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Perturbation of embedded eigenvalues: A general class of exactly soluble models in Fock spaces*

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ABSTRACT: Perturbation problem of embedded eigenvalues is considered for operators with infinite degrees of freedom acting in the tensor product of $L^2(\boldsymbol{R})$ and the Boson Fock space over a Hilbert space $\mathscr{H}$. A general class of operators for which the problem is "exactly soluble" is constructed. In the case $\mathscr{H}=L^2(\boldsymbol{R}^d)$, the class contains the Hamiltonians of standard models of a one dimensional quantum harmonic oscillator coupled quadratically to a quantum scalar field on the $d+1$ dimensional spacetime and gives a unification for those models.

I. Introduction

Perturbation problem of embedded eigenvalues for operators with infinite degrees of freedom arises in quantum field theory (QFT) in a natural way and to solve the problem is of great importance. For example, the radiation theory including the theory of the Lamb shift and the shape of spectral lines of an atom, which is usually formulated in terms of quantum electrodynamics with a formal perturbation theory, may be regarded as one of the most interesting examples for such perturbation problem (see, e. g., [10, 12, 19] for the physical aspects and [1, 4, 18] for some attempts to construct a mathematically rigorous theory). Besides applications to QFT, the problem has mathematical interests also in its own right as an extension of perturbation theory of embedded eigenvalues in the case of operators with finite degrees of freedom, typically finite dimensional Schrödinger operators. In fact, Hamiltonians arising in QFT may be considered as infinite dimensional Schrödinger operators.

Experiences in the case of finite dimensional Schrödinger operators (e. g., [24]) and in some quantum field models [2, 3, 4] show that embedded eigenvalues have a tendency to be unstable in the sense that they disappear under perturbation. On the other hand, we have a counter exam-

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ple [5] in the case of infinite degrees of freedom, in which the instability of embedded eigenvalues depends on the range of the parameters of the relevant operator (In this example, we note also that, after a "renormalization" of a parameter, the embedded eigenvalues become unstable independently of the range of the parameters. The same kind of phonomenon occurs also in the quantum field models considered in [2, 4]. These facts suggest that, in the case of infinite degrees of freedom, any relation between "renormalization" and instability of embedded eigenvalues may exist). In this situation, it is interesting to ask of to what extent the instability of embedded eigenvalues is general in the case of infinite degrees of freedom.

In this paper we consider the problem with operators in the Hilbert space

$$(1.1)\qquad \mathscr{F}=L^2(\boldsymbol{R})\otimes\mathscr{F}_s(\mathscr{H}),$$

where $\mathscr{F}_s(\mathscr{H})$ is the Boson (or symmetric) Fock space over a Hilbert space $\mathscr{H}$(see (2.1)). We construct a general class of operators for which the perturbation problem is exactly soluble and which is favorable to the instability of embedded eigenvalues. The novelty of our work is in (1) that the class of operators is so general that, in the special case $\mathscr{H}=L^2(\boldsymbol{R}^d)$, it contains the Hamiltonians of standard models of a one dimensional quantum harmonic oscillator coupled quadratically to a quantum scalar field on the $d+1$ dimensional space-time (e.g., [2-5]) and hence it gives a mathematical unification for those models and, related to (1), in (2) that we have found a class of exactly soluble models of a quantum harmonic oscillator coupled quadratically to an "abstract quantum scalar field" and clarified their intrinsic mathematical structures.

To describe the main idea of our work, let

$$(1.2)\qquad h_0=\frac{1}{2}\left(-\frac{d^2}{dq^2}+\omega_0^2q^2-\omega_0\right),\ q\in\boldsymbol{R},$$

with a constant $\omega_0>0$ and $d\Gamma(h)$ be the second quantization of a non-negative self-adjoint operator h in $\mathscr{H}$([23], [21, §X. 7], [25]). Then we take the non-negative self-adjoint operator

$$(1.3)\qquad H_0=h_0\otimes I+I\otimes d\Gamma(h)$$

as the unperturbed part of each operator in the class, where I denotes identity. If the continuous spectrum of h is not empty, then H_0 has embedded eigenvalues coming from the eigenvalues $\{n\omega_0\}_{n=0}^{\infty}$ of h_0. Thus, H_0 serves as an unperturbed operator for perturbation problem of embedded eigenvalues in $\mathscr{F}$. As for the perturbation part H_I, we shall take a general class of operators quadratic in id/dq, q, and the boson creation and annihilation operators in $\mathscr{F}_s(\mathscr{H})$(see (3.4)). Then the main point (Theorem 3.1) is to show that, under some conditions, the perturbed operator $H=H_0+H_I$ is unitarily equivalent to the operator $d\,\Gamma(h)+E_0$ *acting in* $\mathscr{F}_s(\mathscr{H})$ with a constant $E_0\in\boldsymbol{R}$. The result implies that, if the spectrum of h is purely continuous, then H has no eigenvalues greater than E_0 and hence all the non-zero eigenvalues of H_0 disappear under the perturbation, that is, they are unstable. This result is in accordance with the above mentioned "general" nature of embedded eigenvalues.

Our method in the present paper consists of abstract and elaborated versions of the methods used in the previous papers [2-5] and clarifies "intrinsic" mathematical structures lying in those models in [2-5].

This paper is organized as follows: Section II is devoted to preliminaries, where some fundamental estimates and facts in $\mathscr{F}_s(\mathscr{H})$ are given. In Section III we define the class of operators H and prove the main result mentioned above. In Section IV we prove the existence of the ground state of H, which is assumed to obtain the main result in Section III. This is done by extending Berezin's method [11] to the present case. In Section V we give a general scheme to generate the class of operator H. In Sections VI and VII, considering the case $\mathscr{H}=L^2(\boldsymbol{R}^d)$, we show that there exists a variety of operators H which contains standard model Hamiltonians with quadratic interactions as mentioned above. In the last section we give briefly remarks on other aspects of the model H.

II. Preliminaries — some estimates and facts in an abstract Boson Fock space

The Boson (or symmetric) Fock space $\mathscr{F}_s(\mathscr{H})$ over a complex Hilbert space $\mathscr{H}$ is defined by

$$(2.1)\qquad \mathscr{F}_s(\mathscr{H})=\bigoplus_{n=0}^{\infty} S_n\mathscr{H}^n,$$

where $S_n\mathscr{H}^n$ is the symmetric n-fold tensor product of $\mathscr{H}$ with the convention $S_0\mathscr{H}^0=\boldsymbol{C}$ ([20, p. 53]). We denote by $\mathscr{F}_0$ the subspace of "finite particle vectors" in $\mathscr{F}_s(\mathscr{H})$, which is spanned by vectors $\Psi=\{\Psi\}_{n=0}^{\infty}\in\mathscr{F}_s(\mathscr{H})$ such that $\Psi^{(n)}=0$ for all but finitely many n ([21, §X. 7, p. 208]). The subspace $\mathscr{F}_0$ is dense in $\mathscr{F}_s(\mathscr{H})$.

In what follows, we shall denote by $D(A)$ the domain of operator A.

Let $b(f)$, $f\in\mathscr{H}$, be the annihilation operator in $\mathscr{F}_s(\mathscr{H})$ [21, §X. 7, p. 208], which is a closed linear operator with $D(b(f))\supset\mathscr{F}_0$ and leaves $\mathscr{F}_0$

invariant satisfying the canonical commutation relations

(2.2) $[b(f), b(g)^*]=(f, g)_{\mathscr{H}} I,$

(2.3) $[b(f), b(g)]=0=[b(f)^*, b(g)^*],\ f, g\in\mathscr{H},$

on $\mathscr{F}_0$, where $[A, B]=AB-BA$ and $(\ ,\ )_{\mathscr{H}}$ is the inner product of $\mathscr{H}$ (We use the convention that $b(f)$ and $(f, g)_{\mathscr{H}}$ are complex linear in f respectively).

Let h be a non-negative self-adjoint operator in $\mathscr{H}$ with Ker $h=\{0\}$ so that h^{-1} is also non-negative self-adjoint. We denote by $d\Gamma(h)$ the second quantization of h([23], [21, §X. 7, p. 208]), which is non-negative self-adjoint in $\mathscr{F}_s(\mathscr{H})$.

Let $X=C^\infty(h)\cap C^\infty(h^{-1})$ and $\mathscr{H}_\alpha$, $\alpha\in\boldsymbol{R}$, be the completion of X in the norm

(2.4) $\|f\|_\alpha=\|h^{\alpha/2}f\|_{\mathscr{H}}\ \ f\in X.$

Henceforth, we write simply as $\|\cdot\|_0=\|\cdot\|$ and $(\cdot, \cdot)_0=(\cdot, \cdot)$.

A fundamental estimate is given by the following lemma.

LEMMA 2.1. *For all* $f\in\mathscr{H}_{-1}\cap\mathscr{H}$ *and* $\Psi\in D(d\Gamma(h)^{1/2})$, *the following estimates hold :*

(2.5) $\|b(f)\Psi\|\leq\|f\|_{-1}\|d\Gamma(h)^{1/2}\Psi\|$

(2.6) $\|b(f)^*\Psi\|\leq\|f\|_{-1}\|d\Gamma(h)^{1/2}\Psi\|+\|f\|\,\|\Psi\|$

The estimates (2.5) and (2.6) are abstract versions of those known for concrete models of *massless* boson quantum fields [15, 1-5]. We omit proof of (2.5) and (2.6), since it can be done in the same way as in the case of concrete models.

We denote by Φ_0 the Fock vacuum in $\mathscr{F}_s(\mathscr{H}): \Phi_0^{(0)}=1,\ \Phi_0^{(n)}=0,\ n\geq1.$

LEMMA 2.2. *Let* $\mathscr{F}_{0,b}^\infty$ *be the subspace spanned by finite linear combination of vectors of the form* $b(f_1)^*\ldots b(f_n)^*\Phi_0$, $n\geq0$, $f_i\in C^\infty(h)\cap C^\infty(h^{-1})$, $j=1, \ldots, n$. *Then,* $\mathscr{F}_{0,b}^\infty$ *is a core for* $d\Gamma(h)^\alpha$ *for every* $\alpha>0$.

PROOF: It is obvious that $\mathscr{F}_{0,b}^\infty$ is dense in $\mathscr{F}_s(\mathscr{H})$ and $d\Gamma(h)$ maps $\mathscr{F}_{0,b}^\infty$ into itself. Further, the one parameter unitary group $V_t=\exp(itd\Gamma(h))$, $t\in\boldsymbol{R}$, leaves $\mathscr{F}_{0,b}^\infty$ invariant with $V_t d\Gamma(h)=d\Gamma(h)V_t$ on $\mathscr{F}_{0,b}^\infty$, since V_t can be written as $V_t=\Gamma(e^{ith})$ (see [23, 21, 25] for the definition of $\Gamma(A)$ with contraction operators A). It is obvious also that $(d/dt)V_t\Psi=id\Gamma(h)V_t\Psi$ for all Ψ in $\mathscr{F}_{0,b}^\infty$. Therefore, by Chernoff's lemma [14, Lemma 2.1], we conclude that, for all integers $n\geq1$, $d\Gamma(h)^n$ is essentially self-adjoint on $\mathscr{F}_{0,b}^\infty$. On the other hand, it is not so diflcult to see that, for every non-negative self-adjoint operator A in a Hilbert space and $\alpha\in$

$(0, n)$, A^{a} is essentially self-adjoint on each core for A^{n}. Applying this fact to the present case, we get the desired result. □

We shall denote by $b(f)^{\#}$ either $b(f)$ or $b(f)^{*}$.

LEMMA 2. 3. *For all $n \geq 1$ and $f \in \mathscr{H}_{-1} \cap \mathscr{H}_{2n}$, $b(f)^{\#}$ maps $D(d\Gamma(h)^{(2n+1)/2})$ into $D(d\Gamma(h)^{n})$ and the following commutation relations hold :*

$$(2.7)\qquad [d\Gamma(h)^{n}, b(f)^{*}]\Psi=\sum_{k=1}^{n} C_{k} b(h^{k}f)^{*} d\Gamma(h)^{n-k}\Psi$$

$$(2.8)\qquad [d\Gamma(h)^{n}, b(f)]\Psi=-\sum_{k=1}^{n} C_{k} d\Gamma(h)^{n-k} b(h^{k}f)\Psi$$

for all Ψ in $D(d\Gamma(h)^{(2n+1)/2})$.

PROOF: We prove the lemma only for the case of $b(f)^{*}$. We first consider the case $n=1$. Let $\mathscr{F}_{0,b}^{\infty}$ be given as in Lemma 2. 2. Then, it is easy to see that, for all Ψ in $\mathscr{F}_{0,b}^{\infty}$, $b(f)^{*}\Psi$ is in $D(d\Gamma(h))$ and $d\Gamma(h)\Psi$ is in $D(b(f)^{*})$ with

$$(2.9)\qquad d\Gamma(h)b(f)^{*}\Psi=b(f)^{*}d\Gamma(h)\Psi+b(hf)^{*}\Psi.$$

Using this relation and the estimate (2.6), we have

$$\|d\Gamma(h)b(f)^{*}\Psi\| \leq \mathrm{const}\|(d\Gamma(h)+1)^{3/2}\Psi\|.$$

By a limiting argument using this estimate and Lemma 2. 2, one can see that (2.9) extends to all Ψ in $D(d\Gamma(h)^{3/2})$ with the result that $b(f)^{*}\Psi$ is in $D(d\Gamma(h))$ for all Ψ in $D(d\Gamma(h)^{3/2})$. Thus the case $n=1$ is proved. We next assume that the assertion holds up to $n=m\geq 1$. Let f be in $\mathscr{H}_{-1} \cap \mathscr{H}_{2(m+1)}$. Then, for all Ψ in $\mathscr{F}_{0,b}^{\infty}$, $b(f)^{*}\Psi$ is in $D(d\Gamma(h)^{m+1})$. Using the induction hypothesis, we see that (2.7) holds with $n=m+1$ and we have the estimate

$$\|d\Gamma(h)^{m+1}b(f)^{*}\Psi\| \leq c\|(d\Gamma(h)+1)^{(2m+3)/2}\Psi\|,$$

where c is a positive constant. Then, by a limiting argument similar to the case $n=1$, we conclude that, for all Ψ in $D(d\Gamma(h)^{(2m+3)/2})$, $b(f)^{*}\Psi$ is in $D(d\Gamma(h)^{m+1})$ and (2.7) holds with $n=m+1$. □

We next consider fractional powers of $d\Gamma(h)$.

LEMMA 2. 4. *Let $0<a<1$ and f be in $\mathscr{H}_{-1} \cap \mathscr{H}_{2}$. Then, $b(f)^{\#}$ maps $D(d\Gamma(h)^{(2a+1)/2})$ into $D(d\Gamma(h)^{a})$ and, for all Ψ in $D(d\Gamma(h)^{(2a+1)/2})$ and $\varepsilon>0$, we have*

$$(2.10)\qquad [(d\Gamma(h)+\varepsilon)^{a}, b(f)^{\#}]\|\Psi\| \leq \varepsilon^{a-1}c_{a}(\|f\|_{1}\|d\Gamma(h)^{1/2}\Psi\|+\|f\|_{2}\|\Psi\|),$$

where

$$c_\alpha=\frac{\sin\pi\alpha}{\pi}\int_0^\infty d\lambda\frac{\lambda^\alpha}{(\lambda+1)^2}.$$

PROOF: Throughout the proof, we ses $A=d\Gamma(h)+\varepsilon$. Applying a general formula for fractional powers of a positive self-adjoint operator to A(e.g., [17, pp. 281-286], [20, p. 317]), we have

$$(2.11)\qquad A^\alpha\Psi=d_\alpha\int_0^\infty d\lambda\lambda^{\alpha-1}(A+\lambda)^{-1}A\Psi$$

for all Ψ in $D(d\Gamma(h))$ with $d_\alpha=\sin\pi\alpha/\pi$. Let Ψ be in $D(d\Gamma(h)^{3/2})$. Then, by Lemma 2.3, $b(f)^{\#}\Psi$ is in $D(d\Gamma(h))$. Hence $A^\alpha b(f)^{\#}\Psi$ is given by the right hand side of (2.11) with Ψ replaced by $b(f)^{\#}\Psi$. Further, by (2.5) and (2.6), we see that $\|\lambda^{\alpha-1}b(f)^{\#}(A+\lambda)^{-1}A\Psi\|$ is integrable on $(0,\infty)$ and hence that $b(f)^{\#}A^\alpha\Psi$ is given by the right hand side of (2.11) with $b(f)^{\#}(A+\lambda)^{-1}$ in place of $(A+\lambda)^{-1}$. Combining these formulae, we get

$$(2.12)\qquad [A^\alpha, b(f)^{\#}]\Psi=d_\alpha\int_0^\infty d\lambda\,\lambda^{\alpha-1}\{[(A+\lambda)^{-1}, b(f)^{\#}]A +(A+\lambda)^{-1}[A, b(f)^{\#}]\}\Psi.$$

It is easy to see that

$$[(A+\lambda)^{-1}, b(f)^*]=-(A+\lambda)^{-1}b(hf)^*(A+\lambda)^{-1}$$

on $D(d\Gamma(h)^{1/2})$. By Lemma 2.3, we have

$$[A, b(f)^*]=b(hf)^*$$

on $D(d\Gamma(h)^{3/2})$. Substituting these relations into (2.12), we get

$$[A^\alpha, b(f)^*]\Psi=d_\alpha\int_0^\infty d\lambda\,\lambda^\alpha(A+\lambda)^{-1}b(hf)^*(A+\lambda)^{-1}\Psi.$$

Using (2.6), we have

$$\|(A+\lambda)^{-1}b(hf)^*(A+\lambda)^{-1}\Psi\|\le(\lambda+\varepsilon)^{-2}(\|f\|_1\|d\Gamma(h)^{1/2}\Psi\|+\|f\|_2\|\Psi\|).$$

Thus, we get (2.10) with $b(f)^{\#}=b(f)^*$ and $\Psi\in D(d\,\Gamma(h)^{3/2})$. The case $b(f)^{\#}=b(f)$ can be proved similarly. Next, let Ψ be in $D(d\Gamma(h)^{(2\alpha+1)/2})$. Then, by a limiting argument using the fact that $D(d\Gamma(h)^{3/2})$ is a core for $d\Gamma(h)^{(2\alpha+1)/2}$ (cf. Lemma 2.2) and (2.10) proved for $\Psi\in D(d\Gamma(h)^{3/2})$, we see that $b(f)^{\#}\Psi$ is in $D((d\Gamma(h)+\varepsilon)^\alpha)=D(d\Gamma(h)^\alpha)$ and that (2.10) extends to all Ψ in $D(d\Gamma(h)^{(2\alpha+1)/2})$. □

LEMMA 2.5. *Let $0<\alpha<1$ and $n\ge0$ be an integer. Let f be in $\mathscr{H}_{-1}\cap\mathscr{H}_{2(n+1)}$. Then, $b(f)^{\#}$ maps $D(d\Gamma(h)^{(2n+2\alpha+1)/2})$ into $D(d\Gamma(h)^{n+\alpha})$.*

PROOF: Let Ψ be in $D(d\Gamma(h)^{(2n+3)/2})$. Then, by Lemma 2.3, $b(f)^{\#}\Psi$ (resp. $d\Gamma(h)^{n+\alpha}\Psi$) is in $D(d\Gamma(h)^{n+1})$(resp. $D(b(f)^{\#})$) and we have, using (2.7), (2.8), (2.5) and (2.6)

$$(2.13)\qquad \begin{aligned}[A^{n+\alpha}, b(f)^{\#}] &\leq \text{const.}\|(A+1)^{n+(1/2)}\Psi\|,\\ &\leq \text{const.}\|(A+1)^{n+\alpha+(1/2)}\Psi\|,\end{aligned}$$

where $A=d\Gamma(h)+\varepsilon$, $\varepsilon>0$. Then, by a limiting argument using the equation $A^{n+\alpha}b(f)^{\#}\Psi=b(f)^{\#}A^{n+\alpha}\Psi+[A^{n+\alpha}, b(f)^{\#}]\Psi$ and the estimate (2.13), we can get the desired result. □

In what follows, we *assume that there exists a conjugation* J *on* $\mathscr{H}$, which is an antilinear isometry on $\mathscr{H}$ with $J^2=I$, *and that* J *commutes with* h. For f in $\mathscr{H}$, we define $\bar{f}\in\mathscr{H}$ by

$$(2.14)\qquad \bar{f}=Jf.$$

We next define operators quadratic in $b^{\#}$. Let K be a self-adjoint Hilbert-Schmidt operator on $\mathscr{H}$. Then K has a canonical expansion [20, pp. 203-204]:

$$(2.15)\qquad K=\sum_n \lambda_n(\cdot, \phi_n)\phi_n,$$

where $\{\phi_n\}_n$ is an orthonormal set in $\mathscr{H}$ and $\sum_n\lambda_n^2<\infty$. We define

$$(2.16)\qquad \langle b|K|b\rangle=\sum_n \lambda_n b(\bar{\phi}_n)b(\phi_n)$$

$$(2.17)\qquad \langle b^*|K|b\rangle=\sum_n \lambda_n b(\phi_n)^* b(\phi_n).$$

It is easy to check that these operators are well-defined on $\mathscr{F}_0$ and independent of the choice of representation of K such as (2.15). In the case where K is Hilbert-Schmidt, but, non-self-adjoint, we write as

$$K=K_1+iK_2$$

with K_1 and K_2 being self-adjoint and Hilbert-Schmidt and define the operator $\langle b^{\#}|K|b^{\#}\rangle$ by

$$(2.18)\qquad \langle b^{\#}|K|b^{\#}\rangle=\langle b^{\#}|K_1|b^{\#}\rangle+i\langle b^{\#}|K_2|b^{\#}\rangle.$$

To generalize the definition of quadratic operators in $b^{\#}$, we introduce a class of bounded linear operators on $\mathscr{H}$. We denote by $\boldsymbol{B}(\mathscr{H})$ the space of all bounded linear operators on $\mathscr{H}$. Let $K\in\boldsymbol{B}(\mathscr{H})$ be of the form

$$(2.19)\qquad Kf=\int_{\boldsymbol{R}} dE(\lambda)(f, \phi_\lambda)\psi_\lambda,$$

where $E(\lambda)$ is a signed Borel measure on $\boldsymbol{R}$ and $\lambda\to\phi_\lambda$(resp. ψ_λ) is an

$\mathscr{H}_1$-valued measurable function on $\boldsymbol{R}$.

DEFINITION 2.6. *Let K be given by* (2.19). *We say that K is in the class $\mathscr{K}(h)$ if and only if the following are satisfied :*

(a) *For all $\lambda \in \boldsymbol{R}$, ϕ_λ and ψ_λ are in $\mathscr{H}_{-1} \cap \mathscr{H}_2$.*

(b) *For $\alpha, \beta = \pm 1, 0, 2$,*

$$\int_{\boldsymbol{R}} d|E(\lambda)| \, \|\phi_\lambda\|_\alpha \|\psi_\lambda\|_\beta < \infty.$$

For K in $\mathscr{K}(h)$, we define the operators $\langle b|K|b\rangle$ and $\langle b^*|K|b\rangle$ by

$$\langle b|K|b\rangle = \int_{\boldsymbol{R}} dE(\lambda) b(\bar{\phi}_\lambda) b(\psi_\lambda), \tag{2.20}$$

$$\langle b^*|K|b\rangle = \int_{\boldsymbol{R}} dE(\lambda) b(\phi_\lambda)^* b(\psi_\lambda), \tag{2.21}$$

which are a priori well-defined on $\mathscr{F}_0$ and independent of the choice of representation of K such as (2.19). Similarly, we define the operators $\langle b^*|K|b^*\rangle$ and $\langle b|K|b^*\rangle$ so that

$$\langle b|K|b\rangle^* = \langle b^*|K^*|b^*\rangle, \tag{2.22}$$
$$\langle b^*|K|b\rangle^* = \langle b^*|K^*|b\rangle. \tag{2.23}$$

LEMMA 2.7. *Let K be in $\mathscr{K}(h)$. Then, $\langle b^\#|K|b^\#\rangle$ is defined on $D(d\Gamma(h))$ and the following estimate holds :*

$$\|\langle b^\#|K|b^\#\rangle\Psi\| \le c\|(d\Gamma(h)+1)\Psi\|, \quad \Psi \in D(d\Gamma(h)), \tag{2.24}$$

with a constant $c>0$. Further, for all Ψ in $D(d\Gamma(h)^2)$ and Φ in $D(d\Gamma(h)^{1/2})$, we have

$$|(\Phi, [d\Gamma(h), \langle b^\#|K|b^\#\rangle]\Psi)| \le d\|(d\Gamma(h)+1)^{1/2}\Phi\| \, \|(d\Gamma(h)+1)^{1/2}\Psi\| \tag{2.25}$$

with a constant $d>0$.

PROOF: We prove the lemma only for the case $b^\# = b$. The other cases can be treated similarly. By Lemmas 2.4 and 2.1, we see that $b(f)b(g)$ is defined on $D(d\Gamma(h))$ for all f and g $\mathscr{H}_{-1} \cap \mathscr{H}_2$ and we have, using (2.5) and (2.10),

$$\begin{aligned}\|b(f)b(g)\Psi\| \le c(&\|f\|_{-1}\|g\|_1 + \|f\|_{-1}\|g\|_2 + \|f\|_{-1}\|g\|_{-1}) \\ &\times \|(d\Gamma(h)+I)\Psi\|, \quad \Psi \in D(d\Gamma(h)),\end{aligned}$$

with a constant $c>0$. Therefore, for every K in $\mathscr{K}(h)$, $\int d|E(\lambda)| \, \|b(\bar{\phi}_\lambda) b(\psi_\lambda)\Psi\|$ converges and (2.24) holds with $b^\# = b^*$.

To prove (2.25), we note that, for Ψ in $D(d\Gamma(h)^2)$,

$$[d\Gamma(h), \langle b|K|b\rangle]\Psi = -\int dE(\lambda)\{b(h\bar{\phi}_\lambda)b(\psi_\lambda)+b(\bar{\phi}_\lambda)b(h\psi_\lambda)\}\Psi.$$

Therefore, for Φ in $D(d\Gamma(h)^{1/2})$, we have

$$|(\Phi, [d\Gamma(h), \langle b|K|b\rangle]\Psi)| \leq \int d|E(\lambda)|(\|b(h\bar{\phi}_\lambda)^*\Phi\|\ \|b(\psi_\lambda)\Psi\| + \|b(\bar{\phi}_\lambda)^*\Phi\|\ \|b(h\psi_\lambda)\Psi\|).$$

Then, using (2.5) and (2.10), we get (2.25) with $b^\# = b$. □

LEMMA 2.8. *Let K be in $\mathscr{K}(h)$. Then, for all Ψ in $D(d\Gamma(h)^{3/2})$, $\langle b^\#|K|b^\#\rangle\Psi$ is in $D(b(f)^\#)$, $f\in\mathscr{H}$, and*

$$(2.26)\qquad [\langle b|K|b\rangle, b(f)^*]\Psi = \{b(K\bar{f})+b(\overline{(K^*f)})\}\Psi,$$
$$(2.27)\qquad [\langle b|K|b\rangle, b(f)]\Psi = 0.$$

PROOF: We first prove (2.26) and (2.27) on $\mathscr{F}^{\infty}_{0,b}$ by using the canonical commutation relations (2.2) and (2.3), and then extend them to equations on $d\Gamma(h)^{3/2}$ by estimates (2.25), (2.5) and (2.6). □

Finally, as an application, we prove the essential self-adjointness of a general quadratic operator in the Hilbert space $\mathscr{F}$ given by (1.1).

Let $p=-id/dq$, $q\in\boldsymbol{R}$, be the generalized derivative in $L^2(\boldsymbol{R})$ and put

$$(2.28)\qquad a = (2\omega_0)^{-1/2}(\omega_0 q + ip)$$

with a constant $\omega_0>0$. Then, the operator a leaves $C_0^\infty(\boldsymbol{R})$ invariant satisfying the canonical commutation relation

$$(2.29)\qquad [a, a^*] = I$$

on $C_0^\infty(\boldsymbol{R})$.

Let $\lambda_0\in\boldsymbol{C}$ be a constant, K and M be in $\mathscr{K}(h)$ (Definition 2.6), and f and g be in $\mathscr{H}_{-1}\cap\mathscr{H}_2$. Then, we consider the following quadratic operator L acting in $\mathscr{F}$:

$$(2.30)\qquad \begin{aligned} L = {} & H_0 + \lambda_0 a^2\otimes I + \bar{\lambda}_0 a^{*2}\otimes I \\ & + I\otimes\langle b|K|b\rangle + I\otimes\langle b|K|b\rangle^* + I\otimes\langle b^*|M|b\rangle + I\otimes\langle b^*|M|b\rangle^* \\ & + a\otimes(b(f)^*+b(g)) + a^*\otimes(b(f)+b(g)^*), \end{aligned}$$

where H_0 is given by (1.3).

PROPOSITION 2.9. *The operator L is defined on $D(H_0)$ and essentially self-adjoint on every core for H_0.*

PROOF: The operator h_0 is written as

$$h_0 = \omega_0 a^* a$$

and non-negative self-adjoint on $D(p^2)\cap D(q^2)$. For $a^\#$ and h_0, we have estimates similar to (2.5), (2.6) and (2.24). Combining these estimates with (2.5), (2.6) and (2.24), we can show that $D(H_0)\subset D(L)$ and

$$\|L\Psi\|\leq c\|(H_0+1)\Psi\|, \quad \Psi\in D(H_0)$$

with a constant $c>0$. Further, using the commutation relations (2.7), (2.8), (2.29) and the estimate (2.25), one can show that

$$|(L\Psi, H_0\Phi)-(H_0\Psi, L\Phi)|\leq d\|(H_0+1)^{1/2}\Psi\| \, \|(H_0+1)^{1/2}\Phi\|, \qquad \Psi, \Phi\in D(H_0),$$

with a constant $d>0$. It is obvious that L is symmetric on $D(H_0)$. Thus, by the Glimm-Jaffe-Nelson commutator theorem ([16, §19.4], [21, §X. 5, Theorem X. 37]), we get the desired result. □

REMARK: (1) Proposition 2.9 can be extended to quadratic operators in the more general Fock space $L^2(\boldsymbol{R}^n)\otimes\mathscr{F}_s(\mathscr{H}_1)\otimes\cdots\otimes\mathscr{F}_s(\mathscr{H}_m)$.
(2) The operator L is an abstract form of Hamiltonians of models of a one dimensional quantum harmonic oscillator coupled quadratically to a quantum scalar field (e. g., [2-5]). See Section VII below.

III. The main theorem

In this section we consider a class of quadratic operators of the form (2.30) and prove, under some conditions, that each operator in the class is unitarily equivalent to $d\Gamma(h)+E_0$ *acting in* $\mathscr{F}_s(\mathscr{H})$ with a constant $E_0\in\boldsymbol{R}$. The operator L given by (2.30) can be viewed as an operator obtained by a perturbation of H_0. As we already mentioned in the Introduction, the operator L serves as an operator giving a perturbation problem of embedded eigenvalues in the case of infinite degrees of freedom. Our main result (Theorem 3.1) gives a sufficient condition for the possible embedded eigenvalues of H_0 to disappear under the perturbation $L-H_0$.

Let J be a conjugation as in Section II. For a bounded linear operator A on $\mathscr{H}$, we define $\bar{A}$ by

$$(3.1) \qquad \bar{A}=JAJ.$$

Let h be as in Section II and a self-adjoint operator $K\in\mathscr{K}(h)$ (Definition 2.6) be given. Let $V, W\in\boldsymbol{B}(\mathscr{H})$ and $f_0, g_0\in D(h)$ such that $WhV^*\in K(h)$

$$(3.2) \qquad u=Whf_0+\bar{V}h\bar{g}_0\in\mathscr{H}_{-1}\cap\mathscr{H}_2$$

and

(3.3) $\quad v = Whg_0 + \bar{V}h\bar{f}_0 \in \mathscr{H}_{-1} \cap \mathscr{H}_2.$

Then we consider the quadratic operator

(3.4)
$$H = H_0 + (f_0, hg_0)a^{*2} + (g_0, hf_0)a^2 + \langle b|\, WhV^*|b\rangle + \langle b|\, WhV^*|b\rangle^* + \langle b^*|K|b\rangle + a(b(u)^* + b(v)) + a^*(b(u) + b(v)^*),$$

where we take the constant ω_0 contained in H_0 as

(3.5) $\quad \omega_0 = (f_0, hf_0) + (g_0, hg_0)$

and we omitted the symbol $\otimes$ for tensor products of operators. H is essentially selfadjoint on every core for H_0 (Proposition 2.9). It is not obvious a priori whether or not H is bounded from below. It may depend on parameters contained in H (see [5] for an example). In the present paper, we assume the following:

(**H**) *H is bounded from below and there exists a constant $c > 0$ such that, for all $\Psi \in D(H_0)$,*

$$\|H_0\Psi\| \le c\|(\hat{H} + I)\Psi\|,$$

where

(3.6) $\quad \hat{H} = H - E_0$

and E_0 is the infimum of the spectrum of H.

REMARK : The assumption (**H**) and Proposition 2.9 imply that $D(H_0) = D(H)$. Hence $D(\hat{H}^{1/2}) = D(H_0^{1/2})$ and

$$\|H_0^{1/2}\Psi\| \le d\|(\hat{H} + 1)^{1/2}\Psi\|, \quad \Psi \in D(\hat{H}^{1/2})$$

with a constant $d > 0$.

We shall further assume the following (**AI**) and (**AII**).

(**AI**)

(3.7) $\quad W^*W - V^*V + (\cdot, f_0)f_0 - (\cdot, g_0)g_0 = I,$

(3.8) $\quad W^*\bar{V} - V^*\bar{W} + (\cdot, \bar{g}_0)f_0 - (\cdot, \bar{f}_0)g_0 = 0.$

(**AII**) *For $\alpha = \pm\frac{1}{2}$, ± 1, $h^{\alpha}Vh^{-\alpha}$ and $h^{\alpha}Wh^{-\alpha}$ can be extended to bounded operators on $\mathscr{H}$ and the equation*

(3.9) $\quad WhW^* + \bar{V}h\bar{V}^* = h + K$

holds on $D(h)$.

We shall denote by $D_{\alpha,\beta}$ the subset of consisting of all vectors Ψ in the domain of all polynomials of $a^{\#}$ and $b(f)^{\#}s$, $f\in\mathscr{H}_{\alpha}\cap\mathscr{H}_{\beta}$.

For $f\in\mathscr{H}$ such that Vf, $Wf\in\mathscr{H}_{-1}$, we define

$$B(f)=b(\overline{Vf})^{*}+b(Wf)+(f,g_0)a^{*}+(f,f_0)a, \tag{3.10}$$

which is well-defined on $D(H_0^{1/2})$ by estimates (2.5) and (2.6).

We note that (**AII**) implies that, for all $f\in\mathscr{H}_{-1}$(resp. $\mathscr{H}_2$), Vf, $Wf\in\mathscr{H}_{-1}$(resp. $\mathscr{H}_2$).

The main result in this section is the following.

THEOREM 3.1. *Assume* (**H**), (**AI**) *and* (**AII**). *Suppose that there exists a non-zero vector* Ω *in* $D_{-1,2}\cap D(H_0^{1/2})$ *such that*

(i) *For all* f *in* $\mathscr{H}_{-1}\cap\mathscr{H}_2$,

$$B(f)\Psi=0.$$

(ii) *The subspace* $F_{0,B}$ *spanned by vectors* $B(f_1)^{*}\cdots B(f_n)^{*}\Omega$, $n\geq 0$, $f_j\in\mathscr{H}_{-1}\cap\mathscr{H}_2$, $j=1,\cdots,n$, *is dense in* $\mathscr{F}$.

Then, there exists a unitary map $U:\mathscr{F}\to\mathscr{F}_s(\mathscr{H})$ *such that*

(i) $U\Omega=\Phi_0$.

(ii) *For all* f *in* $\mathscr{H}_{-1}\cap\mathscr{H}_2$,

$$UB(f)U^{-1}=b(f)$$

on $UF_{0,B}$.

(iii) $UHU^{-1}=d\Gamma(h)+E_0$.

In particular, we have

$$H\Omega=E_0\Omega \tag{3.11}$$

and

$$\sigma(H)=E_0+\sigma(d\Gamma(h)),\ \sigma_p(H)=E_0+\sigma_p(d\Gamma(h)),$$
$$\sigma_{ac}(H)=E_0+\sigma_{ac}(d\Gamma(h)),\ \sigma_{sing}(H)=E_0+\sigma_{sing}(d\Gamma(h)),$$

where σ (*resp.* σ_p, σ_{ac}, σ_{sing}) *denotes the* (*resp. point, absolutely continuous, singular continuous*) *spectrum.*

REMARKS: (1) Let us consider, for example, the case that $\sigma(h)=\sigma_{ac}(h)=[m,\infty)$ with some $m\geq 0$, $\sigma_p(h)=\phi$ and $\sigma_{sing}(h)=\phi$. In this case, H_0 has infinitely many embedded eigenvalues. On the other hand, we have $\sigma(d\Gamma(h))=\{0\}\cup[m,\infty)$, $\sigma_p(d\Gamma(h))=\{0\}$. Therefore, in this case, the above theorem shows that, under the perturbation $H-H_0$, all the non-zero embedded eigenvalues of H_0 disappear if the assumption is satisfied. Thus, the embedded eigenvalues of H_0 are unstable under the perturbation. On the other hand, as we shall see in Lemma 3.2, the transformation $\{b, b^*, a, a^*\}\to\{B, B^*\}$ given by (3.10) is a *Bogoliubov* (or *canonical*) *transformation* (e.g. [11]) and, under the assumption of Theorem 3.1, it is *improper* in the sense that there exist no unitary operators $U_0:\mathscr{F}\to\mathscr{F}$ such that

$$U_0 b(f) U_0^{-1}=B(f),\ f\in\mathscr{H}_{-1}\cap\mathscr{H}.$$

In fact, if such U_0 exists, then the vector

$$\Omega_\xi=U_0\xi\otimes\Phi_0\in\mathscr{F}$$

satisfies the equation

$$B(f)\Omega_\xi=0,\ f\in\mathscr{H}_{-1}\cap\mathscr{H}_0,$$

for all $\xi\in L^2(\boldsymbol{R})$, since $b(f)\Omega_0=0$ for all $f\in\mathscr{H}$. But, this contradicts Lemma 3.5 below. Thus the instability of the embedded eigenvalues is associated with an improper Bogoliubov transformation.
(2) Eq. (3.11) shows that Ω is a ground state of H. As we shall see in Lemma 3.5 below, it is unique up to constant multiples.

To prove Theorem 3.1, we prepare some lemmas.

LEMMA 3.2. *Assume* (**AI**) *and* (**AII**). *Then, for all f and g in $\mathscr{H}_{-1}\cap\mathscr{H}_2$, we have*

(3.12) $[B(f), B(g)^*]=(f, g)I,$
(3.13) $[B(f), B(g)]=0$

on $D(H_0)\cup D_{-1,2}$.

PROOF: By (**AII**), we have that, for all $f\in\mathscr{H}_{-1}\cap\mathscr{H}_2$, Wf, $Vf\in\mathscr{H}_{-1}\cap\mathscr{H}_2$. Therefore, by Lemma 2.5, $B(f)^\# B(g)^\#$ can be expanded on $D(H_0)\cup D_{-1,2}$. Then, direct computations using the canonical commutation relations (2.2), (2.3), (2.29) as well as (3.7) and (3.8) give (3.12) and (3.13).

LEMMA 3.3. *Assume* (**AI**) *and* (**AII**). *Then, for all $f\in\mathscr{H}_{-1}\cap\mathscr{H}_2$ and $\Psi\in D(H_0^{3/2})$, we have*

(3.14) $[H, B(f)]\Psi=-B(hf)\Psi.$

PROOF: By Lemmas 2.3, 2.8, and (**AII**), the left hand side of (3.14) is well-defined and computed by using the canonical commutation relations and the assumptions (**AI**) and (**AII**). Since the computation is straightforward (but somewhat lengthy), we omit the details. □

LEMMA 3.4. *Assume* (**H**), (**AI**) *and* (**AII**). *Then, for all f in*

$\mathscr{H}_{-1}\cap\mathscr{H}$ *and* $t\in\boldsymbol{R}$, *we have*

(3.15) $\quad e^{itH}B(f)e^{-itH}\Psi=B(e^{-ith}f)\Psi,\ \Psi\in D(H_0^{1/2}).$

PROOF: We first prove (3.15) for f satisfying

$$\sum_{n=0}^{\infty}\frac{\|h^n f\|t^n}{n!}<\infty,\ t\in\boldsymbol{R},$$

and in the sense of sesquilinear form on $D(H_0^{1/2})\times D(H_0^{3/2})$ and then use a limiting argument. □

LEMMA 3.5. *Under the assumption of Theorem* 3.1, *every vector* Ψ_0 *in* $\mathscr{F}$ *satisfying the equation* $B(f)\Psi_0=0$, $f\in\mathscr{H}_{-1}\cap\mathscr{H}_2$, *is a constant multiple of* Ω.

PROOF: Suppose that Ψ_0 and Ω are linearly independent. Then without loss of generality, we can assume $(\Psi_0,\Omega)=0$, which, in turn, implies that Ψ_0 is orthogonal to all vectors in $F_{0,B}$. Since $F_{0,B}$ is dense in $\mathscr{F}$ by assumpion, it follows that $\Psi_0=0$, but this is a contradiction.

LEMMA 3.6. *Under the assumption of Theorem* 3.1, Ω *is an eigenvector of* H.

PROOF: Taking $\Psi=\Omega$ in (3.15) and using the condition $B(g)\Omega=0$, $g\in\mathscr{H}_{-1}\cap\mathscr{H}_2$, we get

$$B(f)e^{-itH}\Omega=0,\ f\in\mathscr{H}_{-1}\cap\mathscr{H}_2.$$

Therefore, by Lemma 3.5, we have

$$e^{-itH}\Omega=\lambda(t)\Omega$$

with a function $\lambda(t)$, $t\in\boldsymbol{R}$. Since $\{e^{-itH}\}_{t\in\boldsymbol{R}}$ is a strongly continuous one parameter unitary group, $\lambda(t)$ has to be of the form

$$\lambda(t)=e^{-itE}$$

with a real constant E. Then, it follows that Ω is in $D(H)$ and $H\Omega=E\Omega$. □

PROOF OF THEOREM 3.1: Let $F_{0,b}$ be the subspace spanned by vectors $b(f_1)\cdots b(f_n)^*\Phi_0$, $n\geq 0$, $f_j\in\mathscr{H}_{-1}\cap\mathscr{H}_2$, $j=1,\cdots,n$, which is dense in $\mathscr{F}_s(\mathscr{H})$. We define the operator $U: F_{0,B}\to F_{0,b}$ by

$$UB(f_1)^*\cdots B(f_n)^*\Omega=b(f_1)^*\cdots b(f_n)^*\Phi_0,\ f_1,\cdots,f_n\in\mathscr{H}_{-1}\cap\mathscr{H}_2,$$

and extending by linearity to $F_{0,B}$. By virtue of Lemma 3.2 and the condition $B(f)\Omega=0$, $f\in\mathscr{H}_{-1}\cap\mathscr{H}_2$, U maps $F_{0,B}$ onto $F_{0,b}$ isometrically.

Therefore, U extends to a unitary map from $\mathscr{F}$ onto $\mathscr{F}_s(\mathscr{H})$, since $F_{0,B}$ (resp. $F_{0,b}$) is dense in $\mathscr{F}$ (resp. $\mathscr{F}_s(\mathscr{H})$). Then, by Lemmas 3.4 and 3.6, we have

$$Ue^{itH}\Psi = e^{it(d\Gamma(h)+E)}U\Psi \quad \Psi\in F_{0,B}$$

with a constant $E\in\boldsymbol{R}$, which extends to all $\Psi\in\mathscr{F}$. Hence part (iii) follows (Note that $d\Gamma(h)\geq 0$ and $d\Gamma(h)\Phi_0=0$). Parts (i) and (ii) follow from the definition of U. □

IV. Existence and uniqueness of the ground state

In this section we shall show that, under some additional conditions, there exists a unique vector (ground state) Ω which possesses all the properties described in the assumption of Theorem 3.1. The basic idea is to solve the equation $B(f)\Omega=0$, $f\in\mathscr{H}_{-1}\cap\mathscr{H}_2$. In order to do that, we first determine Ker W. In what follows, we assume $f_0\neq 0$.

LEMMA 4.1. *Suppose that Eq.* (3.7) *and*

(4.1) $\quad WW^*-\bar{V}\bar{V}^*=I,$

(4.2) $\quad VW^*-\bar{W}\bar{V}^*=0,$

(4.3) $\quad Wf_0=\overline{Vg_0},\ Wg_0=\overline{Vf_0},$

hold and that $\|f_0\|\neq\|g_0\|$. *Then,* dim Ker $W=1$ *and* Ker W *is spanned by*

(4.4) $$w_0=u_0-\frac{(u_0,g_0)}{1+(v_0,g_0)}v_0,$$

with

(4.5) $$u_0=(1+V^*V)^{-1}f_0,\quad v_0=(1+V^*V)^{-1}g_0.$$

Further, the relation

(4.6) $$1+\frac{|(u_0,g_0)|^2}{1+(v_0,g_0)}=(u_0,f_0)$$

holds.

We remark that, if Eq. (4.1) holds, then $WW^*=(1+\bar{V}\bar{V}^*)$ is invertible, so that Ran $W=\mathscr{H}$.

PROOF OF LEMMA 4.1: We first show by reductio ad absurdum that Ker $W\neq\{0\}$. Suppose that Ker $W=\{0\}$. Then, by the above remark, W^{-1} exists and bounded on $\mathscr{H}$. Hence we have from (4.3)

$$g_0=W^{-1}\bar{V}\bar{W}^{-1}Vg_0.$$

Using (4.2), we see that

$$\bar{V}\bar{W}^{-1}V = W - (W^*)^{-1}.$$

Therefore we get $W^{-1}(W^*)^{-1}g_0=0$ and hence $g_0=0$, which yields from (4.3) $Wf_0=0$ and hence $f_0=0$. But this contradicts the original assumption $f_0 \neq 0$. Thus we conclude that Ker $W \neq \{0\}$.

We next prove that dim Ker $W \leq 1$. Let f be in Ker W. Then, by (3.7),

(4.7) $\quad f = \alpha u_0 + \beta v_0$

with some constants α and β. It $g_0 = \lambda f_0$, $\lambda \in \boldsymbol{C}$, then $f = (\alpha + \lambda\beta) u_0$ and obviously dim Ker $W=1$.

Suppose that f_0 and g_0 be linearly independent and $f_1, f_2 \in$ Ker W be such that

$$f_1 = \alpha_1 u_0 + \beta_1 v_0, \quad f_2 = \alpha_2 u_0 + \beta_2 v_0$$

with $\alpha_1\beta_2 - \beta_1\alpha_2 \neq 0$. These equations imply that $Wu_0=0$ and $Wv_0=0$, which, combined with (3.7), give the relation

$$(v_0, f_0) f_0 - [1 + (v_0, g_0)] g_0 = 0, \quad 1 + (v_0, g_0) \neq 0.$$

This equation implies that f_0 and g_0 are linearly dependent, which, however, is a contradiction. Thus, we get dim Ker W≤ 1. Eq. (4.6) follows from (4.7) and (3.7). □

Henceforth, throughout this section, we take the assumption of Lemma 4.1 for granted.

As already mentioned, (4.1) implies Ran $W = \mathscr{H}$, so that the operator

(4.8) $\quad W_\perp = W \upharpoonright (\text{Ker } W)^\perp$

is invertible. We put

(4.9) $\quad X = W_\perp^{-1}$.

By (4.4) and (4.6), we have

(4.10) $\quad (w_0, f_0) = 1$.

We set

(4.11) $\quad F = \overline{Vw_0}$,

and

(4.12) $\quad a = (w_0, g_0)$.

Let

$$C = VX - (\cdot, X^* f_0)\bar{F} \tag{4.13}$$

and

$$A = C + \frac{\bar{\alpha}}{1-|\alpha|^2} P_F, \tag{4.14}$$

where

$$P_F = (\cdot, F)\bar{F}. \tag{4.15}$$

REMARK: By (3.7) and (4.10), we have

$$1-|\alpha|^2 = (w_0, (I+V^*V)w_0) > 0. \tag{4.16}$$

LEMMA 4.2. *Suppose that* (4.1)-(4.3), (3.7) *and* (3.8) *hold. Then we have*

$$C^* = \bar{C} \tag{4.17}$$

$$A^* = \bar{A}. \tag{4.18}$$

Further, the estimate

$$\|A\| < 1 \tag{4.19}$$

holds on the operator norm of A.

PROOF: The relation (4.17) is checked by direct computations using (3.7). Eq. (4.18) follows from (4.14), (4.17) and the fact that $P_F^* = \bar{P}_F$.

To prove (4.19), we first note that A is written as

$$A = VX - \frac{1}{1-|\alpha|^2}(\cdot, X^*(1+V^*V)w_0)\bar{F},$$

which follows from (3.7), (3.8) and (4.10). Using (3.7) again, we see that A^*A can be written as

$$A^*A = I - X^*YX,$$

where

$$\begin{aligned} Y = I &- (\cdot, f_0)f_0 + (\cdot, g_0)g_0 \\ &+ \beta\{(\cdot, u_0)V^*Vw_0 + (\cdot, V^*Vw_0)u_0\} \\ &- \beta^2(\cdot, u_0)\|Vw_0\|^2 u_0 \end{aligned}$$

with $\beta = (1-|\alpha|^2)^{-1}$. Then, it is not so difficult to show that, for all f in $\mathscr{H}$, $(f, X^*YXf) - \|Xf\|^2 \geq 0$. Thus we get

$$\|Af\|^2 + \|Xf\|^2 \leq \|f\|^2.$$

Since $\|Xf\| \geq C\|f\|$ for all $f \in \mathscr{H}$ with a constant $C>0$, (4.19) follows. □

We are now ready to prove the main result in this section:

THEOREM 4.3. *Assume that* $f_0 \neq 0$ *and* $\|f_0\| \neq \|g_0\|$. *Suppose that* (4.1)-(4.3), (3.7) *and* (3.8) *hold and that* VX *is Hilbert-Schmidt on* $\mathscr{H}$. *Then, there exists a unique* (*up to constant multiples*) *vector* Ω *in* $D_{0,0}$ *such that* (i) *for all* $f \in \mathscr{H}$, $B(f)\Omega = 0$; (ii) *The subspace* $F_{0,B}$ *is dense in* $\mathscr{F}$.

PROOF: For a vector Ω in $D_{0,0}$, the equation $B(f)\Omega=0$, $f \in \mathscr{H}$, is equivalent to equations

$$[b(F)^* + (w_0, g_0)a^* + a]\Omega = 0, \tag{4.20}$$

$$[b(\overline{VXf}^*) + b(f) + (Xf, g_0)a^* + (Xf, f_0)a]\Omega = 0, \quad f \in \mathscr{H}. \tag{4.21}$$

Let ϕ_0 and Φ_0 be the Fock vacuum in $L^2(\boldsymbol{R})$ and $\mathscr{F}_s(\mathscr{H})$ respectively:

$$a\phi_0 = 0, \quad b(f)\Phi_0 = 0, \quad f \in \mathscr{H}, \tag{4.22}$$

and put

$$\Omega_0 = \phi_0 \otimes \Phi_0. \tag{4.23}$$

Let $\mathscr{F}_{\mathrm{fin}}$ denote the dense subspace (in $\mathscr{F}$) spanned by vectors $b(f_1)^* \cdots b(f_n)^* a^{*m}\Omega_0$, $f_j \in \mathscr{H}$, $n, m \geq 0$. We introduce the operator

$$\Lambda = \frac{1}{2}\langle b^*|C|b^*\rangle + b(F)^* a^* + \frac{1}{2}\alpha a^{*2}, \tag{4.24}$$

which is well-defined on $\mathscr{F}_{\mathrm{fin}}$, since C is Hilbert-Schmidt by assumption. It is not so iifficult to see that, for all $n \geq 1$, $\langle b^*|C|b^*\rangle^n$ is defined on $\mathscr{F}_{\mathrm{fin}}$ and hence we can define the vector

$$\Omega_N = \sum_{n=0}^{N} \frac{(-\Lambda)^n \Omega_0}{n!}. \tag{4.25}$$

Then, by a generalization of Berenzin's continual integral method [11], we can show that Ω_N converges strongly to a vector Ω in $\mathscr{F}$ as $N \to \infty$ with

$$\|\Omega\|^2 = \frac{(1-|\alpha|^2)}{\|w_0\|^2}[\det(1-A^*A)]^{-1/2} \neq 0,$$

where we have used (4.16)-(4.19) and $\det(\cdot)$ denotes the determinant (e.g., [22, 26])(Note that A^*A is trace class on $\mathscr{H}$, since A is Hilbert-Schmidt on $\mathscr{H}$). In the same way, one can show that Ω is in $D_{0,0}$. Then, using the commutation relations

$$[b(f), \Lambda] = b(\overline{Cf})^* + (f, F)a^*, \quad [b(f)^*, \Lambda] = 0,$$

$$[a, \Lambda]=\alpha a^*+b(F)^*,\ [a^*, \Lambda]=0,$$

and (4.22), one can easily check that (4.20) and (4.21) are satisfied. Thus, condition (i) is proved.

To prove that condition (ii) is satisfied, we note that $b(f)^\#$ and $a^\#$ can be written in terms of $B(f)^\#$ as

(4.26) $\quad b(f)=B(W^*f)-B(V^*\bar{f})^*,$

(4.27) $\quad a=(\|f_0\|^2-|g_0\|^2)^{-1}(B(f_0)-B(g_0)^*),$

which follow from (3.10) and (4.1)-(4.3). Therefore, $F_{0,B}$ is a subspace left invariant by $b(f)^\#$'s$(f\in\mathscr{H}_{-1}\cap\mathscr{H}_2)$ and $a^\#$. Hence the closure of $F_{0,B}$ is left invariant by the unitary operators $\exp[i(b(f)+b(f)^*)t]$, $\exp(b(f)^*-b(f))t$, $\exp(iqt)$ and $\exp(ipt)$, $t\in\boldsymbol{R}$. But such a non-trivial closed subspace have to be the whole space $\mathscr{F}$ (cf. [11]). Thus, $F_{0,B}$ is dense in $\mathscr{F}$.

Since we have proved that $F_{0,B}$ is dense in $\mathscr{F}$, the uniqueness of the vector Ω follows in the same way as in the proof of Lemma 3.5. □

Finally we consider conditions under which the vector Ω in Theorem 4.3 is in $D(H_0^{1/2})$.

LEMMA 4.4. *In addition to the assumption of Theorem 4.3, suppose that hVX defines a Hilbert-Schmidt operator on $\mathscr{H}$ and that hVh^{-1} is bounded on $D(h^{-1})$. Then, the vector Ω in Theorem 4.3 is in $D(H_0)$.*

PROOF: Let Λ and Ω_N be given by (4.24) and (4.25) respectively. Then, by the additional assumption and the fact $H_0\Omega_0=0$, we see that Ω_N is in $D(H_0)$ and

$$H_0\Omega_N=\sum_{n=1}^{N}\frac{(\langle b^*|hC|b^*\rangle+b(hF)^*a^*+\omega_0 b(F)^*a^*+\alpha\omega_0 a^{*2})\Lambda^{n-1}\Omega_0}{(n-1)!}.$$

(Note that, by the additional assumption, F is in $D(h)$ and hC defines a Hilbert-Schmidt operator on $\mathscr{H}$. See (4.11) and (4.13)). Then, using Berezin's continual integral method [11], one can show that $H_0\Omega_N$ converges strongly as $N\to\infty$. By the closedness of H_0, it follows that Ω is in $D(H_0)$ and $H_0\Omega_N\to H_0\Omega(N\to\infty)$. □

V. A general construction of the quadruple $\{W, V, f_0, g_0\}$

In this section, we give a general method to construct the quadruple $\{V, W, f_0, g_0\}$ which possesses the properties assumed in Sections III and IV.

Let $\mathscr{H}$ and h be as in the preceding sections. Let $T\in\boldsymbol{B}(\mathscr{H})$ and

$Q\neq 0\in\mathscr{H}$ be given with the following properties:

(T. 1) $T^*T=I$.

(T. 2) For some $\alpha\geq 0$, T maps $D(h^{\pm\alpha})$ into $D(h^{\mp\alpha})$ and the operators

$$T_\alpha\equiv h^{-\alpha}Th^\alpha\restriction D(h^\alpha)$$

and $T_{-\alpha}\restriction D(h^{-\alpha})$ can be extended to bounded operators on $\mathscr{H}$ (We denote the unique extension of $T_{\pm\alpha}$ on $\mathscr{H}$ by the same symbol).

(T. 3) For $\beta=\frac{1}{2}$, 1, $h^{-\beta}T^{\#}_{\pm\alpha}h^\beta\restriction D(h^\beta)$ defines a bounded linear operator on $\mathscr{H}$.

(T. 4) $T^*_{-\alpha}hT_{-\alpha}+T^*_\alpha hT_\alpha=2(h+K)$
with a self-adjoint operator $K\in\mathscr{K}(h)$.

(T. 5) The operator $T^*_{-\alpha}hT_{-\alpha}-T^*_\alpha hT_\alpha+T^*_\alpha hT_{-\alpha}-T^*_{-\alpha}hT_\alpha$ defines a bounded linear operator in $\mathscr{K}(h)$.

(T. 6) $T^*h^{1\pm 2\alpha}T=\bar{T}^*h^{1\pm 2\alpha}\bar{T}$,
$T^*hT=\bar{T}^*h\bar{T}$.

(Q. 1) For $\beta=0, 2$, $h^{\beta\pm\alpha}Q\in\mathscr{H}$.

(TQ. 1) $TT^*+M(\cdot, Q)Q=I$
with a constant $M>0$.

(TQ. 2) $T\bar{T}^*+M(\cdot, \bar{Q})Q=I+S$,
with $S\in\boldsymbol{B}(\mathscr{H})$ satisfying

$$h^\alpha Sh^{-\alpha}=h^{-\alpha}Sh^\alpha$$

on $\mathscr{H}_{-\alpha}\cap\mathscr{H}_\alpha$.

(TQ. 3) For $\beta=\pm 2\alpha$

$$T^*h^\beta Q=\bar{T}h^\beta\bar{Q}$$

as a vector equation in $\mathscr{H}$.

We note that (T. 1) and (TQ. 1) imply

(TQ. 4) $T^*Q=0$.

PROPOSITION 5. 1. *Let T and Q be as above and let*

$$V=\frac{1}{2}(T^*_{-\alpha}-T^*_\alpha), \tag{5.1}$$

$$W=\frac{1}{2}(T^*_{-\alpha}+T^*_\alpha), \tag{5.2}$$

$$f_0=\frac{1}{2}e^{i\theta}M^{1/2}(\tilde{h}^\alpha+\tilde{h}^{-\alpha})Q \tag{5.3}$$

$$g_0=\frac{1}{2}e^{-i\theta}M^{1/2}(\tilde{h}^\alpha-\tilde{h}^{-\alpha})Q \tag{5.4}$$

where $\tilde{h}=h/\varkappa$ *with a constant* $\varkappa>0$ *and* $\theta\in\boldsymbol{R}$ *is a constant. Then, the quadruple* $\{V, W, f_0, g_0\}$ *possesses the following properties* (a)-(f) *:*

(a) $WhV^*\in\mathscr{K}(h)$.

(b) $f_0, g_0\in D(h)$.

(c) *Condition* (**AII**) *in Section III holds.*

(d) *The vectors* u *and* v *defined by* (3.2) *and* (3.3) *are in* $\mathscr{H}_{-1}\cap\mathscr{H}_2$.

(e) *Eqs.* (3.7)-(3.8) *and* (4.1)-(4.3) *hold.*

(f) $f_0\neq 0$ *and* $\|f_0\|\neq\|g_0\|$.

PROOF: (a) It is a simple computation to see that

$$(5.5)\qquad WhV^*=\frac{1}{4}(T_{-\alpha}^*hT_{-\alpha}-T_\alpha^*hT_\alpha+T_\alpha^*hT_{-\alpha}-T_{-\alpha}^*hT_\alpha)$$

which, by (T. 5), is in $\mathscr{K}(h)$.

(b) This follows from the definition of f_0 and g_0 and (Q. 1).

(c) This follows from (T. 2), (T. 3), (T. 4) and (T. 6).

(d) This follows from Part (c) and (Q. 1).

(e) By direct computations: Roughly speaking, we have (TQ. 1), (TQ. 2)⇒(3.7), (3.8). (T. 1), (T. 6)⇒(4.1), (4.2). (TQ. 3)⇒(4.3).

(f) By direct computations, we have

$$\|f_0\|^2-\|g_0\|^2=M\|Q\|^2,$$

which implies the desired result. □

As for Eq. (3.5), we need only to take ω_0 as

$$(5.6)\qquad \omega_0=\frac{1}{2}M\varkappa\{(Q, \tilde{h}^{1+2\alpha}Q)+(Q, \tilde{h}^{1-2\alpha}Q)\}.$$

It is easy to check that the right hand side of (5.6) is equal to $(f_0, hf_0)+(g_0, hg_0)$ and hence that Eq. (3.5) holds.

VI. Existence of $\{T, Q\}$

In this section, considering the case $\mathscr{H}=L^2(\boldsymbol{R}^d)$, we show that there exists a class of $\{T, Q\}$ possessing the properties (T. 1)-(T. 6), (Q. 1) and (TQ. 1)-(TQ. 3) in the last section.

Throughout this section, we take $\mathscr{H}=L^2(\boldsymbol{R}^d)$, $d\in\boldsymbol{N}$.

Let ω_1 be a non-negative, strictly monotone increasing, continuously differentiable function on $(0, \infty)$ such that $\omega_1(t)\to\infty$ as $t\to\infty$. We put

$$(6.1)\qquad \lim_{t\downarrow 0}\omega_1(t)=m\geq 0.$$

The function ω_1 gives a rotation invariant function ω on $\boldsymbol{R}^d$ by

(6.2) $\omega(k)=\omega_1(|k|),\ k\in\boldsymbol{R}^d.$

We take the operator h in $\mathscr{H}$ to be the multiplication operator by the function ω^r with some constant $r>0$:

(6.3) $(hf)(k)=\omega(k)^r f(k),$

and define the conjugation J on $\mathscr{H}$ by

(6.4) $(Jf)(k)=\overline{f(k)},\ f\in\mathscr{H},$

where $\bar{f}$ denotes the complex conjugate of f. Obviously, J commutes with h.

For a real-valued function ρ in $\mathscr{H}$, we define a function $\Phi_\rho(z)$ of the complex variable z by

(6.5) $$\Phi_\rho(z)=\int dk\frac{\rho(k)^2}{z-\omega(k)},$$

which is well-defined in the cut plane

(6.6) $\boldsymbol{C}_m=\boldsymbol{C}\backslash[m,\infty)$

and analytic there. Then we introduce the functions $D^{(\mu)}(z)$, $\mu=0,1$, by

(6.7) $D^{(\mu)}(z)=-a_1^{(\mu)}z+a_0^{(\mu)}+z^\mu\Phi_\rho(z),$

where $a_\nu^{(\mu)}$, $\mu,\nu=0,1$, are real constants such that $a_1^{(\mu)}>0$, $\mu=0,1$, and $a_0^{(1)}\neq0$. For simplicity, we assume

(CI) $D^{(\mu)}(z)\neq0,\ z\in\boldsymbol{C}_m.$

REMARK: It is easy to see that zeros of $D^{(\mu)}(z)$ in $\boldsymbol{C}_m$ are in $(-\infty, m)$ and that $D^{(\mu)}(t)$ is monotone decreasing in $t\in(-\infty,m)$. Thus **(CI)** is equivalent to the condition $d_m^{(\mu)}\equiv\lim_{t\uparrow m}D^{(\mu)}(t)\geq0$. Hence **(CI)** implies $a_0^{(\mu)}>0$, $\mu=0,1$.

Let

(6.8) $u_1(t)=\omega_1'(t)^{1/2}t^{-(d-1)/2},\ t>0,$

and put

(6.9) $u(k)=u_1(|k|),\ k\in\boldsymbol{R}^d.$

In addition to **(CI)**, we further assume the following **(CII)**-**(CIV)**:

(CII) $\sup_{\substack{\varepsilon>0\\ t\in[m,\infty)}}|\Phi_\rho(t\pm i\varepsilon)|<\infty.$

(**CIII**) $\inf_{\substack{\varepsilon>0 \\ t\in[m,\infty)}} |D^{(\mu)}(t\pm i\varepsilon)|>0,\ \mu=0, 1.$

(**CIV**) The functions $u^{-1}\rho$ and $u^{-1}\omega\rho$ are in $L^\infty(\boldsymbol{R}^d)$.

Conditions (**CII**)-(**CIV**) are not empty. For example, functions

$$\omega_1(t)=(t^2+m^2)^{1/2},$$
$$\rho(k)=\frac{\lambda}{1+|k|^\alpha},\ \lambda>0$$

with $\alpha\geq(d/2)+1$, $d\geq 2$, satisfy (**CII**)-(**CIV**). General sufficient conditions for (**CII**) and (**CIII**) to hold have been given in the Appendices in [9].

For each $\varepsilon>0$, we define a linear operator G_ε by

$$(6.10)\qquad (G_\varepsilon f)(k)=\int dk'\frac{u(k)u(k')f(k')}{\omega(k)-\omega(k')+i\varepsilon}.$$

In [5], it was proved that G_ε is a skew-symmetric bounded linear operator on $\mathscr{H}$ and that the strong limit

$$(6.11)\qquad G=s-\lim_{\varepsilon\downarrow 0} G_\varepsilon$$

exists on $\mathscr{H}$.

One can show also that the limits

$$(6.12)\qquad D_\pm^{(\mu)}(t)=\lim_{\varepsilon\downarrow 0} D^{(\mu)}(t\pm i\varepsilon),\ \mu=0, 1,$$

exist for a.e. $t\in\boldsymbol{R}$, which, by assumption (**CIII**), cannot be zero. Then we define the function

$$(6.13)\qquad Q^{(\mu)}(k)=\frac{\rho(k)}{D_+^{(\mu)}(\omega(k))},\ \mu=0, 1.$$

Let $T^{(\mu)}$ be the operator given by

$$(6.14)\qquad T^{(\mu)}f=f-\omega^\mu Q^{(\mu)}u^{-1}Gu^{-1}\rho f,$$

which is bounded and linear on $\mathscr{H}$ by (**CIII**) and (**CIV**).

Our aim is to show that $\{T^{(\mu)}, Q^{(\mu)}\}$ given by (6.14) and (6.13) possesses the properties of $\{T, Q\}$ in the last section.

LEMMA 6.1. *Let $f^{(\mu)}(z)(\mu=0, 1)$ be a meromorphic function in $\boldsymbol{C}$ with poles $a_1, \cdots, a_N$ in $\boldsymbol{C}_m$ and with no poles in $[m, \infty)$. Suppose that*

$$A^{(\mu)}\equiv\lim_{z\to\infty}\frac{z^{1-\mu}f^{(\mu)}(z)}{D^{(\mu)}(z)}$$

exists. Then

(6.15) $$\int dk|Q^{(\mu)}(k)|^2 f^{(\mu)}(\omega(k))$$
$$=\delta_{m,0}\delta_{\mu,1}\frac{f^{(\mu)}(0)}{a_0^{(1)}}-A^{(\mu)}+\sum_{n=1}^{N}\mathrm{Res}\left(\frac{f^{(\mu)}}{z^\mu D^{(\mu)}},\ a_n\right),$$

where $\mathrm{Res}(f^{(\mu)}/z^\mu D^{(\mu)},\ a_n)$ *denotes the residue of* $f^{(\mu)}/z^\mu D^{(\mu)}$ *at* $z=a_n$.

PROOF: We give a proof of (6.15) only for the case $m=0$ (The case $m>0$ is similarly proved). Let $0<\varepsilon<\delta<\infty$ be sufficiently small and $L>0$ be large enough so that the points a_n, $n=1,\cdots,N$, are in the interior of the curve

$$\Gamma=\{\delta e^{i\theta}|-2\pi+\alpha_\varepsilon\leq\theta\leq-\alpha_\varepsilon\}\cup\{x+i\varepsilon|\delta\cos\alpha_\varepsilon\leq x\leq L\cos\beta_\varepsilon\}$$
$$\cup\{Le^{i\theta}|\beta_\varepsilon\leq\theta\leq 2\pi-\beta_\varepsilon\}\cup\{x-i\varepsilon|\delta\cos\alpha_\varepsilon\leq x\leq L\cos\beta_\varepsilon\}$$

with the anti-clockwise orientation, where $\alpha_\varepsilon=\sin^{-1}(\varepsilon/\delta)$, $\beta_\varepsilon=\sin^{-1}(\varepsilon/L)\in(0,\pi/2)$. Then, by Cauchy's theorem, we have

$$I_\mu=\int_\Gamma\frac{f^{(\mu)}(z)}{z^\mu D^{(\mu)}(z)}dz=2\pi i\sum_{n=1}^{N}\mathrm{Res}\left(\frac{f^{(\mu)}(z)}{z^\mu D^{(\mu)}(z)},\ a_n\right).$$

On the other hand, I_μ is written as

$$I_\mu=\sum_{j=1}^{3}I_\mu^{(j)}$$

with

$$I_\mu^{(1)}=\int_{-\alpha_\varepsilon}^{-2\pi+\alpha_\varepsilon}d\theta\frac{i(\delta e^{i\theta})^{1-\mu}f^{(\mu)}(\delta e^{i\theta})}{D^{(\mu)}(\delta e^{i\theta})},$$
$$I_\mu^{(2)}=\int_{\delta\cos\alpha_\varepsilon}^{L\cos\beta_\varepsilon}dx\left\{\frac{f^{(\mu)}(x+i\varepsilon)}{(x+i\varepsilon)^\mu D^{(\mu)}(x+i\varepsilon)}-\frac{f^{(\mu)}(x-i\varepsilon)}{(x-i\varepsilon)^\mu D^{(\mu)}(x-i\varepsilon)}\right\},$$
$$I_\mu^{(3)}=i\int_{\beta_\varepsilon}^{2\pi-\beta_\varepsilon}d\theta\frac{(Le^{i\theta})^{1-\mu}f^{(\mu)}(Le^{i\theta})}{D^{(\mu)}(Le^{i\theta})}.$$

By (**CII**), (**CIII**) and the regularity of $f^{(\mu)}(z)$ in $\boldsymbol{C}\backslash\{a_1,\cdots,a_N\}$, we see that

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}I_\mu^{(1)}=\delta_{\mu,1}(-2\pi i)\frac{f^{(1)}(0)}{a_0^{(1)}},$$
$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}I_\mu^{(2)}=\int_0^L dx\frac{f^{(\mu)}(x)}{x}\left\{\frac{1}{D_+^{(\mu)}(x)}-\frac{1}{D_-^{(\mu)}(x)}\right\}.$$

By using the deminated convergence theorem, one can show that

$$\lim_{L\to\infty}\lim_{\varepsilon\downarrow 0}I_\mu^{(3)}=2\pi iA^{(\mu)}.$$

Thus, we get

$$\frac{1}{2\pi i}\int_0^\infty dx \frac{f^{(\mu)}(x)}{x}\left\{\frac{1}{D_+^{(\mu)}(x)}-\frac{1}{D_-^{(\mu)}(x)}\right\}$$
$$=\delta_{\mu,1}\frac{f^{(1)}(0)}{a_0^{(1)}}-A^{(\mu)}+\sum_{n=1}^N \mathrm{Res}\left(\frac{f^{(\mu)}(z)}{z^\mu D^{(\mu)}(z)}, a_n\right).$$

It is easy to see that the LHS is in fact equal to that of (6.15). □

LEMMA 6.2. *For each $\mu=0, 1$, the following hold :*

(a) $T^{(\mu)*}T^{(\mu)}=I$.
(b) $T^{(\mu)}T^{(\mu)*}+M^{(\mu)}(\cdot, Q^{(\mu)})Q^{(\mu)}=I$,

where

(6.16) $\quad M^{(\mu)}=a_{1-\mu}^{(\mu)}$.

PROOF: Part (a) is proved by applying Lemma 6.1. Part (b) is derived by direct computations. Cf. the proof of [2, Lemma 4.9]. □

Let

(6.17) $\quad (S^{(\mu)}f)(k)=2\pi i\omega(k)^\mu\rho(k)u(k)^{-2}Q^{(\mu)}(k)\int_{S^{d-1}} dS(\theta)f(|k|\theta),$

where S^{d-1} is the $d-1$ sphere and $dS(\theta)$ is the surface integral element on it.

LEMMA 6.3. *For each $\mu=0, 1$, $S^{(\mu)}\in \boldsymbol{B}(\mathscr{H})$.*

PROOF: By using the Schwartz inequality, we see that

$$\|S^{(\mu)}f\|^2\le C\|\omega^\mu u^{-2}\rho Q^{(\mu)}f\|^2$$

with a constant $C>0$. It follows from the conditions (**CIII**) and (**CIV**) that $\omega^\mu u^{-2}\rho Q$ is in $L^\infty(\boldsymbol{R}^d)$. Therefore, $S^{(\mu)}$ is bounded on $\mathscr{H}$.

LEMMA 6.4. *Let*

(6.18) $\quad F^{(\mu)}(k)=\dfrac{D_+^{(\mu)}(\omega(k))}{D_-^{(\mu)}(\omega(k))}, \ \mu=0, 1.$

Then, for each $\mu=0, 1$, we have

(6.19) $\quad \bar{T}^{(\mu)}f=F^{(\mu)}T^{(\mu)}f+(I-F^{(\mu)}+\bar{S}^{(\mu)})f$

for all f in $\mathscr{H}$.

PROOF: Direct computation using the distributional equations

$$\frac{1}{\omega(k)-\omega(k')\pm i0}=\mathrm{P}\frac{1}{\omega(k)-\omega(k')}\mp i\pi\delta(\omega(k)-\omega(k')),$$

where P denotes the principal value. □

LEMMA 6.5. *For every rotaion invariant measurable function f on $\boldsymbol{R}^d$ and for each $\mu=0, 1$, we have*

$$\bar{T}^{(\mu)*} f \bar{T}^{(\mu)} g = T^{(\mu)*} f T^{(\mu)} g \tag{6.20}$$

for all g in $\mathscr{H}$ such that $fT^{(\mu)}g$ is in $\mathscr{H}$ and

$$\bar{T}^{(\mu)*} f \bar{Q}^{(\mu)} = T^{(\mu)*} f Q^{(\mu)} \tag{6.21}$$

provided that $fQ^{(\mu)}$ is in $\mathscr{H}$.

PROOF: We first note that, if f is real, then $T^{(\mu)*}fT^{(\mu)}$ and $\bar{T}^{(\mu)*}f\bar{T}^{(\mu)}$ are reality preserving, which follows by direct computation as in Lemma 6.4. Therefore, it is sufficient to prove (6.20) for the case that f and g are real. In that case, using (6.19), we can prove

$$(g, \bar{T}^{(\mu)*} f \bar{T}^{(\mu)} g) = (g, T^{(\mu)*} f T^{(\mu)} g).$$

Then, the polarization identity gives (6.20).

Eq. (6.21) follows from (6.19) and the following identities:

$$F^{(\mu)} Q^{(\mu)} = \bar{Q}^{(\mu)}, \quad Q^{(\mu)} - \bar{Q}^{(\mu)} = S^{(\mu)} \bar{Q}^{(\mu)}.$$

□

LEMMA 6.6. *For each $\mu=0, 1$, Eq. (TQ. 2) in Section V holds with $\{T, Q\}=\{T^{(\mu)}, Q^{(\mu)}\}$ and $S=S^{(\mu)}$ given by (6.17).*

PROOF: Direct computations using (6.19). □

In what follows we fix a constant $\alpha \geq 0$ and assume the following:

(**CV**) For $\beta=-\frac{1}{2}, 2$, $\omega^{r(\beta\pm\alpha)}\rho \in \mathscr{H}$.

(**CVI**) For $\beta=0, \frac{1}{2}, 1$, $\omega^{\pm r(\beta-\alpha)+\mu}u^{-1}\rho$ and $\omega^{\pm r(\beta+\alpha)+\mu}u^{-1}\rho \in L^\infty(\boldsymbol{R}^d)$.

PROPOSITION 6.7. *For each $\mu=0, 1$, $\{T, Q\}=\{T^{(\mu)}, Q^{(\mu)}\}$ possesses the properties (T. 1)-(T. 3), (T. 6), (Q. 1) and (TQ. 1)-(TQ. 3) in Section V, where the constant M in (TQ. 1) is taken as $M=M^{(\mu)}$ given by (6.16).*

PROOF: Roughly speaking, we have

Lemma 6.2 (a) (resp. (b))⇒(T. 2)(resp. (TQ. 1)), (6.19)⇒(TQ. 2), (6.20)⇒(T. 6), (6.21)⇒(TQ. 3), (**CV**)⇒(Q. 1), (**CVI**)⇒(T. 2), (T. 3). □

It still remains to prove (T. 4) and (T. 5). In order to do that, we have to compute explicitly the relevant operators.

For each $\mu=0,1$, and $\beta\in[-\mu,2-\mu]$, we introduce the operator $R_\beta^{(\mu)}$ by

$$R_\beta^{(\mu)}f=\frac{\sin\pi\beta}{\pi}\int_0^\infty d\lambda\frac{\lambda^{\beta+\mu}}{D^{(\mu)}(-\lambda)}(f,e_\lambda)e_\lambda \tag{6.22}$$

with

$$e_\lambda(k)=\frac{\rho(k)}{\lambda+\omega(k)}. \tag{6.23}$$

LEMMA 6.8. *Let δ_1 and δ_2 be real constants. Suppose that, for $s=0,1$, $t=-r/2,0,r$, and $j=1,2$, the functions $\omega^{r\delta_j+t-s}\rho$ are in $\mathscr{H}$. Then, the operator $h^{\delta_1}R_\beta^{(\mu)}h^{\delta_2}$ is in $\mathscr{K}(h)$ (Definition 2.6).*

PROOF: By the assumption, $h^{\delta_j}e_\lambda$ is in $\mathscr{H}_{-1}\cap\mathscr{H}_2$ for all $\lambda\geq 0$ and we have

$$h^{\delta_1}R_\beta^{(\mu)}h^{\delta_2}f=\frac{\sin\pi\beta}{\pi}\int_0^\infty d\lambda\frac{\lambda^{\beta+\mu}}{D^{(\mu)}(-\lambda)}(f,h^{\delta_2}e_\lambda)h^{\delta_1}e_\lambda.$$

By the assumption (**CI**) for $D^{(\mu)}(z)$, we have for all $\lambda\geq 0$

$$|D^{(\mu)}(-\lambda)|\geq c>0$$

with a constant c and $D^{(\mu)}(-\lambda)=O(\lambda)$ as $\lambda\to\infty$. Hence we have

$$\begin{aligned}\|h^{\delta_1}R_\beta^{(\mu)}h^{\delta_2}f\| &\leq\frac{1}{\pi}\int_0^\infty d\lambda\frac{\lambda^{\beta+\mu}}{|D^{(\mu)}(-\lambda)|}\|h^{\delta_2}e_\lambda\|\,\|h^{\delta_1}e_\lambda\|\,\|f\|\\ &\leq\frac{c}{\pi}\int_0^1 d\lambda\,\lambda^{\beta+\mu}\|\omega^{r\delta_2-1}\rho\|\,\|\omega^{r\delta_1-1}\rho\|\,\|f\|\\ &\quad+\frac{1}{\pi}\int_1^\infty d\lambda\frac{\lambda^{\beta+\mu-2}}{|D^{(\mu)}(-\lambda)|}\|h^{\delta_2}\rho\|\,\|h^{\delta_1}\rho\|\,\|f\|\\ &<\infty.\end{aligned}$$

Hence $h^{\delta_1}R_\beta^{(\mu)}h^{\delta_2}$ is bounded on $\mathscr{H}$. In the same way, we can show that

$$\int_0^\infty d\lambda\frac{\lambda^{\beta+\mu}}{|D^{(\mu)}(-\lambda)|}\|h^{\delta_1}e_\lambda\|_p\|h^{\delta_2}e_\lambda\|_q<\infty$$

for $p,q=\pm 1,2$. Thus, $h^{\delta_1}R_\beta^{(\mu)}h^{\delta_2}$ is in $\mathscr{K}(h)$. □

LEMMA 6.9. *Let $-\mu\leq r\beta\leq 2-\mu$. Then, for all $f\in D(h^\beta)$ such that $T^{(\mu)}f\in D(h^\beta)$, we have*

$$T^{(\mu)*}h^\beta T^{(\mu)}f=h^\beta f+d_{r\beta}^{(\mu)}(f,\rho)\rho+(-1)^{\mu-1}R_{r\beta}^{(\mu)}f, \tag{6.24}$$

where

(6.25) $$d_{r\beta}^{(\mu)}=\frac{\delta_{\mu,0}\delta_{r\beta,2}}{a_1^{(0)}}+\frac{\delta_{\mu,1}\delta_{r\beta,1}}{a_1^{(1)}}.$$

PROOF: Let f be as above and g be in $\mathscr{H}$. Then, we have

$$\begin{aligned}(T^{(\mu)}g, h^\beta T^{(\mu)}f)=&(g, h^\beta f)-(g, h^\beta\omega^\mu Q^{(\mu)}u^{-1}Gu^{-1}\rho f)\\&-(\omega^\mu Q^{(\mu)}u^{-1}Gu^{-1}\rho g, h^\beta f)\\&+\lim_{\varepsilon\downarrow 0}I_\varepsilon^{(\mu)},\end{aligned}$$

where

$$I_\varepsilon^{(\mu)}=\int dk'dk''\rho(k')\rho(k'')g(k')\overline{f(k'')}K_\beta^{(\mu)}(k', k'')$$

with

$$K_\beta^{(\mu)}(k', k'')=\int dk\frac{|Q^{(\mu)}(k)|^2\omega(k)^{r\beta+2\mu}}{(\omega(k)-\omega(k')+i\varepsilon)(\omega(k)-\omega(k'')-i\varepsilon)}.$$

We compute $K_\beta^{(\mu)}(k', k'')$ by using Lemma 6.1. Let $0<\delta\equiv r\beta<1$ first. Then we can write as

$$\omega(k)^\delta=\frac{\sin\pi\delta}{\pi}\int_0^\infty d\lambda\frac{\lambda^{\delta-1}\omega(k)}{(\lambda+\omega(k))}.$$

Using Fubini's theorem, we have

$$K_\beta^{(\mu)}(k', k'')=\frac{\sin\pi\delta}{\pi}\int_0^\infty d\lambda\,\lambda^{\delta-1}J^{(\mu)}(\lambda, k', k'')$$

with

$$J^{(\mu)}(\lambda, k', k'')=\int dk\frac{|Q^{(\mu)}(k)|^2\omega(k)^{2\mu+1}}{(\omega(k)-\omega(k')+i\varepsilon)(\omega(k)-\omega(k'')-i\varepsilon)(\lambda+\omega(k))}$$

The function $J^{(\mu)}(\lambda, k', k'')$ can be computed via Lemma 6.1 and we get (6.24). The other cases can be proved similarly. □

Lemma 6.9 immediately gives

PROPOSITION 6.10. *Let $-\mu\le r(1\pm 2\alpha)\le 2-\mu$. Then the following operator equations hold:*

(6.26) $$\begin{aligned}&T_{-\alpha}^{(\mu)*}hT_{-\alpha}^{(\mu)}+T_\alpha^{(\mu)*}hT_\alpha^{(\mu)}\\&=2h+d_{r(1+2\alpha)}^{(\mu)}(\cdot, h^{-\alpha}\rho)h^{-\alpha}\rho+d_{r(1-2\alpha)}^{(\mu)}(\cdot, h^\alpha\rho)h^\alpha\rho\\&\quad+(-1)^{\mu-1}(h^{-\alpha}R_{r(1+2\alpha)}^{(\mu)}h^{-\alpha}+h^\alpha R_{r(1-2\alpha)}^{(\mu)}h^\alpha).\end{aligned}$$

(6.27) $$\begin{aligned}&T_{-\alpha}^{(\mu)*}hT_{-\alpha}^{(\mu)}-T_\alpha^{(\mu)*}hT_\alpha^{(\mu)}+T_\alpha^{(\mu)*}hT_{-\alpha}^{(\mu)}-T_{-\alpha}^{(\mu)*}hT_\alpha^{(\mu)}\\&=d_{r(1-2\alpha)}^{(\mu)}(\cdot, h^{-\alpha}\rho)h^{-\alpha}\rho-d_{r(1-2\alpha)}^{(\mu)}(\cdot, h^\alpha\rho)h^\alpha\rho+d_r^{(\mu)}(\cdot, h^{-\alpha}\rho)h^\alpha\rho\\&\quad-d_r^{(\mu)}(\cdot, h^\alpha\rho)h^{-\alpha}\rho+(-1)^{\mu-1}(h^{-\alpha}R_{r(1+2\alpha)}^{(\mu)}h^{-\alpha}\\&\quad-h^\alpha R_{r(1-2\alpha)}^{(\mu)}h^\alpha+h^\alpha R_r^{(\mu)}h^{-\alpha}-h^{-\alpha}R_r^{(\mu)}h^\alpha).\end{aligned}$$

In particular, (T. 4) and (T. 5) in Section V hold with $\{T, Q\}=$

$\{T^{(\mu)}, Q^{(\mu)}\}$, $\mu=0, 1$.

Thus we have proved that $\{T^{(\mu)}, Q^{(\mu)}\}$ defined by (6.14) and (6.13) possesses the properties (T. 1)-(T. 6), (Q. 1), and (TQ. 1)-(TQ. 3) in Section V.

In Section IV, we needed for the proof of the existence of the ground state that $VW_{\perp}^{-1}$ and $hVW_{\perp}^{-1}$ are Hilbert-Schmidt on $\mathscr{H}$. Obviously, this condition is satisfied if V and hV is Hilbert-Schmidt. A criterion for the latter condition can be derived by expressing V explicitly as follows. We first define the kernel $A_{\delta}(k, k')(\delta>0)$ by

$$(6.28) \qquad \omega(k)^{2\delta}-\omega(k')^{2\delta}=(\omega(k)-\omega(k'))A_{\delta}(k, k').$$

Let $V^{(\mu)}$ denote the operator V given by (5.1) with $T=T^{(\mu)}$. Then, by direct computation, we have

$$(6.29) \qquad (V^{(\mu)}f)(k)=\int V^{(\mu)}(k, k')f(k')dk'$$

with

$$(6.30) \qquad V^{(\mu)}(k, k')=-\frac{\rho(k)A_{r\alpha}(k, k')\omega(k)^{\mu}\overline{Q^{(\mu)}(k')}}{2[\omega(k)\omega(k')]^{r\alpha}}$$

Thus, we get the following criterion for $V^{(\mu)}$ and $hV^{(\mu)}$ to be Hilbert-Schmidt on $\mathscr{H}$:

LEMMA 6. 11. *The operator $V^{(\mu)}$ (resp. $hV^{(\mu)}$) is Hilbert-Schmidt on $\mathscr{H}$ if and only if $V^{(\mu)}(k, k')$(resp. $\omega(k)^{r}V^{(\mu)}(k, k')$) is a function in $L^2(\boldsymbol{R}^d\times\boldsymbol{R}^d)$.*

VII. Examples

In this section we show by explicit computations that the class of the quadratic operators H defined via $\{T^{(\mu)}, Q^{(\mu)}\}$ given in Section VI contains the Hamiltonians of standard models of a one dimensional quantum harmonic oscillator coupled quadratically to a quantum scalar field. Throughout this section, the same notation is used as that in Section VI.

We first enumerate some lemmas need for the computation of H.

LEMMA 7. 1. *Suppose that, for a non-negative integer n, $\omega^n\rho$ is in $\mathscr{H}$. Then, we have*

$$(7.1) \qquad T^{(\mu)}\omega^n\rho=(a_0^{(\mu)}-a_1^{(\mu)}\omega)\omega^nQ^{(\mu)}+\sum_{j=1}^{n}(\rho, \omega^{n-j}\rho)\omega^{\mu+j-1}Q^{(\mu)}$$

and

$$(7.2) \qquad (Q^{(\mu)}, \omega^{n+1}Q^{(\mu)})=\frac{1}{a_1^{(\mu)}}\{a_0^{(\mu)}(Q^{(\mu)}, \omega^nQ^{(\mu)})$$

$$+\sum_{j=1}^{n}(\rho,\ \omega^{n-j}\rho)(Q^{(\mu)},\ \omega^{n+j-1}Q^{(\mu)})\}.$$

PROOF: Eq. (7.1) follows from direct computations. Taking the quantity $(Q,\ T^{(\mu)}\omega^n\rho)$ and using the fact $T^{(\mu)*}Q^{(\mu)}=0$ [TQ. 4], we get (7.2). □

LEMMA 7.2. *Suppose that $\omega\rho$ is in $\mathscr{H}$. Then, we have*

$$T^{(\mu)*}\omega Q^{(\mu)}=-\frac{\rho}{a_1^{(\mu)}},\quad \mu=0,\ 1. \tag{7.3}$$

PROOF: Similar to the proof of Lemma 6.9. □

Let $W^{(\mu)}$ (resp. $V^{(\mu)}$, $f_0^{(\mu)}$, $g_0^{(\mu)}$) be given by (5.1) (resp. (5.2), (5.3), (5.4)) with $T=T^{(\mu)}$, $Q=Q^{(\mu)}$, $h=\omega^r$, $M=M^{(\mu)}$, $x=x^{(\mu)}$, and $\theta=\theta^{(\mu)}$. We put

$$u^{(\mu)}=w^{(\mu)}hf_0^{(\mu)}+\bar{V}^{(\mu)}h\bar{g}_0^{(\mu)},$$
$$v^{(\mu)}=w^{(\mu)}hg_0^{(\mu)}+\bar{V}^{(\mu)}h\bar{f}_0^{(\mu)}.$$

(see (3.2) and (3.3)).

LEMMA 7.3.

(a) $(r,\ \alpha)=\left(\frac{1}{2},\ \pm\frac{1}{2}\right)$, *we have*

$$u^{(\mu)}=-\frac{e^{i\theta^{(\mu)}}M^{(\mu)}h^{-1/2}\rho}{bx^{(\mu)1/2}a_1^{(\mu)}},\quad u^{(\mu)}=\mp\frac{e^{-i\theta^{(\mu)}}M^{(\mu)}h^{-1/2}\rho}{2x^{(\mu)1/2}a_1^{(\mu)}}$$

(b) *For $(r,\ \alpha)=(1,\ 0)$, we have*

$$u^{(\mu)}=-\frac{e^{i\theta^{(\mu)}}M^{(\mu)}\rho}{a_1^{(\mu)}},\quad v^{(\mu)}=0.$$

PROOF: By direct computations, we have

$$\begin{pmatrix} e^{-i\theta^{(\mu)}}u^{(\mu)} \\ e^{i\theta^{(\mu)}}v^{(\mu)} \end{pmatrix}=\frac{M^{(\mu)}x^{(\mu)}}{2}\{\tilde{h}^{-\alpha}T^{(\mu)*}\tilde{h}^{1+2\alpha}Q^{(\mu)}+\tilde{h}^{\alpha}T^{(\mu)*}\tilde{h}^{1-2\alpha}Q^{(\mu)}\}.$$

Then, Lemma 7.2 gives the desired result. □

LEMMA 7.4.

(a) *Let $(r,\ \alpha)=\left(\frac{1}{2},\ \pm\frac{1}{2}\right)$.*

Then, for each $\mu=0,\ 1$, we have

$$(f_0^{(\mu)},\ hg_0^{(\mu)})=0, \tag{7.4}$$
$$(f_0^{(\mu)},\ hf_0^{(\mu)})+(g_0^{(\mu)},\ hg_0^{(\mu)})=x^{(\mu)} \tag{7.5}$$

with

$$\chi^{(\mu)}=\left(\frac{a_0^{(\mu)}}{a_1^{(\mu)}}\right)^{1/2}.$$

(b) *Let* $(r, \alpha)=(1, 0)$. *Then, for each* $\mu=0, 1$, *we have* (7.4) *and*

$$(f_0^{(\mu)}, hf_0^{(\mu)})+(g_0^{(\mu)}, hg_0^{(\mu)})=\frac{a_0^{(\mu)}}{a_1^{(\mu)}}$$

independently of $\chi^{(\mu)}$.

PROOF: Direct computations using (7.2). □

We are now ready to present examples of the quadratic operator H given by (3.4). We give only three examples.

EXAMPLE 1: We take $\mu=0$, $\theta^{(0)}=\pi$, $(r, \alpha)=(1, 0)$, $a_1^{(0)}=1$, and $a_0^{(0)}=\omega_0$. Then, using the preceding lemmas, we see that H is written as

$$H=H_0+ab(\rho)^*+a^*b(\rho).$$

This is the Hamiltonian of the so-called *RWA* oscillator [5].

EXAMPLE 2: We take $\mu=0$, $\theta^{(0)}=\pi$, $(r, \alpha)=\left(\frac{1}{2}, \frac{1}{2}\right)$, $\chi^{(0)}=\omega_0$. Then we have

$$H=H_0+q\cdot\phi(\rho),$$

where

$$q=\frac{1}{\sqrt{2\omega_0}}(a^*+a)$$

is the position operator in $L^2(\boldsymbol{R})$(see (2.28)) and

$$\phi(f)=\frac{1}{\sqrt{2}}(b(h^{-1/2}f)^*+b(h^{-1/2}f)),\ h^{-1/2}f\in\mathscr{H}$$

is the "time zero scalar field" in $\mathscr{F}_s(\mathscr{H})$. In particular, if we take $\omega(k)=|k|^2$ and $d=3$, then the operator H is the Hamiltonian of the model considered in [2, 3].

EXAMPLE 3: We take $\mu=1$, $(r, \alpha)=\left(\frac{1}{2}, \frac{1}{2}\right)$, $\chi^{(1)}=\omega_0$, $a_0^{(1)}=\omega_0^2a_1^{(1)}$, $M^{(1)}=1$, and $\theta^{(1)}=0$. Then, we have

$$H=H_0+\frac{\omega_0^2}{2}(q-\phi(\rho))^2-\frac{\omega_0^2}{2}q^2.$$

This operator may be regarded as a scalar field version of the

Hamiltonian of a harmonically bound electron minimally coupled to a quantized radiation field [4].

In all the examples above, one can easily see that, under suitable regularity conditions for ρ and a choice of the space dimension d, the operators $V^{(\mu)}$ and $hV^{(\mu)}$ are Hilbert-Schmidt (see Lemma 6.11). One can also prove that, under suitable conditions for the parameters or by "renormalizations", each Hamiltonian satisfies the condition **(H)** in Section III.

In the same way as in the above examples, we can generate a variety of quadratic Hamiltonians for a system of a one dimensional quantum harmonic oscillator coupled to a quantum scalar field. These Hamiltonians are isospectral to $d\Gamma(\omega^r)$ up to constant addition.

VIII. Other aspects

In this section we follow the notation in Sections III and IV.

8.1. The n-point functions of the oscillator.

From a point of view of QFT, it is interesting to investigate properties of the n-point functions of the oscillator

$$W_n(t_1, \cdots, t_n)=(q(t_1)\cdots q(t_n)\Omega, \Omega)$$

where $q(t)$ is the time evolution of q by H:

$$q(t)=e^{itH}qe^{-itH}, \ t\in \boldsymbol{R}.$$

It is easy to see that $W_{2n-1}\equiv 0$, $n\in \boldsymbol{N}$, and W_{2n} is written as a sum of n products of W_2, which is computed as

$$W_2(t, s)=c(F, e^{i(t-s)h}F), \ t, s\in \boldsymbol{R},$$

with

$$c=\frac{1}{2\omega_0(\|f_0\|^2-\|g_0\|^2)^2}, \qquad F=f_0-g_0.$$

(Use (3.15) and (4.27).)

In view of quantum statistical mechanics (e.g., [13]), by a standard method, we can construct a quasi free equilibrium state ω_β at any finite temperature $1/\beta$ of the quantum system governed by the Hamiltonian H such that the two-point correlation function

$$W_2^\beta(t, s)\equiv \omega_\beta(q(t)q(s))$$

is given as

$$W_2^\beta(t, s)=c(F, (e^{\beta h}-1)^{-1}(e^{[\beta+i(t-s)]h}+e^{-i(t-s)h})F).$$

Note that $W_2^\beta(t, s) \to W_2(t, s)$ as $\beta \to \infty$ (the zero temperature limit).

Concerning the examples in Section VI, the long-time behavior of $W_2(t, s)$ and $W_2^\beta(t, s)$ has been analyzed in detail in [9].

8. 2. Supersymmetric embedding.

Recently rigorous analysis has been undertaken on supersymmetric embedding of "ordinary" quantum field models [7] (For a formal theory, see [6] and for quantum mechanical (finite degrees of freedom) cases, see references in [7]). A non-negative self-adjoint operator A is said to be supersymmetrically embeddable if it is unitarily equivalent to a reduced part of a supersymmetric Hamiltonian [7]. Under the assumption of Theorem 3.1, $\hat{H}$ is supersymmetrically embeddable. This follows from the fact that $\hat{H}$ is unitarily equivalent to $d\,\Gamma(h)$ acting in $\mathscr{F}_s(\mathscr{H})$ and that $d\Gamma(h)$ is a reduced part of a supersymmetric Hamiltonian [8]. The explicit construction of a supersymmetric quantum theory in which $\hat{H}$ is embedded can be done in the same way as in [7] (See [8] for a more general construction).

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