a parabolic network, an admissible superharmonic function q (that is, a superharmonic function q on X that has a harmonic minorant outside a finite set in X) is said to be a *pseudo-potential* [2, p. 88] if and only if its g.h.m. on $X \setminus e$ is of the form $\alpha H_e + b$, where b is a bounded harmonic function.

3. Decomposition Theorem

In order to prove the Laurent decomposition in an infinite network in the form given below, we need the following lemmas.

Lemma 3.1 *If F is circled, then $B = X \setminus \overset{\circ}{F}$ has ∂F as its boundary ∂B.*

Proof. Note $B = (X \setminus F) \cup (\partial F)$. Let $z \in \partial F$. Then, for some $y \in \overset{\circ}{F}$, $y \sim z$ (since F is circled); thus $z \in \partial F \subset B$. But a neighbour y of z is not in B; hence $z \in \partial B$. Conversely, let $b \in \partial B$. Then $b \sim a$ for some $a \in \overset{\circ}{F}$. Since $a \in \overset{\circ}{F}$ and $a \sim b$, we should have $b \in F \setminus \overset{\circ}{F}$, that is $b \in \partial F$. Consequently $\partial B = \partial F$. □

Note B is also circled. The interior of B is $X \setminus F$.

Lemma 3.2 *Let E be a circled set, $E \subset F$, and $A = F \setminus \overset{\circ}{E}$. Then $\partial A = \partial F \cup \partial E$.*

Proof. Let $z \in \partial F$. Then z has a neighbour in $X \setminus F$. Consequently $z \in A$ but a neighbour of z is outside $A \Rightarrow z \in \partial A \Rightarrow \partial F \subset \partial A$. Let $y \in \partial E$. Then y has a neighbour in $\overset{\circ}{E}$ (since E is circled). Then $y \in A$ but has a neighbour outside $A \Rightarrow y \in \partial A \Rightarrow \partial E \subset \partial A$. Hence $\partial F \cup \partial E \subset \partial A$. Let now $a \in \partial A$, then $a \in A$ and a has a neighbour b outside A, that is $b \in X \setminus F$ or $b \in \overset{\circ}{E}$. If $b \in X \setminus F$, since $a \sim b \Rightarrow a \in \partial F$ by Lemma 3.1. If $b \in \overset{\circ}{E}$, since $b \sim a$, $a \in \partial E$. Thus if $a \in \partial A$ then $a \in \partial F$ or $a \in \partial E$, that is $\partial A \subset \partial E \cup \partial F$. $\square$

Lemma 3.3 *Let X be hyperbolic and u be subharmonic on X. If there exists a potential p on X such that $u(x) \leq p(x)$ outside a finite set A, then $u(x) \leq 0$ on X.*

Proof. Since A is finite, for a large $\alpha > 1$, $u(x) \leq \alpha p(x)$ for $x \in A$. Hence $u(x) \leq \alpha p(x)$ for all $x \in X$. Since u is subharmonic and αp is a potential on X, we have $u(x) \leq 0$. $\square$

Theorem 3.4 *Let X be an infinite network. Let F be an arbitrary subset of X and E be a finite circled set such that $E \subset \overset{\circ}{F}$. Suppose h is a harmonic function on $F \setminus \overset{\circ}{E}$. Then there exists a harmonic function s on $X \setminus \overset{\circ}{E}$ and a harmonic function t on F such that $h = s - t$ on $F \setminus \overset{\circ}{E}$. Moreover,*

i) *if X is hyperbolic, then s and t are uniquely determined if we impose the restriction $|s| \leq p$ outside a finite set, where p is a positive potential on X.*
ii) *if X is parabolic, then s and t are uniquely determined up to an additive constant, if we impose the restriction that $s - \alpha H_e$ is bounded outside a finite set, for some constant α.*

Proof. Given that E is circled, $E \subset \overset{\circ}{F}$. Then by Lemma 3.2, $\partial(F \setminus \overset{\circ}{E}) = \partial F \cup \partial E$. Given also that h is harmonic on $F \setminus \overset{\circ}{E}$. Extend h on E by taking the Dirichlet solution with boundary value h on ∂E.

Let $-t(x) = h(x) + \sum_{z \in \partial E} \Delta h(z) \varphi_z(x)$ for $x \in F$ where $\varphi_z(x)$ is a superharmonic function defined on X (which is taken g_z if X is hyperbolic and q_z if X is parabolic) such that $\Delta \varphi_z(x) = -\delta_z(x)$ for all x in X.

Then $t(x)$ is harmonic in F. Write $s(x) = -\sum_{z \in \partial E} \Delta h(z) \varphi_z(x)$. Then $s(x)$ is defined on X and harmonic at every vertex $x \notin \partial E$; in particular s is harmonic at every vertex in $X \setminus E$ which is the interior set of $X \setminus \overset{\circ}{E}$. Thus s is harmonic in $X \setminus \overset{\circ}{E}$. Moreover $h(x) = s(x) - t(x)$ for any $x \in F \setminus \overset{\circ}{E}$.

Now, for the uniqueness of decomposition stated in the theorem:

i) Suppose X is hyperbolic. Then each $\varphi_z(x)$ in the expression for $s(x)$ can be taken as a potential $g_z(x)$ in X with point support $\{z\}$. Hence

$|s(x)| \leq \sum_{z \in \partial E} |\Delta h(z)| g_z(x) = p(x)$. It is clear that $p(x)$ is a potential on X. As for the uniqueness of decomposition, suppose $h = s_1 - t_1$ is another such representations with $|s_1| \leq q$ outside a finite set where q is a potential on X. Then, the function u defined on X such that

$$u = \begin{cases} s - s_1 & X \setminus \overset{\circ}{E} \\ t - t_1 & F \end{cases}$$

is a well-defined harmonic function on X such that $|u| \leq p + q$ outside a finite set. Hence by Lemma 3.3, we have $u \equiv 0$. This proves that $s = s_1$ and $t = t_1$.

ii) If X is parabolic, let $s(x) = -\sum_{z \in \partial E} \Delta h(z) \varphi_z(x)$ obtained as above is a pseudo-potential q_z on X. Let $\alpha = -\sum_{z \in \partial E} \phi(z) \Delta h(z)$. Then

$$s(x) + \alpha H_e(x) = -\sum_{z \in \partial E} [q_z(x) + \phi(z) H_e(x)] \Delta h(z).$$

Since $q_z(x) + \phi(z) H_e(x)$ is bounded, we see that $s(x) + \alpha H_e(x)$ is bounded on $X \setminus \overset{\circ}{E}$. Suppose $h(x) = s_1(x) - t_1(x)$ is another representations on $F \setminus \overset{\circ}{E}$.

$$u = \begin{cases} s - s_1 & X \setminus \overset{\circ}{E} \\ t - t_1 & F \end{cases}.$$

is a well-defined harmonic function on X. Since $u + (\alpha - \alpha_1) H_e = (s + \alpha H_e) - (s_1 + \alpha_1 H_e)$ is bounded outside a finite set, there exist a constant $M > 0$ and a finite set A such that $-M \leq u + (\alpha - \alpha_1) H_e \leq M$ on $X \setminus A$. If $\alpha > \alpha_1$, then $u \leq u + (\alpha - \alpha_1) H_e \leq M$ on $X \setminus A$, since $H_e \geq 0$. It follows that u is bounded above on X, since A is finite set. Since u is harmonic and X is parabolic, u is a constant. Similarly, we see that u is constant if $\alpha < \alpha_1$. If $\alpha = \alpha_1$, u is bounded on X. Since X is parabolic, u is constant. Therefore u is constant in any case. Since H_e is unbounded, we have $\alpha = \alpha_1$. $\square$

Corollary 3.5 *Let X be an infinite network. Suppose v is a superharmonic function outside a finite set. Then v can be written as*

i) $v = q + p_1 - p_2$ *outside a finite set if X is hyperbolic*;

ii) $v = q + b(x) + \alpha H_e$ *outside a finite set if X is parabolic,* *where q is a superharmonic function (respectively harmonic) function on X if v is superharmonic (respectively harmonic), p_1 and p_2 are potentials on X with finite harmonic support in* i); *and b is bounded harmonic function outside a finite set and α is finite (α is not unique if q is superharmonic, α is uniquely determined if q is harmonic).*

Proof. Let v be defined outside a finite set A. Choose a finite circled set E and a finite set F such that $A \subset \overset{\circ}{E} \subset F$. Let h be the Dirichlet solution in $F \setminus \overset{\circ}{E}$ with boundary values v. Replacing v by h in $F \setminus \overset{\circ}{E}$, we can assume that v is defined in $X \setminus \overset{\circ}{E}$ and harmonic on $F \setminus \overset{\circ}{E}$. Then by the above theorem, $v = s - t$ on $F \setminus \overset{\circ}{E}$ where s is harmonic on $X \setminus \overset{\circ}{E}$ and $s(x) = -\sum_{z \in \partial E} \Delta v(z) \varphi_z(x)$; and t is harmonic on F.

$$\text{Let } q = \begin{cases} -s + v & \text{on } X \setminus \overset{\circ}{E} \\ -t & \text{on } F. \end{cases}$$

Note q is well-defined on X. For in the common part $F \setminus \overset{\circ}{E}$, $s(x) - v(x) = t(x)$. Consequently, $q(x)$ is superharmonic on X (and $q(x)$ is harmonic if v is harmonic). On $X \setminus \overset{\circ}{E}$,

i) If X is hyperbolic,

$$\begin{aligned} v(x) - q(x) &= s(x) \\ &= \sum_{\partial E} [\Delta v(z)]^+ \varphi_z(x) - \sum_{\partial E} [\Delta v(z)]^- \varphi_z(x) \\ &= p_1(x) - p_2(x) \text{ on } X \setminus \overset{\circ}{E}, \end{aligned}$$

where p_1 and p_2 are potentials on X with finite harmonic support.

ii) If X is parabolic,

$$\begin{aligned} v(x) - q(x) &= s(x) \\ &= -\sum_{\partial E} \Delta v(z)[\varphi_z(x) + \phi(z) H_e(x)] + \left[\sum_{\partial E} \phi(z) \Delta v(z)\right] H_e(x) \\ &= b(x) + \alpha H_e(x) \text{ on } X \setminus \overset{\circ}{E} \text{ (see Theorem 3.4 ii)} \end{aligned}$$

where $b(x)$ is bounded harmonic function outside a finite set and α is finite. □

4. Decomposition theorem for Biharmonic functions

In a hyperbolic Riemannian manifold R, a positive kernel $Q_y(x) = Q(x, y)$ is called a biharmonic Green function with biharmonic support at y if $\Delta^2 Q_y(x) = \delta_y(x)$, $\forall x \in R$. Sario et al. [7] give different sufficient conditions for the existence of the biharmonic Green function on a hyperbolic Riemannian manifold. A similar study considering polyharmonic Green functions on an infinite tree with positive potentials is carried out in Anandam and Bajunaid [3]. In this section, we consider biharmonic functions, biharmonic Green functions etc. in an infinite network X when there are positive potentials on X and some related results when there is no positive potential on X.

In [6, Theorem 4.2], Bajunaid et al. prove the following: Let T be an infinite tree that is recurrent without any terminal vertex. Fix a vertex e. Then there exists a function K biharmonic off e such that for any function f on T which is biharmonic outside a finite set one has the representation $f = \beta K + B + L$ where β is a constant, B is biharmonic on T and L is a function for which the Laplacian is constant on all sectors sufficiently far from e. This result is analogous to the case of $\mathbb{R}^2$ in Bajunaid and Anandam [5, Theorem 16]. In what follows we consider the representation of biharmonic functions in an infinite network similar to the (Laurent) representation of biharmonic functions defined on an annulus in $\mathbb{R}^n$, $n \geq 5$ and some of its consequences.

Definition 4.1 Let X be an infinite network. A real valued function u on $E \subset X$, is said to be biharmonic on E if there exists a harmonic function h on E such that $\Delta u(x) = h(x)$ for every $x \in \overset{\circ}{E}$.

Definition 4.2 A potential q in an infinite network X is said to be a bipotential if and only if $(-\Delta)q = p$ where p is a potential in X.

Definition 4.3 For a fixed z in X, a potential $Q_z(x)$ in X is said to be biharmonic Green function with biharmonic support $\{z\}$ if and only if $(-\Delta)Q_z(x) = g_z(x)$ where $g_z(x)$ is the harmonic Green function with harmonic support z.

Remark Let $\{E_n\}$ be an exhaustion of X and E_n's are finite circled sets. Let $z \in \overset{\circ}{E}_1$. Let $g_z^{(n)}$ be the harmonic Green function on E_n. Let $\Delta Q_z^{(n)} = -g_z^{(n)}$ in $\overset{\circ}{E}_n$, where $Q_z^{(n)}$ is defined on E_n, $Q_z^{(n)} = 0$ on ∂E_n so that $Q_z^{(n)}$ is a potential on E_n. Since $g_z^{(n)} \leq g_z^{(n+1)}$ on E_n, $Q_z^{(n)} \leq Q_z^{(n+1)}$ on $\overset{\circ}{E}_n$. Let $Q_z(x) = \lim_{n\to\infty} Q_z^{(n)}(x)$ for $x \in X$. If $Q_z(x) = \infty$ for some x, then $Q \equiv \infty$. If $Q_z(x) < \infty$ for some x in X, then Q is called the biharmonic Green function on X with point singularity at z. If the biharmonic Green function exists on X, then since $Q_z^{(n)} \to Q_z$, $\Delta Q_z(x) = \lim_n \Delta Q_z^{(n)}(x) = \lim_n[-g_z^{(n)}(x)] = -g_z(x)$ for each x. (Note $\Delta Q_z^{(n)}(x) = -g_z^{(n)}(x)$ is valid for all $n \geq m$ if $x \in \overset{\circ}{E}_m$.)

Definition 4.4 An infinite network X is said to be a bihyperbolic network if there exists the biharmonic Green function on X.

Theorem 4.5 *X is a bihyperbolic network if and only if there exists a bipotential.*

Proof. Suppose the biharmonic Green function $Q_y(x)$ exists on X. Then $(-\Delta)Q_y(x) = g_y(x)$, where $g_y(x)$ is the harmonic Green function on X. Since $Q_y(x)$ and $g_y(x)$ are potentials on X, then X is a bihyperbolic network.

Conversely, let X be a bihyperbolic network. That is $(-\Delta)q = p$ where p and q are potentials on X. Let z be a fixed vertex. Then for some λ, $g_z(x) \leq \lambda p(y)$ for any $y \in X$ (Domination Principle [2, p. 26]). Now

$$\begin{aligned} \lambda q(x) &= \lambda \sum_{y\in X} g_y(x)p(y) \\ &\geq \sum_{y\in X} g_y(x) g_z(y) \\ &= Q_z(x). \end{aligned}$$

Hence $Q_z(x)$ is a well-defined potential on X such that $(-\Delta)Q_z(x) = g_z(x)$ and $(-\Delta)^2 Q_z(x) = \delta_z(x)$. □

Lemma 4.6 *Let f be a real-valued function defined on a finite set E in an infinite network X. Then there exists u on X such that $\Delta u(x) = -f(x)$ for every vertex $x \in E$.*

Proof. Let $\varphi_y(x)$ be a superharmonic function on X such that $\Delta\varphi_y(x) =$

$-\delta_y(x)$ for all $x \in E$. Let f be defined on X by giving values 0 outside E. Then $u(x) = \sum_{y \in X} f(y)\varphi_y(x)$ is a well-defined function on X such that $\Delta u(x) = -f(x)$ for all $x \in E$. $\square$

Lemma 4.7 *Let X be a bihyperbolic network. If f is any real-valued function such that $|f| \leq u$ where u is a potential on X with finite harmonic support, then there exists a uniquely determined real-valued function v such that $\Delta v = -f$ on X, and $|v|$ is dominated by a potential on X.*

Proof. Since X is a bihyperbolic network, there exist potentials p and q on X such that $\Delta q = -p$ on X. Let u be a potential with finite harmonic support A. Then $u(x) = \sum_{z \in X} g_z(x)\mu(z)$ with $\mu(z) = -\Delta u(z)$. Hence μ is non-negative and vanishes outside a finite set. We have

$$\begin{aligned}
\sum_{y \in X} |f(y)| g_y(x) &\leq \sum_{y \in X} u(y) g_y(x) \\
&= \sum_{y \in X} \Bigg(\sum_{z \in X} g_z(x) g_y(x) \Bigg) \mu(z) \\
&= \sum_{z \in X} Q_z(x) \mu(z) \\
&= s(x) < \infty
\end{aligned}$$

where $s(x)$ is a potential. Let $v(x) = \sum_{y \in X} f(y) g_y(x)$. Then $\Delta v = -f$ and $|v(x)| \leq s(x)$.

As for the uniqueness, suppose $\Delta v_1 = -f$ on X and $|v_1|$ is dominated by a potentials in X. Then $v - v_1$ is harmonic on X and is dominated by a potential in X. Consequently, $v - v_1 = 0$. $\square$

Theorem 4.8 *Let F be an arbitrary set in a bihyperbolic network X. Let E be a finite circled set such that $E \subset \overset{\circ}{F}$. Let b be a biharmonic function on $F \setminus \overset{\circ}{E}$. Then there exist a biharmonic function on $X \setminus \overset{\circ}{E}$ and a biharmonic function v on F such that $b = u - v$ on $F \setminus \overset{\circ}{E}$. We can choose u so that there exists a potential p on X such that on $X \setminus \overset{\circ}{E}$, $|u|$ and $|\Delta u|$ are dominated by p. With this restriction the decomposition of b is unique.*

Proof. Choose a finite set F_1 such that $E \subset \overset{\circ}{F_1} \subset \overset{\circ}{F}$. Since b is biharmonic on $F_1 \setminus \overset{\circ}{E}$, by the definition there exists a harmonic function h on $F_1 \setminus \overset{\circ}{E}$ such that $\Delta b(x) = h(x)$ for every x in the interior of $F_1 \setminus \overset{\circ}{E}$. Then by Theorem 3.4, there exist a harmonic function s on $X \setminus \overset{\circ}{E}$ and a harmonic function t on F_1 such that $h = s - t$ on $F_1 \setminus \overset{\circ}{E}$. Since X is a bihyperbolic (and hence a hyperbolic) network we can assume that s is defined on X by giving values 0 on $\overset{\circ}{E}$. Note that s has been chosen (Theorem 3.4) such that $|s| \leq p_0$ on X where p_0 is a potential with finite harmonic support. Hence there exists u_1 on X such that $\Delta u_1 = s$ on X (Lemma 4.7); in particular, note that for a potential p_1 on X, $|u_1| \leq p_1$ on X and u_1 is biharmonic on $X \setminus \overset{\circ}{E}$. Similarly assuming t is defined on X by giving values 0 outside F_1, there exists v_1 on X such that $\Delta v_1 = t$ on X (Lemma 4.6); note that v_1 is biharmonic on F_1. Hence for every x in the interior of $F_1 \setminus \overset{\circ}{E}$, $\Delta b(x) = h(x) = s(x) - t(x) = \Delta(u_1 - v_1)(x)$. Hence on $F_1 \setminus \overset{\circ}{E}$, $b - (u_1 - v_1)$ is a harmonic function H. Again using Theorem 3.4, write $H = u_2 - v_2$ on $F \setminus \overset{\circ}{E}$ where u_2 is harmonic on $X \setminus \overset{\circ}{E}$ dominated by a potential p_2 on X and v_2 is harmonic on F. Then $b = (u_1 + u_2) - (v_1 + v_2) = u - v$ on $F_1 \setminus \overset{\circ}{E}$, where $u = u_1 + u_2$ is biharmonic on $X \setminus \overset{\circ}{E}$ and $v = v_1 + v_2$ is biharmonic on F_1. Therefore $b = u - v$ in $F_1 \setminus \overset{\circ}{E}$ where u is biharmonic at each vertex in $X \setminus \overset{\circ}{E}$ and v is biharmonic in F_1. Define

$$v' = \begin{cases} u - b & \text{in } F \setminus \overset{\circ}{E} \\ v & \text{in } F_1 \end{cases}$$

v' is well-defined in F. For the common part $F_1 \setminus \overset{\circ}{E}$, $u - b = v$. Note that v' is biharmonic in F and $u - v' = b$ in $F \setminus \overset{\circ}{E}$. Note that on $F \setminus \overset{\circ}{E}$ $|u|$ and $|\Delta u|$ are dominated by the potential $p = p_1 + p_2$ on X.

Suppose $b = u^* - v^*$ is another such representation. Then

$$B = \begin{cases} u - u^* & \text{on } X \setminus \overset{\circ}{E} \\ v' - v^* & \text{on } F \end{cases}$$

is biharmonic on X such that $|B|$ and $|\Delta B|$ are dominated by a potential P in X, outside a finite set. By Lemma 3.3 we can assume that $|B|$ and $|\Delta B|$ are dominated by P on X. Since ΔB is harmonic and $|\Delta B| \leq P$ on X, then $\Delta B = 0$ and hence B is harmonic on X; and $|B| \leq P$ so that $B = 0$. Thus $u = u^*$ and $v' = v^*$. □

Remark Let X be an infinite tree, parabolic or hyberbolic. Assume that every non-terminal vertex in X has at least two non-terminal vertices as neighbours. Then given any harmonic function h on a subset E of X (that is h is defined on E and $\Delta h = 0$ on $\overset{\circ}{E}$) there exists a harmonic function u on X such that $u = h$ on E ([2, p. 113]). Consequently, given any real-valued function $f \geq 0$ on X, there exists a superharmonic function s on X such that $\Delta s = -f$ on X ([2, Theorem 5.1.4]). In this case, we prove the following: Let F be an arbitrary set in an infinite tree X in which every non-terminal vertex has at least two non-terminal neighbours. There may or may not be positive potentials on X. Let E be a finite circled set, $E \subset \overset{\circ}{F}$. Let b be a biharmonic function defined on $F \setminus \overset{\circ}{E}$. Then there exist a biharmonic function u on $X \setminus \overset{\circ}{E}$ and a biharmonic function v on F such that $b = u - v$ on $F \setminus \overset{\circ}{E}$.

Corollary 4.9 *Let X be a bihyperbolic network. Let b be a biharmonic function defined outside a finite set in X. Then there exists a unique biharmonic function B on X such that if $f = b - B$, then $|f|$ and $|\Delta f|$ are bounded by a potential p in X.*

Proof. Replace F by X, u by f and v by $-B$ in Theorem 4.8 we can get a biharmonic function B on X such that if $f = b - B$ then $|f|$ and $|\Delta f|$ are bounded by a potential p in X.

As for the uniqueness, note that if B_1 is another such that biharmonic function in X, then $|(B - B_1)|$ and $|\Delta(B - B_1)|$ are bounded by a potential in X outside a finite set. It leads to the conclusion $B = B_1$, as shown in the proof of Theorem 4.8. □

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