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Splitting mixed groups of torsion-free finite rank II

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Abstract. First we introduce the concept of QD-hulls in arbitrary abelian groups. Then we use the concept to give a new characterization of purifiable torsion-free finite rank subgroups of an arbitrary abelian group. Finally we use it to formulate a splitting criterion for mixed groups of torsion-free finite rank.

Key words: Splitting mixed group, QD-hull, purifiable subgroup, vertical subgroup, N-high subgroup, n-th p-overhang.

1. Introduction

It is well-known that there are three main classes of abelian groups: torsion, torsion-free and mixed. Here we study mixed abelian groups of torsion-free finite rank. The basic problem for mixed groups is the splitting problem, i.e., if the maximal torsion subgroup is a direct summand or not. This problem together with lots of variations is an unsettled issue.

In 1917, Levi constructed non-splitting abelian groups, and later Baer partially solved the splitting problem. Numerous authors studied many variations of the splitting problem. Stratton solved the splitting problem for mixed groups of torsion-free rank 1, cf. [6]. Moreover, he studied the splitting problem for torsion-free finite rank modules over discrete valuation rings, cf. [7]. Using the concept of purifiable subgroups, Okuyama presented a splitting criterion for groups of torsion-free rank 1, cf. [3], and later for groups of torsion-free finite rank, cf. [5]. However, those techniques did not apply to countable rank.

Here we first introduce the concept of QD-hulls in abelian groups and use it to develop a new characterization of purifiable torsion-free finite rank subgroups of an abelian group and obtain a splitting criterion for mixed abelian groups of torsion-free finite rank. This improved technique might apply to countable rank to lead to a splitting criterion.

The terminology and notation here, unless explicitly stated, follow Fuchs [1]. Throughout this article, **P** denotes the set of primes, **Z** the ring of

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integers and p a prime.

2. Preliminaries

We recall definitions and properties mentioned in [2] and [4]. We frequently use them in this article. First we recall N-high subgroups.

Definition 2.1 Let N be a subgroup of a group G. Then a subgroup A of G is said to be ***N-high*** in G if A is maximal with respect to $A \cap N = 0$.

The existence of N-high subgroups is guaranteed by Zorn's lemma.

Definition 2.2 A subgroup A of a group G is said to be ***p-neat*** in G if $A \cap pG = pA$. If A is p-neat in G for every $p \in \mathbf{P}$, then A is called ***neat*** in G.

Proposition 2.3 *Let N be a subgroup of a group G. Then a subgroup A of G is N-high in G if and only if*

(1) $A \cap N = 0$,
(2) *A is neat in G,*
(3) *$G[p] = A[p] \oplus N[p]$ for every $p \in \mathbf{P}$, and*
(4) *$G/(A \oplus N)$ is torsion.*

For a proof, see [4, Proposition 2.2].

Corollary 2.4 *A torsion-free subgroup A of a group G is $T(G)$-high in G if and only if*

(1) *A is neat in G and*
(2) *G/A is torsion.*

Proposition 2.5 *Let G be a group. If a $T(G)$-high subgroup H of G is pure in G, then G is splitting.*

Proof. Let H be $T(G)$-high and pure. Then $H \oplus T(G) \subseteq G$. Let $x \in G$. Since $G/(H \oplus T(G))$ is a torsion group by Proposition 2.3(4), there exists $(0 \leq)$ $n \in \mathbf{Z}$ such that $nx \in H \oplus T(G)$. Without loss of generality $nx \in H$ because T is torsion. By purity, there is $y \in H$ such that $nx = ny$. This means that $x - y \in T(G)$ and hence $x \in H + T(G)$. We have $G = H \oplus T(G)$. □

Next we recall almost-dense subgroups.

Definition 2.6 A subgroup A of a group G is said to be ***p-almost-dense*** in G if, for every p-pure subgroup K of G containing A, the torsion part of G/K is p-divisible. Moreover, A is said to be ***almost dense*** in G if A is p-almost-dense in G for every $p \in \mathbf{P}$.

Proposition 2.7 ([2, Proposition 1.3, Proposition 1.4]) *The following are equivalent*:

(1) *A is p-almost-dense [almost dense] in a group G*;
(2) *for all* $(0 \leq)$ $n \in \mathbf{Z}$ [*and all* $p \in \mathbf{P}$],

$$p^n G[p] \subseteq A + p^{n+1} G.$$

Next we recall n-th p-overhang.

Definition 2.8 Let G be a group and A a subgroup of G. For all $(0 \leq)$ $n \in \mathbf{Z}$, we define the ***n-th p-overhang*** of A in G to be the vector space

$$V_{p,n}(G, A) = \frac{(A + p^{n+1}G) \cap p^n G[p]}{(A \cap p^n G)[p] + p^{n+1}G[p]}.$$

The subgroup A is said to be ***p-vertical*** in G if $V_{p,n}(G, A) = 0$ for all $(0 \leq)$ $n \in \mathbf{Z}$. The subgroup A is said to be ***vertical*** in G if A is p-vertical in G for all $p \in \mathbf{P}$.

It is convenient to use the following notations for the numerator and the denominator of $V_{p,n}(G, A)$:

$$A_G^n(p) = (A + p^{n+1}G) \cap p^n G[p] = ((A \cap p^n G) + p^{n+1}G)[p]$$
$$A_n^G(p) = (A \cap p^n G)[p] + p^{n+1}G[p].$$

We immediately obtain the following properties.

Proposition 2.9 *Let G and A be as in Definition* 2.8. *Then the following hold*:

(1) *for any $x \in A_G^n(p) \setminus A_n^G(p)$, we have*

$$h_p(x) = n;$$

(2) *if $x \in A_G^n(p)$, then $h_p^{G/A}(x + A) > n$*;

(3) *if* A *is* p*-almost-dense in* G*, then* $A + p^{n+1}G \supseteq p^nG[p]$*, so* $A_G^n(p) = p^nG[p]$;
(4) *if* A *is torsion-free, then* $A_n^G(p) = p^{n+1}G[p]$;
(5) *if* A *is torsion-free and* p*-almost-dense in* G*, then*

$$V_{p,n}(G, A) = \frac{p^nG[p]}{p^{n+1}G[p]};$$

hence $\dim V_{p,n}(G, A)$ *is the* n*th Ulm-Kaplansky invariant of* G_p;
(6) *if* $G_p = 0$*, then* A *is* p*-vertical in* G.
(7) [2, Lemma 4.1(1)] $V_{p,m+n}(G, A) = V_{p,n}(p^mG, A \cap p^mG)$ *for all* n, $m \geq 0$.

Proposition 2.10 ([2, Proposition 2.2]) *Let* G *be a group and* A *a subgroup of* G*. For* p*-pure subgroup* K *of* G *containing* A,

$$V_{p,n}(G, A) \cong V_{p,n}(K, A)$$

for all $(0 \leq)$ $n \in \mathbf{Z}$.

We present a useful property for verticality.

Proposition 2.11 ([2, Proposition 2.7]) *Let* G *be a group and* A *a subgroup of* G*. Then the following properties are equivalent*:

(1) A *is* p*-vertical in* G;
(2) $(A + p^nG)[p] = A[p] + p^nG[p]$ *for all* $(0 \leq)$ $n \in \mathbf{Z}$.

Next we recall purifiable subgroups.

Definition 2.12 Let G be a group. A subgroup A of G is said to be ***p-purifiable***[***purifiable***] in G if, among the p-pure[pure] subgroups of G containing A, there exists a minimal one. Such a minimal p-pure[pure] subgroup is called a ***p-pure***[***pure***] ***hull*** of A.

We give a relationship between p-purifiability and purifiability.

Proposition 2.13 ([2, Theorem 1.12]) *Let* G *be a group and* A *a subgroup of* G*. Then* A *is purifiable in* G *if and only if, for all* $p \in \mathbf{P}$, A *is* p*-purifiable in* G.

Proposition 2.10 leads to the following intrinsic necessary condition for p-purifiability.

Proposition 2.14 ([2, Proposition 2.3]) *If a subgroup of a group G is p-purifiable in G, then there exists $(0 \leq)\ m \in \mathbf{Z}$ such that $V_{p,n}(G, A) = 0$ for all $n \geq m$.*

Proposition 2.14 and Proposition 2.13 led to the following property.

Proposition 2.15 ([2, Proposition 2.3]) *If a subgroup of a group G is purifiable in G, then, for every $p \in \mathbf{P}$, there exists $(0 \leq)\ m_p \in \mathbf{Z}$ such that $V_{p,n}(G, A) = 0$ for all $n \geq m_p$.*

Proposition 2.16 ([2, Theorem 4.1]) *Let G be a group and A a subgroup of G. Then the following hold.*

(1) *If A is p-purifiable in G, then $A \cap p^n G$ is p-purifiable in $p^n G$ for all $(0 \leq)\ n \in \mathbf{Z}$.*
(2) *If $A \cap p^m G$ is p-purifiable in $p^m G$ for some $(0 \leq)\ m \in \mathbf{Z}$, then A is p-purifiable in G.*

We recall structures of p-pure hulls and pure-hulls.

Proposition 2.17 ([2, Theorem 1.8, Theorem 1.11]) *Let G be a group and A a subgroup of G. Let H be a p-pure [pure] subgroup of G containing A. Then H is a p-pure [pure] hull of A in G if and only if the following three conditions are satisfied*;

(1) *for all $(0 \leq)\ n \in \mathbf{Z}$ [and all $p \in \mathbf{P}$], $p^n H[p] \subseteq A + p^{n+1} H$;*
(2) *H/A is p-primary [torsion];*
(3) *[for every $p \in \mathbf{P}$], there exists $(0 \leq)\ m_p \in \mathbf{Z}$ such that $p^{m_p} H[p] \subseteq A$.*

For p-purifiable [purifiable] torsion-free subgroup, we have the following.

Proposition 2.18 ([5, Corollary 2.12]) *Let G be a group and A a subgroup of G. Suppose that A is p-purifiable[purifiable] torsion-free in G. Let H be a p-pure[pure] hull of A in G. Then the following are equivalent*:

(1) *A is p-vertical[vertical] in G*;
(2) *H is torsion-free.*

Proposition 2.19 ([5, Proposition 3.2]) *Let G be a group and A a subgroup of G. For every $p \in \mathbf{P}$, let $G^{(p)}/A = (G/A)_p$. Then the following hold.*

(1) *$G^{(p)}$ is p-pure in G.*

(2) *Suppose that A is p-purifiable torsion-free in G. Let H be a p-pure hull of A in G. Then $G^{(p)} = H \oplus K$ where K is a subgroup of $G^{(p)}$.*

Proposition 2.20 ([5, Lemma 3.5]) *Let F be a torsion-free group and B a sugroup of F. Suppose that F/B is a p-group. Then* $\dim(F/B)[p] \leq rk(F)$.

3. QD hulls

Let G be a group, A a torsion-free subgroup of G and E/A the maximal divisible subgroup of $T(G/A)$. Let D/A be a $(T(E)\oplus A)/A$-high subgroup of E/A. Then D/A is torsion divisible and D is torsion-free. Such a subgroup D is called a ***QD hull of A in G***. For $p \in \mathbf{P}$, let $E^{(p)}/A = (E/A)_p$. Let $C^{(p)}/A$ be a $(T(E^{(p)}) \oplus A)/A$-high subgroup of $E^{(p)}/A$. Then $C^{(p)}/A$ is a divisible p-group and $C^{(p)}$ is torsion-free. Such a subgroup $C^{(p)}$ is called a ***p-QD hull of A in G***.

From definitions and Proposition 2.3, the following hold.

Proposition 3.1 *Let G be a group and A a torsion-free subgroup of G. Then the following hold.*

(1) *Let D be a QD hull of A in G. Then D is a maximal torsion-free subgroup of G containing A such that D/A is torsion divisible. Further, for every $p \in \mathbf{P}$, let $D^{(p)}/A = (D/A)_p$. Then $D^{(p)}$ is a p-QD hull of A in G for all $p \in \mathbf{P}$.*
(2) *For every $p \in \mathbf{P}$, let $C^{(p)}$ be a p-QD hull of A in G. Then $C^{(p)}$ is a maximal torsion-free subgroup of G containing A such that $C^{(p)}/A$ is a divisible p-group. Further, $\sum_{p\in\mathbf{P}} C^{(p)}$ is a QD hull of A in G.*

Now we mention about p-QD hulls of p-vertical subgroups.

Lemma 3.2 *Let G be a group, A a torsion-free subgroup of G and E/A the maximal divisible subgroup of $T(G/A)$. Then the following hold.*

(1) *If A is p-vertical in G, then E is p-vertical in G. Hence all p-QD hulls of A and all QD-hulls of A in G are p-vertical in G. Moreover, $E[p] \subseteq p^\omega G[p]$.*
(2) *If A is vertical in G, then E is vertical in G. Hence all p-QD hulls of A and all QD-hulls of A in G are vertical in G.*

Proof. (1) Suppose that A is p-vertical in G. By Proposition 2.11, we have

$(A + p^n G)[p] = p^n G[p]$ for every $(0 \leq)\ n \in \mathbf{Z}$. Let $x \in (E + p^n G)[p]$. Then we can write $x = d + p^n g$ for some $d \in E$ and $g \in G$. Since E/A is divisible, there exist $d' \in E$ and $a \in A$ such that $d = p^n d' + a$ and so

$$x = a + p^n(d' + g) \in (A + p^n G)[p] = p^n G[p].$$

Hence E is p-vertical in G. Let E' be a subgroup of E. Then

$$(E' + p^n G)[p] \subseteq (E + p^n G)[p] = p^n G[p].$$

Hence E' is p-vertical in G. Let $y \in E[p]$. For every $(0 \leq)\ n \in \mathbf{Z}$, there exist $a_n \in A$ and $d_n \in E$ such that $y = a_n + p^n d_n$. Since A is p-vertical in G, we have

$$y = a_n + p^n d_n \in (A + p^n G)[p] = p^n G[p].$$

Hence $y \in p^\omega G[p]$. (2) By Definition 2.8 and (1), it is obvious. □

Next we mention about purifiability of p-QD hulls.

Lemma 3.3 *Let G be a group, A a p-vertical torsion-free subgroup of G and D a p-QD hull of A in G. If D is p-purifible in G, then A is p-purifiable in G.*

Proof. Let K be a p-pure hull of D in G. By Lemma 3.2, D is p-vertical in G. Hence, by Proposition 2.18, K is torsion-free. Hence A is p-purifiable in G, because all subgroups are purifiable in torsion-free groups. □

Lemma 3.4 *Let G be a group and A a p-vertical torsion-free subgroup of G. If A is p-purifiable in G, then all p-QD hulls of A in G are p-purifiable in G.*

Proof. Let $G^{(p)}/A = (G/A)_p$, E/A the maximal divisible subgroup of $G^{(p)}/A$ and D a p-QD hull of A in G. Then there exists a divisible subgroup D'/A of E/A such that

$$E/A = D/A \oplus D'/A,$$

$$(D'/A)[p] = ((T(E) \oplus A)/A)[p] = (E[p] \oplus A)/A. \tag{3.5}$$

Suppose that A is p-purifiable in G and let H be a p-pure hull of A in

G. Since A is p-vertical in G, H is torsion-free by Proposition 2.18. By Proposition 2.19(2), we have

$$G^{(p)} = H \oplus G_p.$$

Let D_1/A be the maximal divisible subgroup of H/A. Then we have

$$H/A = D_1/A \oplus H_1/A, \quad H_1/A < H/A$$

and

$$G^{(p)}/A = D_1/A \oplus H_1/A \oplus (G_p \oplus A)/A. \tag{3.6}$$

We will prove that

$$G^{(p)}/A = D/A \oplus H_1/A \oplus (G_p \oplus A)/A.$$

Since H_1/A is a reduced p-group, we have

$$D/A \cap (H_1/A \oplus (G_p \oplus A)/A) = 0. \tag{3.7}$$

Let $\overline{x} \in (G^{(p)}/A)[p]$. By (3.6), we can write

$$\overline{x} = \overline{d_1} + \overline{h_1} + \overline{g_0},$$
$$\overline{d_1} \in (D_1/A)[p], \quad \overline{h_1} \in (H_1/A)[p], \quad \overline{g_0} \in (G[p] \oplus A)/A.$$

By (3), we have $\overline{d_1} = \overline{d} + \overline{d'}$ for some $\overline{d} \in (D/A)[p]$, $\overline{d'} \in (E[p] \oplus A)/A$. It follows that

$$\overline{x} = \overline{d} + \overline{h_1} + \overline{d'} + \overline{g_0},$$
$$\overline{d} \in (D/A)[p], \quad \overline{h_1} \in (H_1/A)[p], \quad \overline{d'} + \overline{g_0} \in (G[p] \oplus A)/A.$$

Hence we have

$$(G^{(p)}/A)[p] = (D/A)[p] \oplus (H_1/A)[p] \oplus (G[p] \oplus A)/A. \tag{3.8}$$

By (3.6),

$$H_1/A \oplus (G_p \oplus A)/A \text{ is pure in } G^{(p)}/A. \tag{3.9}$$

By (3.7), (3.8), (3.9), and Proposition 2.3, $(H_1/A) \oplus (G_p \oplus A)/A$ is D/A-high in $G^{(p)}/A$. Since D/A is divisible, we have

$$G^{(p)}/A = D/A \oplus H_1/A \oplus (G_p \oplus A)/A.$$

Hence we also have

$$G^{(p)} = (D + H_1) \oplus G_p.$$

Note that $G^{(p)}$ is p-pure in G by Proposition 2.19(1). Then, by Proposition 2.17, the subgroup $D + H_1$ is a p-pure hull of D in G. □

Proposition 3.10 *Let G be a group and A a vertical torsion-free subgroup of G. Then the following hold.*

(1) *If there exists a QD hull of A in G that is purifiable in G, then A is purifiable in G.*
(2) *If A is purifiable in G, then all QD hulls of A in G are purifiable in G.*

Proof. (1) Let D be a QD hull of A in G and suppose that D is purifiable in G. Let L be a pure hull of D in G. Then, by Lemma 3.2(2) and Proposition 2.18, L is torsion-free. Hence A is purifiable in G, because all subgroups are purifiable in torsion-free groups.

(2) Suppose that A is purifiable in G and let D be a QD hull of A in G. By Lemma 3.2, D is vertical in G. For every $p \in \mathbf{P}$, let $D^{(p)}/A = (D/A)_p$. By Proposition 3.1, $D^{(p)}$ is a p-QD hull of A in G and by Proposition 2.13, A is p-purifiable in G. By Lemma 3.4, $D^{(p)}$ is p-purifiable in G. Let $H^{(p)}$ be a p-pure hull of $D^{(p)}$ in G. Then, by Proposition 2.18, $H^{(p)}$ is torsion-free. Let $p^n g \in H^{(p)} + D$. Then we have $p^n g = h + d$ for some $h \in H^{(p)}$ and $d \in \sum_{p \neq q \in \mathbf{P}} D^{(q)}$. Since $(\sum_{p \neq q \in \mathbf{P}} D^{(q)})/A$ is p-divisible, we have $d = p^n d' + a$ for some $d' \in \sum_{p \neq q \in \mathbf{P}} D^{(q)}$ and $a \in A$. Thus

$$p^n g - p^n d' = h + a \in H^{(p)} \cap p^n G = p^n H^{(p)}.$$

Hence $p^n g \in p^n(H^{(p)} + D)$ and so $H^{(p)} + D$ is p-pure in G. It is obvious that $H^{(p)} + D$ is torsion-free. Since $(H^{(p)} + D)/D$ is a p-group, $H^{(p)} + D$ is a p-pure hull of D in G by Proposition 2.17. Hence D is p-purifiable in G

for all $p \in \mathbf{P}$ and by Proposition 2.13, D is purifiable in G. □

4. Splitting groups of torsion-free finite rank

Definition 4.1 Let G be a group and A a subgroup of G. A is said to be ***eventually p-pure*** in G if there exists a nonnegative integer m such that $A \cap p^m G$ is p-pure in $p^m G$.

Lemma 4.2 *Let G be a group and A a torsion-free finite rank subgroup of G. Suppose that A is p-vertical in G. Then A is p-purifiable in G if and only if all p-QD hulls of A in G are eventually p-pure in G.*

Proof. Let D be any p-QD hull of A in G. By Lemma 3.2, D is p-vertical in G. By Lemma 3.3 and Lemma 3.4, it suffices to prove that D is p-purifiable in G if and only if D is eventually p-pure in G. Suppose that D is p-purifiable in G. Let H be a p-pure hull of D in G. Then, by Proposition 2.18, H is torsion-free. Let $G^{(p)}/A = (G/A)_p$, Then, by Proposition 2.19(2), we have

$$G^{(p)} = H \oplus G_p.$$

Since H/D is a reduced p-group and D is of finite rank, H/D is finite by Proposition 2.20. Let $p^m(H/D) = 0$. Then $D \cap p^m G = p^m H$ is p-pure in $p^m G$ and so D is eventually p-pure in G. Conversely, suppose that D is eventually p-pure in G. By Definition 4.1 and Proposition 2.16(2), D is p-purifiable in G. □

We use Lemma 4.2 to characterize torsion-free finite rank subgroups to be p-purifiable in a given group.

Theorem 4.3 *Let G be a group and A a torsion-free finite rank subgroup of G. Then A is p-purifiable in G if and only if there exists $(0 \leq)$ $m \in \mathbf{Z}$ satisfying the following two conditions.*

(1) *$V_{p,n}(G, A) = 0$ for all $n \geq m$.*
(2) *All p-QD hulls of $A \cap p^m G$ in $p^m G$ are eventually p-pure in $p^m G$.*

Proof. Suppose that A is p-purifiable in G. By Proposition 2.14, there exists $(0 \leq)$ $m \in \mathbf{Z}$ such that $V_{p,n}(G, A) = 0$ for all $n \geq m$. Then $A \cap p^m G$ is p-vertical in $p^m G$ and by Proposition 2.16(1), $A \cap p^m G$ is p-purifiable in $p^m G$. Hence, by Lemma 4.2, all p-QD hulls of $A \cap p^m G$ in $p^m G$ are eventually p-pure in $p^m G$. Conversely, the two conditions are satisfied. By

(1), $A \cap p^m G$ is p-vertical in $p^m G$ and by (2) and Lemma 4.2, $A \cap p^m G$ is p-purifiable in $p^m G$. Hence, by Proposition 2.16(2), A is p-purifiable in G. $\square$

By Proposition 2.13 and Proposition 2.15, we obtain the following.

Corollary 4.4 *Let G be a group and A a torsion-free finite rank subgroup of G. Then A is purifiable in G if and only if, for every $p \in \mathbf{P}$, there exists $(0 \leq)\ m_p \in \mathbf{Z}$ satisfying the following two conditions.*

(1) $V_{p,n}(G, A) = 0$ *for all* $n \geq m_p$.
(2) *All p-QD hulls of $A \cap p^{m_p} G$ in $p^{m_p} G$ are eventually p-pure in $p^{m_p} G$.*

Next we consider splitting mixed groups of torsion-free finite rank. First we give a useful lemma.

Lemma 4.5 *Let G be a group and A a subgroup of G. Suppose that D is a subgroup of G containing A such that D/A is torsion. Let $D^{(p)}/A = (D/A)_p$. Then D is eventually p-pure in G if and only if $D^{(p)}$ is eventually p-pure in G.*

Proof. Suppose that D is eventually p-pure in G. Then, by Definition 4.1, there exists $(0 \leq)\ m \in \mathbf{Z}$ such that $D \cap p^m G$ is p-pure in $p^m G$. We will prove that $D^{(p)} \cap p^m G$ is p-pure in $p^m G$. Let $x \in D^{(p)} \cap p^{m+n} G$ for some $(0 \leq)\ n \in \mathbf{Z}$. Since $D^{(p)} \cap p^{m+n} G \subseteq D \cap p^{m+n} G = p^n (D \cap p^m G)$, we can write $x = p^n d$ for some $d \in D \cap p^m G$. Then we have $d = d_1 + d_2$ for some $d_1 \in D^{(p)}$ and $d_2 \in \sum_{p \neq q \in \mathbf{P}} D^{(q)}$. Since $p^n d_2 = p^n d - p^n d_1 \in D^{(p)} \cap \sum_{p \neq q \in \mathbf{P}} D^{(q)} = A$ and $(\sum_{p \neq q \in \mathbf{P}} D^{(q)}/A)_p = 0$, we have $d_2 \in A$ and so $d \in D^{(p)} \cap p^m G$. Conversely, suppose that $D^{(p)}$ is eventually p-pure in G. There exists $(0 \leq)\ t \in \mathbf{Z}$ such that $D^{(p)} \cap p^t G$ is p-pure in $p^t G$. Let $p^{t+n} g \in D$, $g \in G$ for some $(0 \leq n) \in \mathbf{Z}$. Then we have $p^{t+n} g = g_1 + g_2$ for some $g_1 \in p^{(p)}$ and $g_2 \in \sum_{p \neq q \in \mathbf{P}} D^{(q)}$. Since $(\sum_{p \neq q \in \mathbf{P}} D^{(q)})/A$ is p-divisible, we can write $g_2 = p^{t+n} g_2' + a$ for some $g' \in \sum_{p \neq q \in \mathbf{P}} D^{(q)}$ and $a \in A$. Then we have $p^{t+n}(g - g_2') = g_1 + a \in D^{(p)} \cap p^{t+n} G = p^n (D^{(p)} \cap p^t G)$ and so $p^{t+n} g = p^n (g_1' + p^t g_2')$ for some $g_1' \in D^{(p)} \cap p^t G$. Since $g_1' + p^t g_2' \in D \cap p^t G$, it follows that $p^{t+n} g \in p^n (D \cap p^t G)$. $\square$

We characterize vertical torsion-free finite rank subgroups to be purifiable in a given group.

Corollary 4.6 *Let G be a group and A a torsion-free finite rank subgroup of G. Suppose that A is vertical in G. Then A is purifiable in G if and only if all QD hulls of A in G are eventually p-pure in G for every $p \in \mathbf{P}$.*

Proof. Suppose that A is purifiable in G. Let D be any QD hull of A in G and $D^{(p)}/A = (D/A)_p$ for every $p \in \mathbf{P}$. By Proposition 3.1, $D^{(p)}$ is a p-QD hull of A in G. Since A is p-purifiable in G for all $p \in \mathbf{P}$ by Proposition 2.13, $D^{(p)}$ is eventually p-pure in G by Lemma 4.2. Then, by Lemma 4.5, D is eventually p-pure in G for all $p \in \mathbf{P}$. Conversely, suppose that all QD hulls D' of A in G are eventually p-pure in G for all $p \in \mathbf{P}$. Then, by Proposition 2.16(2), D' is p-purifiable in G for all $p \in \mathbf{P}$ and by Proposition 2.13, D' is purifiable in G. Hence, by Proposition 3.10, A is purifiable in G. □

We need to recall definition of full free subgroups.

Definition 4.7 Let G be a group. A subgroup A of G is said to be *full free* in G if A is free and G/A is torsion.

Now we give an improved characterization of splitting mixed groups of torsion-free finite rank.

Corollary 4.8 *Let G be a mixed group of torsion-free finite rank. Then G is splitting if and only if there exists a full free subgroup A of G satisfying the following two conditions.*

(1) *A is vertical in G.*
(2) *All QD hulls of A in G are eventually p-pure in G for all $p \in \mathbf{P}$.*

Proof. Suppose that G is splitting. Then we have $G = T(G) \oplus F$ for some torsion-free subgroup F of G. Let A be a full free subgroup of F. Then, by Proposition 2.17, F is a pure hull of A in G and by Proposition 2.18, A is vertical in G. Further, since A is purifiable in G, (2) is satisfied by Corollary 4.6. Conversely, suppose that there exists a full free subgroup A of G satisfying the two conditions. By Corollary 4.6, A is purifiable in G. Let H be a pure hull of A in G. By (1) and Proposition 2.18, H is torsion-free and by Corollary 2.4, H is a $T(G)$-high subgroup of G. Therefore, by Proposition 2.5, $G = H \oplus T(G)$. □

Remark 4.9 Let G be a group with $p^\omega G[p] = 0$ and A a p-vertical torsion-free subgroup of G. Let $G^{(p)}/A = (G/A)_p$ and E/A the maximal divisible subgroup of $T(G^{(p)}/A)$. By Lemma 3.2, E is torsion-free. Hence E is the unique p-QD-hull of A in G.

By Corollary 4.8 and Remark 4.9, we obtain the following result about splitting mixed groups whose maximal torsion subgroup are separable.

Corollary 4.10 *Let G be a mixed group of torsion-free finite rank whose maximal torsion subgroup is separable. Then G is splitting if and only if there exists a full free subgroup A of G satisfying the following two conditions.*

(1) *A is vertical in G.*
(2) *Let D/A the maximal divisible subgroup of $T(G/A)$. Then D is eventually p-pure in G for all $p \in \mathbf{P}$.*

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