Certain properties of submanifolds in a Riemannian manifold of constant curvature admitting a conformal Killing vector

By Hidemaro KÔJYÔ

Introduction. In a 3-dimensional Euclidean space E^3 a sphere is characterized by certain special properties of a closed surface. In 1900, H. Liebmann [1][1)] has proved that an ovaloid with constant mean curvature H in E^3 is a sphere. W. Süss [2] generalized this result for a closed convex hypersurface in an n-dimensional Euclidean space E^n. Various generalizations of the condition $H=$const. in the Liebmann-Süss theorem have been studied by many investigators and it is one of the interesting problem in the differential geometry in the large. The interesting results of this problem for a closed orientable hypersurface in E^n were given by T. Bonnesen and W. Fenchel [3], H. Hopf [4], C. C. Hsiung [5], A. D. Alexandrov [6], [7], S. S. Chern [9], S. S. Chern and C. C. Hsiung [39], K. Amur [40], D. J. Stong [41], R. L. Bishop and S. J. Goldberg [42], R. B. Gardner [43] and J. K. Shahin [44]. In the field of these investigations the integral formulas of Minkowski type has played one of the important role.

We consider an ovaloid F in E^3, and let H and K be the mean curvature and the Gauss curvature at a point P of F respectively. Then the integral formula of Minkowski is

$$\iint_F (Kp+H)dA=0\,,$$

where p denotes the oriented distance from a fixed point O in E^3 to the tangent space of F at P and dA is the area element of F at P.

As generalization of this formula for a closed orientable hypersurface in E^n, C. C. Hsiung derived the integral formulas of Minkowski type, and gave certain characterizations of hyperspheres in E^n. Afterward Y. Katsurada [10], [12] generalized more these formulas of Hsiung in a Riemannian manifold, that is, derived the integral formulas of generalized Minkowski type which are valid for a closed orientable hypersurface V^{n-1} in an n-dimensional Riemannian manifold R^n and proved the following theorem:

1) Numbers in brackets refer to the references at the end of the paper.

THEOREM 0.1. (Y. Katsurada) *Let R^n be a Riemannian manifold of constant curvature which admits a vector field ξ^i generating a continuous one-parameter group of conformal transformations in R^n and V^{n-1} a closed orientable hypersurface in R^n such that*

(i) $H_1 = const.$,

(ii) $n_i\xi^i$ *has fixed sign on* V^{n-1}.

Then every point of V^{n-1} is umbilic, where H_1 and n_i denote the first mean curvature of V^{n-1} and covariant component of a unit normal vector of V^{n-1} respectively.

THEOREM 0.2. (Y. Katsurada) *Let R^n be a Riemannian manifold of constant curvature which admits a vector field ξ^i generating a continuous one-parameter group of conformal transformations in R^n and V^{n-1} a closed orientable hypersurface in R^n such that*

(i) $k_1, k_2, \cdots, k_{n-1} > 0$ *on* V^{n-1},

(ii) $H_\nu = const.$ *for any* ν $(1 < \nu \leqq n-2)$,

(iii) $n_i\xi^i$ *has fixed sign on* V^{n-1}.

Then every point of V^{n-1} is umbilic, where k_p $(p=1, 2, \cdots, n-1)$ and H_ν denote principal curvature of V^{n-1} and the ν-th mean curvature of V^{n-1} respectively.

THEOREM 0.3. (Y. Katsurada) *Let R^n be a Riemannian manifold of constant curvature which admits a vector field ξ^i generating a continuous one-parameter group of conformal transformations in R^n and V^{n-1} a closed orientable hypersurface in R^n such that*

(i) $H_1 = const.$,

(ii) $n_i\xi^i$ *has fixed sign on* V^{n-1}.

Then V^{n-1} is isometric to a sphere.

The analogous problems for a closed orientable hypersurface V^{n-1} in R^n have been discussed by A. D. Alexandrov [8], K. Nomizu [45], [46], K. Yano [18], K. Nomizu and B. Smyth [47], R. C. Reilly [48], T. Ôtsuki [26], T. Nagai [16], M. Tani [27], T. Koyanagi [28] and T. Muramori [29]. Most of these investigations are related to the characterization of an umbilical hypersurface in R^n.

Certain generalizations of Theorem 0.1 and Theorem 0.3 for an m-dimensional closed orientable submanifold V^m in R^n $(m \leqq n-1)$ with constant curvature have been studied by Y. Katsurada [13], [14], T. Nagai [13], [15] and the present author [14] and the following theorems were proved:

THEOREM 0.4. (Y. Katsurada and T. Nagai) *Let R_n be a Riemannian manifold of constant curvature which admits a vector field ξ^i generating* *a continuous one-parameter group of concircular transformations in* R^n *and* V^m *a closed orientable submanifold in* R^n *such that*

(i) $H_1 = const.$ *and* $\Gamma''^{A}_{E\ \alpha} = 0$,[2)]

(ii) ξ^i *is contained in the vector space spanned by m independent tangent vectors and* $\underset{E}{n}^i$ *at each point on* V^m.

(iii) $\underset{E}{n}_i \xi^i$ *has fixed sign on* V^m.

Then every point of V^m *is umbilic with respect to Euler-Schouten unit vector* $\underset{E}{n}^i$, *where* $\underset{E}{n}_i$ *denotes covariant components of a unit normal vector which has the same direction with Euler-Schouten vector of* V^m[3)].

THEOREM 0.5. (Y. Katsurada and H. Kôjyô) *Let* R^n *be a Riemannian manifold of constant curvature which admits a vector field* ξ^i *generating a continuous one-parameter group of conformal transformations in* R^n *and* V^m *a closed orientable submanifold in* R^n *such that*

(i) $H_1 = const.$,

(ii) ξ^i *is contained in the vector space spanned by m independent tangent vectors and* $\underset{E}{n}^i$ *at each point on* V^m,

(iii) $\underset{E}{n}_i \xi^i$ *has fixed sign on* V^m.

Then every point of V^m *is umbilic with respect to Euler-Schouten unit vector* $\underset{E}{n}^i$.

THEOREM 0.6. (Y. Katsurada and H. Kôjyô) *Let* R^n *be a Riemannian manifold of constant curvature which admits a vector field* ξ^i *generating a continuous one-parameter group of conformal transformations in* R^n *and* V^m *a closed orientable submanifold in* R^n *such that*

(i) $\underset{E}{k}_1, \underset{E}{k}_2, \cdots, \underset{E}{k}_m > 0$ *on* V^m,

(ii) $\underset{E}{H}_\nu = const.$ *for any* ν $(1 < \nu \leqq m-1)$,

(iii) ξ^i *is contained in the vector space spanned by m independent tangent vectors and* $\underset{E}{n}^i$ *at each point on* V^m,

(iv) $\underset{E}{n}_i \xi^i$ *has fixed sign on* V^m.

Then every point of V^m *is umbilic with respect to Euler-Schouten unit vector* $\underset{E}{n}^i$.

The analogous problems for a closed orientable submanifold V^n in R^{n+p} has been discussed by K. Yano [20], [21], G. D. Ludden [49], D. E. Blair and G. D. Ludden [50], T. Nagai [15], and M. Okumura [33].

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2), 3). With respect to these object we shall find again in §1 of the present paper.

M. Okumura [32] has proved that a closed orientable submanifold of codimension 2 in an odd dimensional sphere with the natural normal contact structure is totally umbilical under certain conditions.

In 1969, S. Tachibana [31] has introduced a notion of a conformal Killing tensor field of degree 2 in R^{n+p} as the generalization of conformal Killing vector field. Furthermore T. Kashiwada [30] has given the definition of a conformal Killing tensor field of degree p ($p\geqq 2$) in R^{n+p}. Recently M. Morohashi [34], [35], [36] has found that a structure tensor of the normal contact structure is a conformal Killing tensor field of degree 2 defined by S. Tachibana. Making use of that fact, M. Morohashi investigated about a submanifold V^n of codimension p in a sphere S^{n+p} and a Riemannian manifold R^{n+p} of constant curvature and showed that the submanifold V^n is totally umbilical under certain conditions. Furthermore he obtained the following theorem:

THEOREM 0.7. (M. Morohashi) *Let R^{n+p} be a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a conformal Killing tensor field $T_{i_1\cdots i_p}$ of degree p and V^n a closed orientable submanifold in R^{n+p} such that*

(i) *the mean curvature vector field H^i of V^n is parallel with respect to the connection induced on the normal bundle,*

(ii) *$T_{i_1\cdots i_p} \underset{n+1}{n^{i_1}} \cdots \underset{n+p}{n^{i_p}}$ has fixed sign on V^n.*

Then every point of V^n is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$, where $\underset{A}{n^i}$ $(A=n+1, \cdots, n+p)$ denote p unit normal vectors of V^n.

However, in the above theorem, if V^n is a submanifold of codimension p in a Riemannian manifold R^{n+p}, then it has been assumed that the ambient space admits a conformal Killing tensor field of degree p.

The purpose of the present paper is to investigate a closed orientable submanifold V^n of codimension p in a Riemannian manifold R^{n+p} of constant curvature admitting a conformal Killing vector field without the assumption that R^{n+p} admits a conformal Killing tensor field of degree p. **§1** is devoted to give notations and fundamental formulas in the theory of submanifolds in a general Riemannian manifold and a Riemannian manifold of constant curvature respectively, and gives some important relations in R^{n+p}.

Let us denote by M^{n+p} a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a vector field ξ^i generating a continuous one-parameter group of conformal transformations in M^{n+p}. In **§2** we give the definition of a conformal Killing tensor field of degree p ($p\geqq 2$) and proves by the mathematical induction that M^{n+p} admits necessarily a conformal Killing tensor field of degree p. In § 3 we derive the integral formulas which are valid for a closed orientable submanifold V^n of M^{n+p}, and making use of the integral formulas and the results in § 2, we shall show that a closed orientable submanifold V^n in M^{n+p} is totally umbilic under some conditions, and from that result we prove that the submanifold V^n is isometric to a sphere S^n. We study in the last section § 4 the analogous problems under weaker conditions than the assumptions in § 3, and show that a closed orientable submanifold V^n in M^{n+p} is umbilical with respect to Euler-Schouten unit vector $\underset{E}{n}^i$.

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§ 1. Notations and fundamental formulas in the theory of submanifolds. Let R^{n+p} $(n+p>2)$ be a $(n+p)$-dimensional Riemannian manifold of class r $(r>2)$ and x^i and g_{ij} be the local coordinates and the positive definite metric tensor of R^{n+p} respectively. We now consider an n-dimensional closed orientable submanifold V^n in R^{n+p} whose local expression is

$$x^i = x^i(u^\alpha), \qquad \begin{array}{l} i=1,2,\cdots,n+p, \\ \alpha=1,2,\cdots,\mathrm{n}, \end{array}$$

where u^α denotes the local coordinates on V^n. We shall henceforth confine ourselves to that Latin indices run from 1 to $n+p$ and Greek indices from 1 to n. If we put

$$B_\alpha{}^i = \frac{\partial x^i}{\partial u^\alpha},$$

then n vectors $B_\alpha{}^i$ are linearly independent vectors tangent to V^n. The Riemannian metric tensor $g_{\alpha\beta}$ on V^n induced from g_{ij} is given by

$$g_{\alpha\beta} = g_{ij} B_\alpha{}^i B_\beta{}^j .$$

We indicate by $\underset{A}{n}^i$ $(A=n+1, n+2, \cdots, n+p)$ the contravariant components of p unit vectors which are normal to V^n and mutually orthogonal. Hence they satisfy the following relations:

$$g_{ij} B_\alpha{}^i \underset{A}{n}^j = 0, \qquad g_{ij} \underset{A}{n}^i \underset{B}{n}^j = \delta_{AB},$$

where δ_{AB} means the Kronecker delta. In this case a set of $n+p$ independent vectors

$$(1.1) \qquad (B_1{}^i, B_2{}^i, \cdots, B_n{}^i, \underset{n+1}{n}^i, \underset{n+2}{n}^i, \cdots, \underset{n+p}{n}^i)$$

determines an ennuple at each point on V^n. We put

$$B^\alpha{}_i = g^{\alpha\beta} g_{ij} B_\beta{}^j, \qquad \underset{A}{n}_i = g_{ij} \underset{A}{n}{}^j,$$

where $g^{\alpha\beta}$ are defined by the equations $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$. Then we have

$$\begin{aligned} g^{ij} &= g^{\alpha\beta} B_\alpha{}^i B_\beta{}^j + \sum_{A=n+1}^{n+p} \underset{A}{n}{}^i \underset{A}{n}{}^j, \\ g_{ij} &= g_{\alpha\beta} B^\alpha{}_i B^\beta{}_j + \sum_{A=n+1}^{n+p} \underset{A}{n}_i \underset{A}{n}_j, \qquad (1.2) \\ \delta^i_j &= B_\alpha{}^i B^\alpha{}_j + \sum_{A=n+1}^{n+p} \underset{A}{n}{}^i \underset{A}{n}_j. \end{aligned}$$

Denoting by the symbol ";" the operation of D-symbol due to van der Waerden-Bortolotti [52], from the definition we have

$$\begin{aligned} B_\alpha{}^i{}_{;\beta} &= (B_\gamma{}^i B^\gamma{}_j)_{;k} B_\alpha{}^j B_\beta{}^k \\ &= \Big(\delta^i_j - \sum_{A=n+1}^{n+p} \underset{A}{n}{}^i \underset{A}{n}_j\Big)_{;k} B_\alpha{}^j B_\beta{}^k \qquad (1.3) \\ &= -\sum_{A=n+1}^{n+p} (\underset{A}{n}_{j;k} B_\alpha{}^j B_\beta{}^k) \underset{A}{n}{}^i, \end{aligned}$$

and

$$\begin{aligned} \underset{A}{n}{}^i{}_{;\alpha} &= \Big(\sum_{B=n+1}^{n+p} \underset{B}{n}{}^i \underset{B}{n}_j\Big)_{;k} \underset{A}{n}{}^j B_\alpha{}^k \\ &= (\delta^i_j - B_\beta{}^i B^\beta{}_j)_{;k} \underset{A}{n}{}^j B_\alpha{}^k \qquad (1.4) \\ &= -(B^\beta{}_{j;k} \underset{A}{n}{}^j B_\alpha{}^k) B_\beta{}^i \end{aligned}$$

by virtue of the last equation in (1.2). Putting $H_{\alpha\beta}{}^i = B_\alpha{}^i{}_{;\beta}$, we call $H_{\alpha\beta}{}^i$ the Euler-Schouten curvature tensor. Therefore if we put $\underset{A}{b}_{\alpha\beta} = H_{\alpha\beta}{}^i \underset{A}{n}_i$, from (1.3) we have

$$H_{\alpha\beta}{}^i = \sum_{A=n+1}^{n+p} \underset{A}{b}_{\alpha\beta} \underset{A}{n}{}^i. \qquad (1.5)$$

We call $\underset{A}{b}_{\alpha\beta}$ the second fundamental tensor with respect to $\underset{A}{n}{}^i$. Transvecting (1.5) with $g^{\alpha\beta}$, we find

$$g^{\alpha\beta} H_{\alpha\beta}{}^i = \sum_{A=n+1}^{n+p} n \underset{A}{H}_1 \underset{A}{n}{}^i, \qquad (1.6)$$

where we put $\underset{A}{H}_1 = \frac{1}{n} g^{\alpha\beta} \underset{A}{b}_{\alpha\beta}$. $\underset{A}{H}_1$ is called the first mean curvature of V^n for the normal vector $\underset{A}{n}{}^i$.

On the other hand, the equation (1.4) may put as follows:

$$\underset{A}{n}{}^i{}_{;\alpha} = \underset{A}{\gamma}{}_\alpha{}^\beta B_\beta{}^i .$$

Multiplying the above equation by $g_{ij}B_\gamma{}^j$ and contracting, we have

$$g_{ij}B_\gamma{}^j \underset{A}{n}{}^i{}_{;\alpha} = \underset{A}{\gamma}{}_\alpha{}^\beta g_{\beta\gamma} .$$

Since we have

$$\begin{aligned} \underset{A}{b}{}_{\gamma\alpha} &= g_{ij}B_\gamma{}^j{}_{;\alpha} \underset{A}{n}{}^i \\ &= -g_{ij}B_\gamma{}^j \underset{A}{n}{}^i{}_{;\alpha} , \end{aligned}$$

by means of the last equation we get

$$\underset{A}{b}{}_{\gamma\alpha} = -\underset{A}{\gamma}{}_\alpha{}^\beta g_{\beta\gamma} .$$

Consequently we obtain

$$\underset{A}{n}{}^i{}_{;\alpha} = -\underset{A}{b}{}_\alpha{}^\gamma B_\gamma{}^i . \tag{1.7}$$

This equation is called the equation of Weingarten.
By virtue of (1.3) and (1.4), after some calculations we find

$$B_\alpha{}^i{}_{;\beta} = \frac{\partial B_\alpha{}^i}{\partial u^\beta} + \Gamma^i_{hj} B_\alpha{}^h B_\beta{}^j - \Gamma'^\gamma_{\alpha\beta} B_\gamma{}^i ,$$

$$\underset{A}{n}{}^i{}_{;\alpha} = \frac{\partial \underset{A}{n}{}^i}{\partial x^j} B_\alpha{}^j + \Gamma^i_{hj} \underset{A}{n}{}^h B_\alpha{}^j - \Gamma''^B_{A\alpha} \underset{B}{n}{}^i ,$$

where Γ^i_{hj} are the Christoffel symbol of the first kind formed with g_{ij} and

$$\Gamma'^\gamma_{\alpha\beta} = \Gamma^i_{hj} B_\alpha{}^h B_\beta{}^j B^\gamma{}_i + \frac{\partial B_\alpha{}^i}{\partial u^\beta} B^\gamma{}_i ,$$

$$\Gamma''^B_{A\alpha} = \frac{\partial \underset{A}{n}{}^i}{\partial x^j} B_\alpha{}^j \underset{B}{n}{}_i + \Gamma^i_{hj} \underset{A}{n}{}^h B_\alpha{}^j \underset{B}{n}{}_i .$$

Since $\underset{A}{n}{}^i \underset{B}{n}{}_i = \delta_{AB}$, from the last relation we can easily find

$$\Gamma''^B_{A\alpha} + \Gamma''^A_{B\alpha} = 0 . \tag{1.8}$$

Let H^i be the mean curvature vector field of V^n. Then H^i is given by

$$\begin{aligned} H^i &= \frac{1}{n} g^{\alpha\beta} H_{\alpha\beta}{}^i \\ &= \frac{1}{n} \sum_{A=n+1}^{n+p} \underset{A}{b}{}_\alpha{}^\alpha \underset{A}{n}{}^i , \end{aligned} \tag{1.9}$$

and H^i is independent of the choice of mutually orthogonal unit normal vectors.

Now we take a unit normal vector $\underset{n+1}{n^i}$ in the direction of the mean curvature vector field H^i. Then the components of the vector $\underset{n+1}{n^i}$ are independent of a change of parameters u^α on V^n, that is, the vector $\underset{n+1}{n^i}$ is determined uniquely at each point on V^n. We may consider the Euler-Schouten unit vector $\underset{E}{n^i}$ as one of $\underset{A}{n^i}$ in (1. 1). Consequently, putting $\underset{E}{n^i} = \underset{n+1}{n^i}$, we take a set of $n+p$ independent vectors

$$(1.10) \qquad (B_1^{\ i}, B_2^{\ i}, \cdots, B_n^{\ i}, \underset{E}{n^i}, \underset{n+2}{n^i}, \cdots, \underset{n+p}{n^i})$$

as an ennuple at each point on V^n.

The first mean curvature of V^n for normal vector $\underset{E}{n^i}$ is the socalled first mean curvature of V^n. Hence we denote it by H_1 without subscript $_E$. In this case, with respect to the ennuple (1. 10) we get from (1. 6)

$$(1.11) \qquad g^{\alpha\beta} H_{\alpha\beta}^{\ \ i} = nH_1 \underset{E}{n^i}.$$

When at each point of V^n the second fundamental tensors $\underset{A}{b_{\alpha\beta}}$ are proportional to the metric tensor $g_{\alpha\beta}$, that is, satisfying the following condition:

$$\underset{A}{b_{\alpha\beta}} = \underset{A}{H_1} g_{\alpha\beta},$$

we call V^n a totally umbilical submanifold. Then we have the following lemma:

LEMMA 1. 1. *A necessary and sufficient condition for V^n to be totally umbilical is that the following relations are satisfied*:

$$\underset{A}{b_{\alpha\beta}} \underset{A}{b^{\alpha\beta}} = \frac{1}{n} (\underset{A}{b_\gamma^{\ \gamma}})^2.$$

PROOF. The above equations follows from the following relations:

$$\left(\underset{A}{b_{\alpha\beta}} - \frac{1}{n} \underset{A}{b_\gamma^{\ \gamma}} g_{\alpha\beta}\right)\left(\underset{A}{b^{\alpha\beta}} - \frac{1}{n} \underset{A}{b_\gamma^{\ \gamma}} g^{\alpha\beta}\right) = \underset{A}{b_{\alpha\beta}} \underset{A}{b^{\alpha\beta}} - \frac{1}{n}\left(\underset{A}{b_\gamma^{\ \gamma}}\right)^2,$$

and the positive definiteness of the Riemannian metric $g_{\alpha\beta}$.

Next we consider the normal bundle of V^n. For a normal vector n^i, if the normal part of $n^i{}_{|\alpha}$ vanishes identically along V^n, then we call that n^i is parallel with respect to the connection induced on the normal bundle. The symbol " $|$ " denotes the operator of covariant derivative along V^n. Thus we have the following lemma:

LEMMA 1. 2. *Let V^n be a submanifold of a Riemannian manifold R^{n+p}. In order that the mean curvature vector field H^i of V^n is parallel with respect to the connection induced on the normal bundle, it is necessary and sufficient that*

$$\frac{\partial \underset{A}{H_1}}{\partial u^\alpha} - \Gamma''^{B}_{A\alpha} \underset{B}{H_1} = 0 .$$

PROOF. Since the assumption of Lemma 1. 2 means that the covariant derivative $H^i{}_{|\alpha}$ of the mean curvature vector field is tangent to the submanifold V^n. Differentiating (1. 9) covariantly we get

$$\begin{aligned} H^i{}_{|\alpha} &= \Big(\sum_{A=n+1}^{n+p} \underset{A}{H_1} \underset{A}{n^i} \Big)_{|\alpha} \\ &= \sum_{A=n+1}^{n+p} \underset{A}{H_1} \underset{A}{n^i}{}_{|\alpha} + \sum_{A=n+1}^{n+p} \underset{A}{H_1}{}_{|\alpha} \underset{A}{n^i} \\ &= - \sum_{A=n+1}^{n+p} \underset{A}{H_1} \underset{A}{b_\alpha{}^\gamma} B_\gamma{}^i + \sum_{A=n+1}^{n+p} \Big(\frac{\partial \underset{A}{H_1}}{\partial u^\alpha} - \Gamma''^{B}_{A\alpha} \underset{B}{H_1} \Big) \underset{A}{n^i} \end{aligned}$$

by virtue of (1. 7). Then we have

$$\frac{\partial \underset{A}{H_1}}{\partial u^\alpha} - \Gamma''^{B}_{A\alpha} \underset{B}{H_1} = 0 .$$

LEMMA 1. 3. *Let V^n be a submanifold of Riemannian manifold R^{n+p}. If the mean curvature vector field H^i of V^n is parallel with respect to the connection induced on the normal bundle, then the mean curvature H_1 of V^n is constant.*

PROOF. The mean curvature H_1 of V^n is given by

$$H_1^2 = \frac{1}{n^2} \sum_{A=n+1}^{n+p} (\underset{A}{b_\alpha{}^\alpha})^2 .$$

Differentiating the above equation covariantly and making use of Lemma 1. 2, we find

$$\begin{aligned} \frac{\partial H_1^2}{\partial u^\alpha} &= \frac{2}{n^2} \sum_{A=n+1}^{n+p} \underset{A}{b_\beta{}^\beta} \frac{\partial \underset{A}{b_\gamma{}^\gamma}}{\partial u^\alpha} \\ &= \frac{2}{n^2} \sum_{A=n+1}^{n+p} \underset{A}{b_\beta{}^\beta} \underset{B}{b_\gamma{}^\gamma} \Gamma''^{B}_{A\alpha} = 0 , \end{aligned}$$

by virtue of (1. 8). This equation shows that H_1^2 is constant.
Consequently from Lemma 1. 2 and Lemma 1. 3, we obtain the following lemma :

LEMMA 1.4. *Let V^n be a submanifold of a Riemannian manifold R^{n+p}. In order that the mean curvature vector field H^i of V^n is parallel with respect to the connection induced on the normal bundle, it is necessary and sufficient that*

$$H_1 = const. \quad and \quad \Gamma''^{E}_{A\alpha} = 0 .$$

PROOF. As we put $\underset{E}{n^i} = \underset{n+1}{n^i}$, we have $\underset{B}{H_1}=0$. $(B \neq E)$

Then from Lemma 1.2, it follows that

$$\frac{\partial H_1}{\partial u^\alpha} = 0 \quad \text{and} \quad \Gamma''^{E}_{A\alpha} H_1 = 0 .$$

Therefore we obtain easily the result.

REMARK. When $p=1$, that is, V^n is a closed orientable hypersurface in R^{n+1}, it is always satisfied that the mean curvature vector field H^i of V^n is parallel with respect to the connection induced on the normal bundle.

We now write the equations of Gauss, Mainardi-Codazzi and Ricci-Kühne:

$$(1.12) \qquad R_{ijkl} B_\alpha{}^i B_\beta{}^j B_\gamma{}^k B_\delta{}^l = R_{\alpha\beta\gamma\delta} - \sum_{A=n+1}^{n+p} \underset{A}{b_{\beta\gamma}} \underset{A}{b_{\alpha\delta}} + \sum_{A=n+1}^{n+p} \underset{A}{b_{\beta\delta}} \underset{A}{b_{\alpha\gamma}} ,$$

$$(1.13) \qquad \underset{A}{b_{\alpha\gamma;\beta}} - \underset{A}{b_{\alpha\beta;\gamma}} = R_{ijkl} B_\alpha{}^i \underset{A}{n^j} B_\gamma{}^k B_\beta{}^l ,$$

$$(1.14) \qquad R_{ijkl} \underset{A}{n^i} \underset{B}{n^j} B_\alpha{}^k B_\beta{}^l = \underset{B}{b_{\gamma\alpha}} \underset{A}{b^\gamma{}_\beta} - \underset{B}{b_{\gamma\beta}} \underset{A}{b^\gamma{}_\alpha} + \Gamma''^{B}_{A\alpha;\beta} - \Gamma''^{B}_{A\beta;\alpha} + \sum_{C=n+1}^{n+p} \Gamma''^{B}_{C\alpha} \Gamma''^{C}_{A\beta} - \sum_{C=n+1}^{n+p} \Gamma''^{B}_{C\beta} \Gamma''^{C}_{A\alpha} ,$$

where R_{ijkl} is the curvature tensor of R^{n+p}.

Let M^{n+p} be a Riemannian manifold of constant curvature. Then the curvature tensor $R^i{}_{jkl}$ of M^{n+p} has the form

$$(1.15) \qquad R^i{}_{jkl} = k(g_{jk}\delta^i_l - g_{jl}\delta^i_k) ,$$

where k is a constant given by $k = \dfrac{R}{(n+p)(n+p-1)}$ and R is the scalar curvature.

If M^{n+p} has the curvature tensor of the form (1.15), then equations (1.12), (1.13) and (1.14) can be rewritten respectively as

$$(1.16) \qquad R_{\alpha\beta\gamma\delta} = k(g_{\beta\gamma} g_{\alpha\delta} - g_{\alpha\gamma} g_{\beta\delta}) + \sum_{A=n+1}^{n+p} \underset{A}{b_{\beta\gamma}} \underset{A}{b_{\alpha\delta}} - \sum_{A=n+1}^{n+p} \underset{A}{b_{\beta\delta}} \underset{A}{b_{\alpha\gamma}} ,$$

$$(1.17) \qquad \underset{A}{b_{\alpha\gamma;\beta}} - b_{\alpha\beta;\gamma} = 0 ,$$

(1.18) $$\underset{B}{b}_{\gamma\alpha}\underset{A}{b}^{\gamma}{}_{\beta}-\underset{B}{b}_{\gamma\beta}\underset{A}{b}^{\gamma}{}_{\alpha}+\Gamma''^{B}_{A\alpha;\beta}-\Gamma''^{B}_{A\beta;\alpha}+\sum_{C=n+1}^{n+p}\Gamma''^{B}_{C\alpha}\Gamma''^{C}_{A\beta}-\sum_{C=n+1}^{n+p}\Gamma''^{B}_{C\beta}\Gamma''^{C}_{A\alpha}=0\,.$$

Let V^n be a submanifold of M^{n+p}. Then from Lemma 1.4 and (1.17) we have

(1.19) $$\underset{A}{b}^{\alpha}{}_{\beta;\alpha}=0\,.$$

When there exist mutually orthogonal unit normal vector fields $\underset{A}{n}^i$ such that $\Gamma''^{B}_{A\alpha}=0$, we call that the connection induced on the normal bundle is trivial. Then J. Erbacher [38] gave the following lemma:

LEMMA 1.15. (J. Erbacher) *Let M^{n+p} be a Riemannian manifold of constant curvature. Then the connection induced on the normal bundle is trivial if and only if the following relation is satisfied*:

$$\underset{B}{b}_{\gamma\alpha}\underset{A}{b}^{\gamma}{}_{\beta}=\underset{B}{b}_{\gamma\beta}\underset{A}{b}^{\gamma}{}_{\alpha}\,.$$

REMARK. When $p=1$, it is always satisfied that the connection induced on the normal bundle is trivial. When $p=2$, the connection induced on the normal bundle is trivial under the condition that the mean curvature vector field H^i of V^n is parallel with respect to the connection induced on the normal bundle.

If the second fundamental tensor $\underset{E}{b}_{\alpha\beta}$ with respect to $\underset{E}{n}^i$ is proportional to the metric tensor $g_{\alpha\beta}$, that is, satisfying $\underset{E}{b}_{\alpha\beta}=\lambda g_{\alpha\beta}$, where λ is a scalar function on V^n, then we say that the submanifold V^n is umbilical with respect to Euler-Schouten unit normal vector $\underset{E}{n}^i$, or simply pseudo-umbilical. Thus we have the following lemma:

LEMMA 1.6. *A necessary and sufficient condition for V^n to be umbilical with respect to Euler-Schouten unit vector $\underset{E}{n}^i$ is that the following relation is satisfied*:

$$\underset{E}{b}_{\alpha\beta}\underset{E}{b}^{\alpha\beta}=nH_1^2\,.$$

PROOF. The above equation follows from the following relation:

$$\left(\underset{E}{b}_{\alpha\beta}-\frac{1}{n}\underset{E}{b}_{\gamma}{}^{\gamma}g_{\alpha\beta}\right)\cdot\left(\underset{E}{b}^{\alpha\beta}-\frac{1}{n}\underset{E}{b}_{\gamma}{}^{\gamma}g^{\alpha\beta}\right)=\underset{E}{b}_{\alpha\beta}\underset{E}{b}^{\alpha\beta}-\frac{1}{n}(\underset{E}{b}_{\gamma}{}^{\gamma})^2$$
$$=\underset{E}{b}_{\alpha\beta}\underset{E}{b}^{\alpha\beta}-nH_1^2\,,$$

and the positive definiteness of the Riemannian metric $g_{\alpha\beta}$.

§2. Conformal Killing tensor fields of a Riemannian manifold M^{n+p} of constant curvature. Let ξ^i be a vector field in R^{n+p} such that

$$\underset{\xi}{\pounds}\, g_{ij} = \xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}, \tag{2.1}$$

where ϕ is a scalar field in R^{n+p} and the symbol $\underset{\xi}{\pounds}$ denotes the operator of Lie derivation with respect to ξ^i. Then ξ^i is called a conformal Killing vector field and a continuous one-parameter group G generated by an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i \delta\tau$$

is called a conformal transformation group. If $\phi = c$ ($c = const.$) in (2.1), then ξ^i is called a homothetic Killing vector field and the group G is called a homothetic transformation group. If ϕ vanishes identically in (2.1), then ξ^i is called a Killing vector field and the group G is called a motion.

As the generalization of conformal Killing vector field (2.1), we shall show the definition of a conformal Killing tensor field.

We shall call a skew symmetric tensor field T_{ij} a conformal Killing tensor field of degree 2 in R^{n+p} if there exists a vector field ρ_i such that

$$T_{ij;k} + T_{kj;i} = 2\rho_j g_{ik} - \rho_k g_{ij} - \rho_i g_{jk}. \tag{2.2}$$

The vector ρ_i is called the associated vector field of T_{ij}. If ρ_i vanishes identically in (2.2), then T_{ij} is called a Killing tensor field of degree 2. (cf. [31])

Furthermore, we shall generalize it to the case of degree p ($p \geqq 2$). A skew symmetric tensor field $T_{i_1 i_2 \cdots i_p}$ is called a conformal Killing tensor field of degree p in R^{n+p}, if there exists a skew symmetric tensor field $\rho_{i_1 i_2 \cdots i_{p-1}}$ such that

$$\begin{aligned} T_{i_1 i_2 \cdots i_p; i} + T_{i i_2 \cdots i_p; i_1} &= 2\rho_{i_2 \cdots i_p} g_{i_1 i} \\ &\quad - \sum_{h=2}^{p} (-1)^h (\rho_{i_1 \cdots \hat{i}_h \cdots i_p} g_{i_h i} + \rho_{i i_2 \cdots \hat{i}_h \cdots i_p} g_{i_h i_1}), \end{aligned} \tag{2.3}$$

where $\hat{i}_h$ means that i_h is omitted. We call $\rho_{i_1 i_2 \cdots i_{p-1}}$ the associated tensor field of $T_{i_1 i_2 \cdots i_p}$. If $\rho_{i_1 i_2 \cdots i_{p-1}}$ vanishes identically in (2.3), then $T_{i_1 i_2 \cdots i_p}$ is called a Killing tensor field of degree p. Especially, if R^{n+p} is a Riemannian manifold of constant curvature, then the associated tensor field of a conformal Killing tensor field of degree p is a Killing tensor field. (cf. [30])

From (2.3) we have

$$T_{i_1\cdots i\cdots i_p;i}+T_{i_1\cdots i\cdots i_p;i_h}=(-1)^{h-1}(T_{i_hi_1\cdots\hat{i}_h\cdots i_p;i}+T_{ii_1\cdots\hat{i}_h\cdots i_p;i_h})$$

$$=(-1)^{h-1}\Big\{2\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{i_hi}-\sum_{l=1}^{h-1}(-1)^{l+1}(\rho_{i_hi_1\cdots\hat{i}_l\cdots\hat{i}_h\cdots i_p}g_{i_li}$$

$$+\rho_{ii_1\cdots\hat{i}_l\cdots\hat{i}_h\cdots i_p}g_{i_li_h})-\sum_{k=h+1}^{p}(-1)^k(\rho_{i_hi_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_p}g_{i_ki}$$

$$+\rho_{ii_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_p}g_{i_hi_k})\Big\}$$

$$=(-1)^{h-1}\Big\{2\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{i_hi}-\sum_{l=1}^{h-1}(-1)^{l+1}(-1)^{h-2}(\rho_{i_1\cdots\hat{i}_l\cdots i_h\cdots i_p}g_{i_li}$$

$$+\rho_{i_1\hat{i}_l\cdots i\cdots i_p}g_{i_hi_l})-\sum_{k=h+1}^{p}(-1)^k(-1)^{h-1}(\rho_{i_1\cdots i_h\cdots\hat{i}_k\cdots i_p}g_{i_ki}$$

$$+\rho_{i_1\cdots i\cdots\hat{i}_k\cdots i_p}g_{i_hi_k})\Big\}$$

$$=-(-1)^h2\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{i_hi}-\sum_{l=1}^{h-1}(-1)^l(-1)^{2(h-1)}(\rho_{i_1\cdots\hat{i}_l\cdots i_h\cdots i_p}g_{i_li}$$

$$+\rho_{i_1\cdots\hat{i}_l\cdots i\cdots i_p}g_{i_li_h})-\sum_{k=h+1}^{p}(-1)^k(-1)^{2(h-1)}(\rho_{i_1\cdots i_h\cdots\hat{i}_k\cdots i_p}g_{ii_k}$$

$$+\rho_{i_1\cdots i\cdots\hat{i}_k\cdots i_p}g_{i_hi_k})\,.$$

Hence we get

$$T_{i_1\cdots i_h\cdots i_p;i}+T_{i_1\cdots i\cdots i_p;i_h}=-(-1)^h2\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{ii_h}$$

$$(2.4)\qquad -\sum_{\substack{l=1\\(l\neq h)}}^{p}(-1)^l(\rho_{i_1\cdots\hat{i}_l\cdots i_h\cdots i_p}g_{i_li}+\rho_{i_1\cdots\hat{i}_l\cdots i\cdots i_p}g_{i_li_h})\,.$$

If P_{ij} is a covariant tensor field, then we have

$$\underset{\xi}{\pounds}(P_{ij;k})-(\underset{\xi}{\pounds}P_{ij})_{;k}=-(\underset{\xi}{\pounds}\Gamma^l_{ki})P_{lj}-(\underset{\xi}{\pounds}\Gamma^l_{kj})P_{il}\,. \qquad \text{(cf. [51])}$$

Applying the above formula to the metric tensor g_{ij}, we obtain

$$(2.5)\qquad \underset{\xi}{\pounds}\Gamma^i_{jk}=\frac{1}{2}g^{il}\Big\{(\underset{\xi}{\pounds}g_{kl})_{;j}+(\underset{\xi}{\pounds}g_{jl})_{;k}-(\underset{\xi}{\pounds}g_{jk})_{;l}\Big\}.$$

Substituting (2.1) into (2.5), we find

$$(2.6)\qquad \underset{\xi}{\pounds}\Gamma^i_{jk}=\delta^i_j\phi_k+\delta^i_k\phi_j-g_{jk}\phi^i\,,$$

where $\phi_i=\phi_{;i}$ and $\phi^i=g^{ij}\phi_j$.
Substituting (2.6) into

$$\underset{\xi}{\pounds}R^i_{jkl}=(\underset{\xi}{\pounds}\Gamma^i_{jk})_{;l}-(\underset{\xi}{\pounds}\Gamma^i_{lk})_{;j}$$

we obtain

(2. 7) $$\underset{\xi}{\pounds} R^i{}_{jkl} = -\delta^i_l \phi_{j;k} + \delta^i_k \phi_{j;l} - g_{jk}\phi^i{}_{;l} + g_{jl}\phi^i{}_{;k} .$$

By contraction with respect to i and l, it follows from (2. 7) that

(2. 8) $$\underset{\xi}{\pounds} R_{jk} = -(n+p-2)\phi_{k;j} - g_{jk}\phi^i{}_{;i} ,$$

where R_{jk} is the Ricci tensor.
Transvecting (2. 8) with g^{jk}, we find

(2. 9) $$\underset{\xi}{\pounds} R = -2(n+p-1)\phi^i{}_{;i} - 2\phi R .$$

When R^{n+p} is an Einstein space, that is,

$$R_{jk} = \frac{R}{n+p} g_{jk} , \qquad R = const.,$$

we have, for a conformal Killing vector field ξ^i,

$$\underset{\xi}{\pounds} R_{jk} = \frac{R}{n+p} \underset{\xi}{\pounds} g_{jk} = \frac{2R}{n+p} \phi g_{jk} , \qquad \underset{\xi}{\pounds} R = 0 .$$

Consequently, from (2. 8) and (2. 9), we get

$$\frac{2R}{n+p}\phi g_{jk} = -(n+p-2)\phi_{k;j} - g_{jk}\phi^i{}_{;i} ,$$

$$(n+p-1)\phi^i{}_{;i} + R\phi = 0 ,$$

respectively. From these relations, it follows that

(2. 10) $$\phi_{;i;j} = -k\phi g_{ij} , \qquad k = \frac{R}{(n+p)(n+p-1)} .$$

Thus if an Einstein space of dimension $n+p$ ($n+p>2$) admits a conformal Killing vector field, then it admits a non-zero scalar function ϕ which satisfies the above equation.

LEMMA 2. 1. *Let R^{n+p} ($n+p>2$) be an Einstein space which admits a conformal Killing vector field ξ^i. Then R^{n+p} admits a Killing vector field.*

PROOF. We put

$$\rho_i = \xi_i + \frac{1}{k}\phi_i , \qquad k = \frac{R}{(n+p)(n+p-1)} .$$

Differentiating the above equation covariantly, by means of (2. 1) and (2. 10) we get

(2. 11) $$\rho_{i;j} + \rho_{j;i} = 0 .$$

REMARK. Since a space of constant curvature is necessarily an Einstein space, a Riemannian manifold of constant curvature admits a Killing vector field.

Let M^{n+p} be a $(n+p)$-dimensional Riemannian manifold of constant curvature.

LEMMA 2.2. *If M^{n+p} admits a conformal Killing vector field ξ^i, then M^{n+p} admits a skew symmetric tensor field T_{ij} of degree 2 such that*

$$T_{ij;k}=k(\rho_j g_{ki}-\rho_i g_{jk}).$$

PROOF. Since M^{n+p} admits a Killing vector field ρ_i by Lemma 2.1, differentiating (2.11) covariantly, we obtain

$$\rho_{i;j;k}+\rho_{j;i;k}=0.$$

From the above equation, we have

$$\rho_{i;j;k}+\rho_{j;i;k}+\rho_{i;k;j}+\rho_{k;i;j}-(\rho_{j;k;i}+\rho_{k;j;i})=0.$$

Then by virtue of Ricci's identity, we get

$$2\rho_{i;j;k}-\rho_h(R^h{}_{jik}+R^h{}_{kij}+R^h{}_{ikj})=0.$$

In consequence of Bianchi's identity the above equation reduces to

$$\rho_{i;j;k}+\rho_h R^h{}_{kji}=0.$$

Then by means of (1.15) the last equation turns to

$$\rho_{i;j;k}=k(\rho_j g_{ki}-\rho_i g_{jk}).$$

If we put $T_{ij}=\rho_{i;j}$, then the above equation is rewritten as follows:

(2.12) $$T_{ij;k}=k(\rho_j g_{ki}-\rho_i g_{jk}).$$

LEMMA 2.3. *Let M^{n+p} be a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a skew symmetric tensor field T_{ij} of degree 2 such that*

$$T_{ij;k}=k(\rho_j g_{ki}-\rho_i g_{jk}).$$

Then M^{n+p} admits a skew symmetric tensor field T_{ijk} of degree 3 such that

$$T_{ijk;l}=k(\rho_{jk} g_{il}-\rho_{ik} g_{jl}+\rho_{ij} g_{kl}),$$

where ρ_{jk} is a skew symmetric tensor field of degree 2 defined by

$$\rho_{jk}=\rho_j\phi_k-\rho_k\phi_j-\phi T_{jk}.$$

PROOF. We put

(2.13) $$T_{ijk}=T_{ij}\phi_k+T_{jk}\phi_i+T_{ki}\phi_j.$$

Then it is clear that T_{ijk} is skew symmetric with respect to all indices. Differentiating (2.13) covariantly, by means of (2.10) and (2.12) we have

$$T_{ijk;l}=k\Big\{(\rho_j\phi_k-\rho_k\phi_j-\phi T_{jk})g_{il}-(\rho_i\phi_k-\rho_k\phi_i-\phi T_{ik})g_{jl}$$
$$+(\rho_i\phi_j-\rho_j\phi_i-\phi T_{ij})g_{kl}\Big\}.$$

Hence if we put

$$\rho_{jk}=\rho_j\phi_k-\rho_k\phi_j-\phi T_{jk},$$

then the last equation turns to

$$T_{ijk;l}=k(\rho_{jk}g_{il}-\rho_{ik}g_{jl}+\rho_{ij}g_{kl}). \tag{2.14}$$

LEMMA 2.4. *Let M^{n+p} be a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a skew symmetric tensor field $T_{i_1\cdots i_{p-1}}$ of degree $p-1$ such that*

$$T_{i_1\cdots i_{p-1};i}=-k\sum_{h=1}^{p-1}(-1)^h\rho_{i_1\cdots\hat{i}_h\cdots i_{p-1}}g_{i_hi}, \tag{2.15}$$

where $\rho_{i_1\cdots\hat{i}_h\cdots i_{p-1}}$ is a skew symmetric tensor field of degree $p-2$. Then M^{n+p} admits a skew symmetric tensor field $T_{i_1\cdots i_p}$ of degree p such that

$$T_{i_1\cdots i_p;i}=-k\sum_{h=1}^{p}(-1)^h\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{i_hi},$$

where $\rho_{i_1\cdots\hat{i}_h\cdots i_p}$ is a skew symmetric tensor field of degree $p-1$ defined by

$$\rho_{i_1\cdots\hat{i}_h\cdots i_p}=\sum_{\substack{k=1\\(h\neq k)}}^{p}(-1)^k\rho_{i_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_p}\phi_{i_k}+\phi T_{i_1\cdots\hat{i}_h\cdots i_p}.$$

PROOF. We put

$$T_{i_1\cdots i_p}=\sum_{h=1}^{p}(-1)^hT_{i_1\cdots\hat{i}_h\cdots i_p}\phi_{i_h}. \tag{2.16}$$

Then it is clear that $T_{i_1\cdots i_p}$ is skew symmetric with respect to all indices. Differentiating (2.16) covariantly we have

$$T_{i_1\cdots i_p;i}=\sum_{h=1}^{p}(-1)^hT_{i_1\cdots\hat{i}_h\cdots i_p;i}\phi_{i_h}+\sum_{h=1}^{p}(-1)^hT_{i_1\cdots\hat{i}_h\cdots i_p}\phi_{i_h;i}.$$

Substituting (2.10) and (2.15) into this equation, we find

$$T_{i_1\cdots i_p;i}=\sum_{h=1}^{p}-(-1)^hk\sum_{\substack{k=1\\(h\neq k)}}^{p}(-1)^k\rho_{i_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_p}\phi_{i_h}g_{i_ki}$$
$$-k\phi\sum_{h=1}^{p}(-1)^hT_{i_1\cdots\hat{i}_h\cdots i_p}g_{i_hi}$$

$$=-k\sum_{h=1}^{p}(-1)^h\Big\{\sum_{\substack{k=1\\(h\neq k)}}^{p}(-1)^k\rho_{i_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_p}\phi_{i_k}+\phi T_{i_1\cdots\hat{i}_h\cdots i_p}\Big\}g_{i_h i}\,.$$

Hence if we put

$$\rho_{i_1\cdots\hat{i}_h\cdots i_p}=\sum_{\substack{k=1\\(h\neq k)}}^{p}(-1)^k\rho_{i_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_p}\phi_{i_k}+\phi T_{i_1\cdots\hat{i}_h\cdots i_p}\,,$$

then the last equation turns to

$$T_{i_1\cdots i_p;i}=-k\sum_{h=1}^{p}(-1)^h\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{i_h i}\,.$$

REMARK. Putting $p=2$ and $p=3$ in (2. 15), we obtain (2. 12) and (2. 14) respectively.

THEOREM 2. 5. *Let M^{n+p} be a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a conformal Killing vector field ξ^i. Then M^{n+p} admits a skew symmetric tensor field $T_{i_1\cdots i_p}$ of degree p such that*

$$T_{i_1\cdots i_p;i}=-k\sum_{h=1}^{p}(-1)^h\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{i_h i}\,.$$

PROOF. We shall prove Theorem 2. 5 by the mathematical induction. By virtue of Lemma 2. 2, Lemma 2. 3 and Lemma 2. 4, we can easily obtain the result.

COROLLARY 2. 6. *Let M^{n+p} be a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a conformal Killing vector field ξ^i. Then M^{n+p} admits a conformal Killing tensor field of degree p.*

PROOF. From (2. 15), we have

$$\begin{aligned}
&T_{i_1 i_2\cdots i_p;i}+T_{i i_2\cdots i_p;i_1}\\
&=-k\sum_{h=1}^{p}(-1)^h\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{i_h i}-k\sum_{\substack{h=1\\(h\neq 1)}}^{p}(-1)^h\rho_{i i_2\cdots\hat{i}_h\cdots i_p}g_{i_h i_1}\\
&=-k\Big\{-\rho_{i_2\cdots i_p}g_{i_1 i}+\sum_{h=2}^{p}(-1)^h\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{i_h i}\Big\}\\
&\quad-k\Big\{-\rho_{i_2\cdots i_p}g_{i_1 i}+\sum_{h=2}^{p}(-1)^h\rho_{i i_2\cdots\hat{i}_h\cdots i_p}g_{i_h i_1}\Big\}\\
&=k\Big\{2\rho_{i_2\cdots i_p}g_{i_1 i}-\sum_{h=2}^{p}(-1)^h(\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{i i_h}+\rho_{i i_2\cdots\hat{i}_h\cdots i_p}g_{i_1 i_h})\Big\}.
\end{aligned}$$

This equation shows that $T_{i_1\cdots i_p}$ is a conformal Killing tensor field of degree p whose associated tensor field is given by $k\rho_{i_2\cdots i_p}$. Therefore by Theorem 2. 5, we get the result.

§3. Certain conditions for V^n to be isometric to a sphere. Let M^{n+p} be a $(n+p)$-dimensional Riemannian manifold of constant curvature admitting a conformal Killing vector field ξ^i and V^n a closed orientable submanifold of codimension p in M^{n+p}. Then by virtue of Corollary 2.6, M^{n+p} admits a conformal Killing tensor field $T_{i_1\cdots i_p}$ of degree p with the associated tensor field $\rho_{i_1\cdots i_{p-1}}$. In this section we assume that the mean curvature vector field H^i of V^n is parallel with respect to the connection induced on the normal bundle and the connection induced on the normal bundle is trivial. Under these restrictions we derive some integral formulas which are valid for V^n in M^{n+p} and give some properties of V^n.

Now we put

$$f = T_{i_1\cdots i_p} \underset{n+1}{n^{i_1}} \cdots \underset{n+p}{n^{i_p}}, \tag{3.1}$$

$$\xi_\alpha = \sum_{h=1}^{p} \underset{n+h}{b_\alpha{}^\beta} T_{i_1\cdots i_h\cdots i_p} \underset{n+1}{n^{i_1}} \cdots B_\beta{}^{i_h} \cdots \underset{n+p}{n^{i_p}}, \tag{3.2}$$

$$\eta_\alpha = \sum_{h=1}^{p} \underset{n+h}{b_\gamma{}^\gamma} T_{i_1\cdots i_h\cdots i_p} \underset{n+1}{n^{i_1}} \cdots B_\alpha{}^{i_h} \cdots \underset{n+p}{n^{i_p}}. \tag{3.3}$$

LEMMA 3.1. *f, ξ_α and η_α are independent of the choice of mutually orthogonal unit normal vectors.*

PROOF. Let $\underset{A}{\tilde{n}^i}$ be another mutually orthogonal unit normal vectors. Then there exists an orthogonal matrix (U_{AB}) satisfying the following relations:

$$\sum_{A=n+1}^{n+p} U_{AB}U_{AC} = \delta_{BC}, \quad \sum_{C=n+1}^{n+p} U_{AC}U_{BC} = \delta_{AB}, \tag{3.4}$$
$$det.(U_{AB}) = 1,$$

and $\underset{A}{\tilde{n}^i}$ can be written as

$$\underset{A}{\tilde{n}^i} = \sum_{B=n+1}^{n+p} U_{AB} \underset{B}{n^i}. \tag{3.5}$$

Therefore we find

$$\begin{aligned}
\tilde{f} &= T_{i_1\cdots i_p} \underset{n+1}{\tilde{n}^{i_1}} \cdots \underset{n+p}{\tilde{n}^{i_p}} \\
&= T_{i_1\cdots i_p} (\sum_{A_1} U_{n+1A_1} \underset{A_1}{n^{i_1}}) \cdots (\sum_{A_p} U_{n+pA_p} \underset{A_p}{n^{i_p}}) \\
&= \sum_{A_1,\cdots,A_p} sgn\begin{pmatrix} n+1, & n+2, & \cdots, & n+p \\ A_1, & A_2, & \cdots, & A_p \end{pmatrix} U_{n+1A_1} \cdots U_{n+pA_p} T_{i_1\cdots i_p} \underset{n+1}{n^{i_1}} \cdots \underset{n+p}{n^{i_p}} \\
&= det.(U_{AB}) T_{i_1\cdots i_p} \underset{n+1}{n^{i_1}} \cdots \underset{n+p}{n^{i_p}} \\
&= f
\end{aligned}$$

by making use of (3. 4), (3. 5) and the skew symmetry of $T_{i_1\cdots i_p}$. The above equation shows that f is independent of the choice of mutually orthogonal unit normal vectors.

Next let $\underset{A}{\tilde{b}}_{\alpha\beta}$ be the second fundamental tensor with respect to $\underset{A}{\tilde{n}}^i$. Then by means of (1. 5) and (3. 5) we have

$$\begin{aligned} B_{\alpha}{}^{i}{}_{;\beta} &= \sum_{A=n+1}^{n+p} \underset{A}{\tilde{b}}_{\alpha\beta} \underset{A}{\tilde{n}}^i = \sum_{A,B=n+1}^{n+p} \underset{A}{\tilde{b}}_{\alpha\beta} U_{AB} \underset{B}{n}^i \\ &= \sum_{B=n+1}^{n+p} \left(\sum_{A=n+1}^{n+p} U_{AB} \underset{A}{\tilde{b}}_{\alpha\beta} \right) \underset{B}{n}^i = \sum_{B=n+1}^{n+p} \underset{B}{b}_{\alpha\beta} \underset{B}{n}^i , \end{aligned}$$

from which we get

$$\sum_{A=n+1}^{n+p} U_{AB} \underset{A}{\tilde{b}}_{\alpha\beta} = \underset{B}{b}_{\alpha\beta} .$$

By virtue of (3. 4) and the above equation, we find

$$\begin{aligned} \sum_{B=n+1}^{n+p} U_{CB} \underset{B}{b}_{\alpha\beta} &= \sum_{B=n+1}^{n+p} U_{CB} \sum_{A=n+1}^{n+p} U_{AB} \underset{A}{\tilde{b}}_{\alpha\beta} \\ &= \sum_{A,B=n+1}^{n+p} U_{CB} U_{AB} \underset{A}{\tilde{b}}_{\alpha\beta} \\ &= \sum_{A=n+1}^{n+p} \delta_{CA} \underset{A}{\tilde{b}}_{\alpha\beta} \\ &= \underset{C}{\tilde{b}}_{\alpha\beta} , \end{aligned}$$

from which we have

$$\sum_{B=n+1}^{n+p} U_{CB} \underset{B}{b}_{\alpha\beta} = \underset{C}{\tilde{b}}_{\alpha\beta} .$$

From (3. 5) and the last equation, we obtain

$$\begin{aligned} \tilde{\xi}_\alpha &= \sum_{h=1}^{p} \underset{n+h}{\tilde{b}}_{\alpha}{}^{\beta} T_{i_1\cdots i_h\cdots i_p} \underset{n+1}{\tilde{n}}^{i_1} \cdots B_\beta{}^{i_h} \cdots \underset{n+p}{\tilde{n}}^{i_h} \\ &= \sum_{k=1}^{p} \underset{n+k}{b}_{\alpha}{}^{\beta} \Big\{ \sum_{h=1}^{p} U_{n+h\, n+k} T_{i_1\cdots i_h\cdots i_p} (\sum_{A_1} U_{n+1A_1} \underset{A_1}{n}^{i_1}) \cdots B_\beta{}^{i_h} \cdots (\sum_{A_p} U_{n+pA_p} \underset{A_p}{n}^{i_p}) \Big\} \\ &= \sum_{k=1}^{p} \underset{n+k}{b}_{\alpha}{}^{\beta} \sum_{h=1}^{p} U_{n+h\, n+k} \sum_{l=1}^{p} (-1)^{h+l} sgn \begin{pmatrix} n+1 \cdots \widehat{n+l} \cdots n+p \\ A_1 \cdots \hat{A}_h \cdots A_p \end{pmatrix} \cdot U_{n+1A_1} \cdots \widehat{U_{n+hA_h}} \cdot \\ &\quad \cdots U_{n+pA_p} T_{i_1\cdots i_l\cdots i_p} \underset{n+1}{n}^{i_1} \cdots B_\beta{}^{i_l} \cdots \underset{n+p}{n}^{i_p} \\ &= \sum_{k=1}^{p} \underset{n+k}{b}_{\alpha}{}^{\beta} \sum_{h=1}^{p} U_{n+h\, n+k} \overline{U}_{n+h\, n+l} T_{i_1\cdots i_l\cdots i_p} \underset{n+1}{n}^{i_1} \cdots B_\beta{}^{i_l} \cdots \underset{n+p}{n}^{i_p} , \end{aligned}$$

where $\widehat{n+l}$, $\hat{A}_h$ and $\widehat{U_{n+hA_h}}$ denotes that $n+l$, A_h and U_{n+hA_h} are omitted respectively and $\overline{U}_{n+h\, n+l}$ means the cofactor of $U_{n+h\, n+l}$ in det. (U_{AB}). Since we have

$$\sum_{h=1}^{p} U_{n+h\, n+k}\overline{U}_{n+h\, n+l} = \begin{cases} det.(U_{AB}), & if\ k=l, \\ 0 \quad , & if\ k\neq l, \end{cases}$$

then we find

$$\tilde{\xi}_\alpha = det.(U_{AB})\cdot \sum_{k=1}^{p} \underset{n+k}{b_\alpha}{}^\beta T_{i_1\cdots i_k\cdots i_p} \underset{n+1}{n^{i_1}}\cdots B_\beta{}^{i_k}\cdots \underset{n+p}{n^{i_p}} = \xi_\alpha .$$

The above equation proves that ξ_α is independent of the choice of mutually orthogonal unit normal vectors. In the same way we can prove that η_α is independent of the choice of mutually orthogonal unit normal vectors. Consequently f, ξ_α and η_α are the scalar function and vector fields on V^n respectively.

Differentiating (3. 2) covariantly we have

$$\begin{aligned} \xi_{\alpha;\beta} = & \sum_{h=1}^{p}\Big(\underset{n+h}{b_\alpha}{}^\gamma{}_{;\beta} + \sum_{A=n+1}^{n+p} \Gamma''{}_{n+h\beta}^{A}\underset{A}{b_\alpha}{}^\gamma\Big) T_{i_1\cdots i_h\cdots i_p}\underset{n+1}{n^{i_1}}\cdots B_\gamma{}^{i_h}\cdots \underset{n+p}{n^{i_p}} \\ & + \sum_{h=1}^{p} \underset{n+h}{b_\alpha}{}^\gamma T_{i_1\cdots i_h\cdots i_p;i} B_\beta{}^i \underset{n+1}{n^{i_1}}\cdots B_\gamma{}^{i_h}\cdots \underset{n+p}{n^{i_p}} \\ & + \sum_{h=1}^{p} \underset{n+h}{b_\alpha}{}^\gamma T_{i_1\cdots i_h\cdots i_l\cdots i_p}\underset{n+1}{n^{i_1}}\cdots \sum_{\substack{l+1 \\ (l\neq h)}}^{p}\Big(\big(-\underset{n+l}{b_\beta}{}^\delta B_\delta{}^{i_l} + \sum_{A=n+1}^{n+p}\Gamma''{}_{n+l\,\beta}^{A}\underset{A}{n^{i_l}}\Big)\cdots B_\gamma{}^{i_h}\cdots \underset{n+p}{n^{i_p}} \\ & + \sum_{h=1}^{p} \underset{n+h}{b_\alpha}{}^\gamma T_{i_1\cdots i_h\cdots i_p}\underset{n+1}{n^{i_1}}\cdots H_{\gamma\beta}{}^{i_h}\cdots \underset{n+p}{n^{i_p}} \end{aligned}$$

by means of (1. 7).

Multiplying the last equation by $g^{\alpha\beta}$, by virtue of (1. 5), (3. 1), our assumption and the skew symmetry of $T_{i_1\cdots i_p}$ we get

$$\begin{aligned} \xi^\alpha{}_{;\alpha} = & \sum_{h=1}^{p} \underset{n+h}{b^{\alpha\gamma}}{}_{;\alpha} T_{i_1\cdots i_h\cdots i_p}\underset{n+1}{n^{i_1}}\cdots B_\gamma{}^{i_h}\cdots \underset{n+p}{n^{i_p}} \\ & + \sum_{h=1}^{p} \underset{n+h}{b^{\alpha\gamma}} T_{i_1\cdots i_h\cdots i_p;i} B_\alpha{}^i \underset{n+1}{n^{i_1}}\cdots B_\gamma{}^{i_h}\cdots \underset{n+p}{n^{i_p}} \\ & - \sum_{\substack{h,l=1 \\ (h\neq l)}}^{p} \underset{n+h}{b^{\alpha\gamma}}\underset{n+l}{b_\alpha}{}^\delta T_{i_1\cdots i_h\cdots i_l\cdots i_p}\underset{n+1}{n^{i_1}}\cdots B_\gamma{}^{i_h}\cdots B_\delta{}^{i_l}\cdots \underset{n+p}{n^{i_p}} \\ & + \sum_{A=n+1}^{n+p} f \underset{A}{b_{\alpha\gamma}}\underset{A}{b^{\alpha\gamma}} . \end{aligned}$$

The above equation turns to

$$\xi^{\alpha}{}_{;\alpha}=\frac{1}{2}\sum_{h=1}^{p}\underset{n+h}{b^{\alpha\gamma}}(T_{i_1\cdots i_h\cdots i_p;i}+T_{i_1\cdots i\cdots i_p;i_h})B_{\alpha}{}^{i}\underset{n+1}{n^{i_1}}\cdots B_{\gamma}{}^{i_h}\cdots\underset{n+p}{n^{i_p}}$$

$$-\sum_{\substack{h,l=1\\(h\neq l)}}^{p}\underset{n+h}{b^{\alpha\gamma}}\underset{n+l}{b_{\alpha}{}^{\delta}}T_{i_1\cdots i_h\cdots i_l\cdots i_p}\underset{n+1}{n^{i_1}}\cdots B_{\gamma}{}^{i_h}\cdots B_{\delta}{}^{i_l}\cdots\underset{n+p}{n^{i}}$$

$$+f\sum_{A=n+1}^{n+p}\underset{A}{b_{\alpha\gamma}}\underset{A}{b^{\alpha\gamma}}$$

$$=\frac{1}{2}\sum_{h=1}^{p}\underset{n+h}{b^{\alpha\gamma}}\Big\{-(-1)^{h}2\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{i_hi}-\sum_{\substack{k=1\\(h\neq k)}}^{p}(-1)^{k}(\rho_{i_1\cdots\hat{i}_k\cdots i_h\cdots i_p}g_{ii_k}$$

$$+\rho_{i_1\cdots\hat{i}_k\cdots i\cdots i_p}g_{i_hi_k})\Big\}B_{\alpha}{}^{i}\underset{n+1}{n^{i_1}}\cdots B_{\gamma}{}^{i_h}\cdots\underset{n+p}{n^{i_p}}$$

$$+f\sum_{A=n+1}^{n+p}\underset{A}{b_{\beta\gamma}}\underset{A}{b^{\beta\gamma}}$$

by virtue of our assumption, (1. 19) and (2. 4). Thus we obtain

$$\xi^{\alpha}{}_{;\alpha}=f\sum_{A=n+1}^{n+p}\underset{A}{b_{\beta\gamma}}\underset{A}{b^{\beta\gamma}}-\sum_{h=1}^{p}(-1)^{h}\underset{n+h}{b_{\beta}{}^{\beta}}\rho_{i_1\cdots\hat{i}_h\cdots i_p}\underset{n+1}{n^{i_1}}\cdots\underset{n+h}{\hat{n}^{i_h}}\cdots\underset{n+p}{n^{i_p}},$$

where $\underset{n+h}{\hat{n}^{i_h}}$ denotes that $\underset{n+h}{n^{i_h}}$ is omitted. Therefore by means of Green's theorem (cf. [25]) we get the following integral formula:

$$\text{(I)}\qquad\int_{V^n}\Big(f\sum_{A=n+1}^{n+p}\underset{A}{b_{\beta\gamma}}\underset{A}{b^{\beta\gamma}}-\sum_{h=1}^{p}(-1)^{h}\underset{n+h}{b_{\beta}{}^{\beta}}\rho_{i_1\cdots\hat{i}_h\cdots i_p}\underset{n+1}{n^{i_1}}\cdots\underset{n+h}{\hat{n}^{i_h}}\cdots\underset{n+p}{n^{i_p}}\Big)dV=0,$$

where dV is the area element of V^n.

Next, differentiating (3. 3) covariantly we have

$$\eta_{\alpha;\beta}=\sum_{h=1}^{p}\Big(\underset{n+h}{b_{\gamma}{}^{\gamma}{}_{;\beta}}+\sum_{A=n+1}^{n+p}\Gamma''{}_{n+h\,\beta}{}^{A}\underset{A}{b_{\gamma}{}^{\gamma}}\Big)T_{i_1\cdots i_h\cdots i_p}\underset{n+1}{n^{i_1}}\cdots B_{\alpha}{}^{i_h}\cdots\underset{n+p}{n^{i_p}}$$

$$+\sum_{h=1}^{p}\underset{n+h}{b_{\gamma}{}^{\gamma}}T_{{}_1\cdots i_h\cdots i_p;i}B_{\beta}{}^{i}\underset{n+1}{n^{i_1}}\cdots B_{\alpha}{}^{i_h}\cdots\underset{n+p}{n^{i_p}}$$

$$+\sum_{h=1}^{p}\underset{n+h}{b_{\gamma}{}^{\gamma}}T_{i_1\cdots i_l\cdots i_h\cdots i_p}\underset{n+1}{n^{i_1}}\cdots\sum_{l=1}^{p}\Big(-\underset{n+l}{b_{\beta}{}^{\delta}}B_{\delta}{}^{i_l}+\sum_{A=n+1}^{n+p}\Gamma''{}_{n+l\,\beta}{}^{A}\underset{A}{n^{i_l}}\Big)\cdots B_{\alpha}{}^{i_h}\cdots\underset{n+p}{n^{i_p}}$$

$$+\sum_{h=1}^{p}\underset{n+h}{b_{\gamma}{}^{\gamma}}T_{i_1\cdots i_h\cdots i_p}\underset{n+1}{n^{i_1}}\cdots H_{\alpha\beta}{}^{i_h}\cdots\underset{n+p}{n^{i_p}},$$

by means of (1. 7).

Multiplying the above equation by $g^{\alpha\beta}$, by virtue of our assumption, (1. 5) and (3. 1) we find

$$\eta^{\alpha}{}_{;\alpha}=\sum_{h=1}^{p}g^{\alpha\beta}\underset{n+h}{b_{\gamma}{}^{\gamma}{}_{;\beta}}T_{i_1\cdots i_h\cdots i_p}\underset{n+1}{n^{i_1}}\cdots B_{\alpha}{}^{i_h}\cdots\underset{n+p}{n^{i_p}}$$

$$+\sum_{h=1}^{p}\underset{n+h}{b_{\gamma}{}^{\gamma}}T_{i_1\cdots i_h\cdots i_p;i}g^{\alpha\beta}B_{\beta}{}^{i}\underset{n+1}{n^{i_1}}\cdots B_{\alpha}{}^{i_h}\cdots\underset{n+p}{n^{i_p}}$$

$$-\sum_{\substack{h,l=1\\(h\neq l)}}^{p} \underset{n+h}{b_{\gamma}}^{\gamma} \underset{n+l}{b}^{\alpha\delta} T_{i_1\cdots i_l\cdots i_h\cdots i_p} \underset{n+1}{n}^{i_1}\cdots B_{\delta}^{i_l}\cdots B_{\alpha}^{i_h}\cdots \underset{n+p}{n}^{i_p}$$

$$+f\sum_{A=n+1}^{n+p}(\underset{A}{b_{\beta}}^{\beta})^2 .$$

The last equation turns to

$$\eta^{\alpha}{}_{;\alpha}=\frac{1}{2}\sum_{h=1}^{p}\underset{n+h}{b_{\gamma}}^{\gamma}(T_{i_1\cdots i_h\cdots i_p;i}+T_{i_1\cdots i\cdots i_p;i_1})g^{\alpha\beta}B_{\beta}^{i}\underset{n+1}{n}^{i_1}\cdots B_{\alpha}^{i_h}\cdots \underset{n+p}{n}^{i_p}$$

$$+f\sum_{A=n+1}^{n+p}(\underset{A}{b_{\beta}}^{\beta})^2$$

$$=\frac{1}{2}\sum_{h=1}^{p}\underset{n+h}{b_{\gamma}}^{\gamma}\Big\{-(-1)^h\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{i_h i}-\sum_{\substack{k=1\\(h\neq k)}}^{p}(-1)^k(\rho_{i_1\cdots\hat{i}_k\cdots i_h\cdots i_p}g_{i_k i}$$

$$+\rho_{i_1\cdots\hat{i}_k\cdots i\cdots i_p}g_{i_h i_k})\Big\}g^{\alpha\beta}B_{\beta}^{i}\underset{n+1}{n}^{i^1}\cdots B_{\alpha}^{i_h}\cdots \underset{n+p}{n}^{i_p}$$

$$+f\sum_{A=n+1}^{n+p}(\underset{A}{b_{\beta}}^{\beta})^2 .$$

by virtue of our assumption, (1. 19) and (2. 4). Thus by means of the skew symmetry of $T_{i_1\cdots i_p}$ we have

$$\eta^{\alpha}{}_{;\alpha}=-n\sum_{h=1}^{p}(-1)^h\underset{n+h}{b_{\gamma}}^{\gamma}\rho_{i_1\cdots\hat{i}_h\cdots i_p}\underset{n+1}{n}^{i_1}\cdots \underset{n+h}{\hat{n}}^{i_h}\cdots \underset{n+p}{n}^{i_p}$$

$$+f\sum_{A=n+1}^{n+p}(\underset{A}{b_{\beta}}^{\beta})^2.$$

Therefore by means of Green's theorem we obtain the following integral formula:

$$(II)\qquad \int_{V^n}\Big(f\sum_{A=n+1}^{n+p}(\underset{A}{b_{\beta}}^{\beta})^2-n\sum_{h=1}^{p}(-1)^h\underset{n+h}{b_{\gamma}}^{\gamma}\rho_{i_1\cdots\hat{i}_h\cdots i_p}\underset{n+1}{n}^{i_1}\cdots \underset{n+h}{\hat{n}}^{i_h}\cdots \underset{n+p}{n}^{i_p}\Big)dV=0 .$$

Eliminating $\displaystyle\int_{V^n}\sum_{h=1}^{p}(-1)^h\underset{n+h}{b_{\beta}}^{\beta}\rho_{i_1\cdots\hat{i}_h\cdots i_p}\underset{n+1}{n}^{i_1}\cdots \underset{n+h}{\hat{n}}^{i_h}\cdots \underset{n+p}{n}^{i_p}\,dV$

from (I) and (II), we obtain

$$(3.6)\qquad \int_{V^n}f\sum_{A=n+1}^{n+p}\Big\{\underset{A}{b_{\alpha\beta}}\underset{A}{b}^{\alpha\beta}-\frac{1}{n}(\underset{A}{b_{\gamma}}^{\gamma})^2\Big\}dV=0 .$$

Hence we have the following theorem:

THEOREM 3. 2. *Let M^{n+p} be a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a vector field ξ^i generating a continuous one-parameter group of conformal transformations in M^{n+p} and V^n a closed orientable submanifold in M^{n+p} such that*

(i) *the mean curvature vector field H^i of V^n is parallel with respect to the connection induced on the normal bundle,*

(ii) *the connection induced on the normal bundle is trivial,*

(iii) *the scalar function f has fixed sign on V^n.*

Then the submanifold V^n is totally umbilical.

PROOF. From (3.6) and our assumption we have

$$\underset{A}{b}_{\alpha\beta}\underset{A}{b}^{\alpha\beta}-\frac{1}{n}(\underset{A}{b}_{\gamma}{}^{\gamma})^2=0\,,$$

because $\underset{A}{b}_{\alpha\beta}\underset{A}{b}^{\alpha\beta}-\frac{1}{n}(\underset{A}{b}_{\gamma}{}^{\gamma})^2$ is non negative. Thus this equation shows that V^n is totally umbilical by means of Lemma 1.1.

REMARK. When $p=1$, that is, V^n is a closed orientable hypersurface in M^{n+1}, Euler-Schouten unit normal vector $\underset{E}{n}^i$ is the unit normal vector n^i of V^n. In this case our assumption (i) and (ii) in Theorem 3.2 is always satisfied. Accordingly when $p=1$, Theorem 3.2 coincides with Theorem 0.1 due to Y. Katsurada.

From the above theorem and the following theorem due to M. Obata [37], we obtain Theorem 3.3.

THEOREM (M. Obata). *Let R^{n+p} $(n+p\geqq 2)$ be a complete Riemannian manifold which admits a non-null function φ such that $\varphi_{;i;j}=-c^2\varphi g_{ij}$ $(c=$ const.). Then R^{n+p} is isometric to a sphere of radius $\frac{1}{c}$.*

THEOREM 3.3. *Let M^{n+p} be a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a vector field ξ^i generating a continuous one-parameter group of conformal transformations in M^{n+p} and V^n a closed orientable submanifold in M^{n+p} such that*

(i) *the mean curvature vector field H^i of V^n is parallel with respect to the connection induced on the normal bundle,*

(ii) *the connection induced on the normal bundle is trivial,*

(iii) *the scalar function f has fixed sign on V^n,*

(iv) *$\phi\neq$ const. along V^n.*

Then the submanifold V^n is isometric to a sphere.

PROOF. In §2, we proved that M^{n+p} admits a non-zero scalar function ϕ which satisfies the equation (2.10). On the other hand, by virtue of Theorem 3.2, every point of V^n is totally umbilic. Since $H_1=$const. and $\underset{A}{H_1}=0$ $(A=n+2,\cdots,n+p)$, we have

(3. 7) $$\underset{E}{b}_{\alpha\beta}=\lambda g_{\alpha\beta}, \qquad (\lambda=const.)$$

(3. 8) $$\underset{A}{b}_{\alpha\beta}=0. \qquad (A=n+2, n+3, \cdots, n+p)$$

Now we have

$$\phi_{;\alpha}=\phi_{;i}B_{\alpha}^{i}.$$

Differentiating the above equation covariantly we have

(3. 9) $$\phi_{;\alpha;\beta}=\phi_{;i;j}B_{\alpha}^{i}B_{\beta}^{j}+\phi_{;i}H_{\alpha\beta}^{i}.$$

From (2. 10), (1. 5), (3. 7), (3. 8) and (3. 9), we obtain

(3. 10) $$\phi_{;\alpha;\beta}=(-k\phi+\lambda\phi_{;i}\underset{E}{n}^{i})g_{\alpha\beta}.$$

Differentiating the scalar $\phi_{;i}\underset{E}{n}^{i}$ covariantly we have

$$(\phi_{;i}\underset{E}{n}^{i})_{;\alpha}=\phi_{;i;j}\underset{E}{n}^{i}B_{\alpha}^{j}+\phi_{;i}\underset{E}{n}^{i}{}_{;\alpha}+\phi_{;i}\Gamma''^{A}_{E\alpha}\underset{A}{n}^{i}.$$

By means of our assumption (i), (1. 7), (1. 8), (2. 10) and (3. 7) we have

$$(\phi_{;i}\underset{E}{n}^{i})_{;\alpha}=-\lambda\phi_{;\alpha}.$$

Hence we get

(3. 11) $$\phi_{;i}\underset{E}{n}^{i}=-\lambda\phi+c. \qquad (c=const.)$$

Substituting (3. 11) into (3. 10) we obtain

(3. 12) $$\phi_{;\alpha;\beta}=\left\{-(k+\lambda^{2})\phi+c\lambda\right\}g_{\alpha\beta}.$$

Here $k+\lambda^2\neq 0$. Because, if $k+\lambda^2=0$, then (3. 12) becomes $\phi_{;\alpha;\beta}=c\lambda g_{\alpha\beta}$ from which $\Delta\phi=nc\lambda$, where Δ means the Laplacian operator on V^n. This is impossible unless $\phi=$const. Thus $k+\lambda^2$ being different from zero, we have, from (3. 12),

(3. 13) $$\left(\phi-\frac{c\lambda}{k+\lambda^{2}}\right)_{;\alpha;\beta}=-(k+\lambda^{2})\cdot\left(\phi-\frac{c\lambda}{k+\lambda^{2}}\right)g_{\alpha\beta}.$$

Therefore we obtain

$$\Delta\left(\phi-\frac{c\lambda}{k+\lambda^{2}}\right)=-n(k+\lambda^{2})\cdot\left(\phi-\frac{c\lambda}{k+\lambda^{2}}\right).$$

Consequently it follows that $k+\lambda^2>0$. Hence, by virtue of M. Obata's theorem, V^n is isometric to a sphere.

REMARK. When $p=1$, Theorem 3.3 coincides with Theorem 0.2 due to Y. Katsurada.

§4. Certain conditions for V^n to be umbilical with respect to $\underset{E}{n}^i$. In this section we study on a closed orientable submanifold V^n of codimension p in a Riemannian manifold M^{n+p} of constant curvature without the condition in §3 that the connection induced on the normal bundle is trivial.

Let M^{n+p} be a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a conformal Killing vector ξ^i. Then by virtue of Corollary 2.6, M^{n+p} admits a conformal Killing tensor field $T_{i_1\cdots i_p}$ of degree p with the associated tensor field $\rho_{i_1\cdots i_{p-1}}$.

We assume that the mean curvature vector field H^i of V^n is parallel with respect to the connection induced on the normal bundle.

Now we put

$$v_\alpha = T_{i_1\cdots i_p} B_\alpha^{i_1} \underset{n+2}{n}^{i_2} \cdots \underset{n+p}{n}^{i_p}, \tag{4.1}$$

$$w_\alpha = \underset{E}{b}_\alpha{}^\gamma T_{i_1\cdots i_p} B_\gamma^{i_1} \underset{n+2}{n}^{i_2} \cdots \underset{n+p}{n}^{i_p}. \tag{4.2}$$

LEMMA 4.1. *The vector v_α and w_α are independent of the choice of mutually orthogonal unit normal vectors.*

PROOF. Let $\underset{A}{\tilde{n}}^i$ $(A=n+2, \cdots, n+p)$ be another $p-1$ mutually orthogonal unit normal vectors orthogonal to $\underset{E}{n}^i = \underset{n+1}{n}^i$. Then there exists an orthogonal matrix (U_{AB}), $(A, B=n+2, \cdots, n+p)$ such that $det.(U_{AB})=1$. Therefore by means of (3.5) and the skew symmetry of $T_{i_1\cdots i_p}$, we find

$$\begin{aligned}
\tilde{v}_\alpha &= T_{i_1\cdots i_p} B_\alpha^{i_1} \underset{n+2}{\tilde{n}}^{i_2} \cdots \underset{n+p}{\tilde{n}}^{i_p} \\
&= T_{i_1\cdots i_p} B_\alpha^{i_1} \Big(\sum_{A_2} U_{n+2\,A_2} \underset{A_2}{n}^{i_2}\Big) \cdots \Big(\sum_{A_p} U_{n+p\,A_p} \underset{A_p}{n}^{i_p}\Big) \\
&= \sum_{A_2, \cdots, A_p} sgn\begin{pmatrix} n+2, & \cdots, & n+p \\ A_2, & \cdots, & A_p \end{pmatrix} U_{n+2\,A_2} \cdots U_{n+p\,A_p} \cdot T_{i_1\cdots i_p} B_\alpha^{i_1} \underset{n+2}{n}^{i_2} \cdots \underset{n+p}{n}^{i_p} \\
&= det.(U_{AB}) T_{i_1\cdots i_p} B_\alpha^{i_1} \underset{n+2}{n}^{i_2} \cdots \underset{n+p}{n}^{i_p} = v_\alpha.
\end{aligned}$$

The above equation shows that v_α is independent of the choice of $p-1$ mutually orthogonal unit normal vectors orthogonal to $\underset{E}{n}^i$. In the same way we can prove that w_α is independent of the choice of mutually orthogonal unit normal vectors. Consequently v_α and w_α are the vector fields on V^n.

Differentiating (4.1) covariantly we have

$$v_{\alpha;\beta}=T_{i_1\cdots i_p;i}B_\beta^{\ i}B_\alpha^{\ i_1}\underset{n+2}{n}^{i_2}\cdots\underset{n+p}{n}^{i_p}$$
$$+T_{i_1\cdots i_p}H_{\alpha\beta}^{\ \ i_1}\underset{n+2}{n}^{i_2}\cdots\underset{n+p}{n}^{i_p}$$
$$+T_{i_1\cdots i_h\cdots i_p}B_\alpha^{\ i_1}\underset{n+2}{n}^{i_2}\cdots\sum_{h=1}^{p}\Big(-\underset{n+h}{b}_\beta^{\ \delta}B_\delta^{\ i_h}+\sum_{A=n+1}^{n+p}\Gamma''^{\ A}_{n+h\,\beta}\underset{A}{n}^{i_h}\Big)\cdots\underset{n+p}{n}^{i_p}$$

by means of (1. 7).
Multiplying the last equation by $g^{\alpha\beta}$, by virtue of our assumption, (1.5) and (3. 1) we get

$$v^\alpha_{\ ;\alpha}=T_{i_1\cdots i_p;i}g^{\alpha\beta}B_\beta^{\ i}B_\alpha^{\ i_1}\underset{n+2}{n}^{i_2}\cdots\underset{n+p}{n}^{i_p}+fg^{\alpha\beta}\underset{A}{b}_{\alpha\beta}$$
$$-\sum_{h=2}^{p}\underset{n+h}{b}^{\alpha\delta}T_{i_1\cdots i_h\cdots i_p}B_\alpha^{\ i_1}\underset{n+2}{n}^{i_2}\cdots B_\delta^{\ i_h}\cdots\underset{n+p}{n}^{i_p}.$$

The above equation turns to

$$v^\alpha_{\ ;\alpha}=\frac{1}{2}(T_{i_1\cdots i_p;i}+T_{ii_2\cdots i_p;i_1})g^{\alpha\beta}B_\beta^{\ i}B_\alpha^{\ i_1}\underset{n+2}{n}^{i_2}\cdots\underset{n+p}{n}^{i_p}$$
$$+nfH_1$$
$$=\frac{1}{2}\Big\{2\rho_{i_2\cdots i_p}g_{i_1 i}-\sum_{h=2}^{p}(-1)^h\cdot(\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{i_h i}+\rho_{ii\cdots\hat{i}_h\cdots i_p}g_{i_h i_1})\Big\}$$
$$\cdot g^{\alpha\beta}B_\beta^{\ i}B_\alpha^{\ i_1}\underset{n+2}{n}^{i_2}\cdots\underset{n+p}{n}^{i_p}+nfH_1$$

by virtue of our assumption, (1. 11), (2. 4) and the skew symmetry of $T_{i_1\cdots i_p}$. Thus we have

$$v^\alpha_{\ ;\alpha}=nfH_1+n\rho_{i_2\cdots i_p}\underset{n+2}{n}^{i_2}\cdots\underset{n+p}{n}^{i_p}.$$

Therefore by means of Green's theorem we get the following integral formula:

$$\text{(I)}\qquad \int_{V^n}(fH_1+\rho_{i_2\cdots i_p}\underset{n+2}{n}^{i_2}\cdots\underset{n+p}{n}^{i_p})dV=0.$$

Next, differentiating (4. 2) covariantly we have

$$w_{\alpha;\beta}=\Big(\underset{E}{b}_\alpha^{\ \gamma}{}_{;\beta}+\sum_{A=n+1}^{n+p}\Gamma''^{\ A}_{E\beta}\underset{A}{b}_\alpha^{\ \gamma}\Big)\cdot T_{i_1\cdots i_p}B_\gamma^{\ i_1}\underset{n+2}{n}^{i_2}\cdots\underset{n+p}{n}^{i_p}$$
$$+\underset{E}{b}_\alpha^{\ \gamma}T_{i_1\cdots i_p;i}B_\beta^{\ i}B_\gamma^{\ i_1}\underset{n+2}{n}^{i_2}\cdots\underset{n+p}{n}^{i_p}$$
$$+\underset{E}{b}_\alpha^{\ \gamma}T_{i_1\cdots i_p}H_{\gamma\beta}^{\ \ i_1}\underset{n+2}{n}^{i_2}\cdots\underset{n+p}{n}^{i_p}$$
$$+\underset{E}{b}_\alpha^{\ \gamma}T_{i_1\cdots i_h\cdots i_p}B_\gamma^{\ i_1}\underset{n+2}{n}^{i_2}\cdots\sum_{h=2}^{p}\Big(-\underset{n+h}{b}_\beta^{\ \delta}B_\delta^{\ i_h}+\sum_{A=n+1}^{n+p}\Gamma''^{\ A}_{n+h\,\beta}\underset{A}{n}^{i_h}\Big)\cdots\underset{n+p}{n}^{i_p},$$

by means of (1. 7).
Multiplying the above equation by $g^{\alpha\beta}$, by virtue of our assumption, (1. 5) and (3. 1) we find

$$\begin{aligned} w^{\alpha}{}_{;\alpha} &= \underset{E}{b}{}^{\alpha\gamma}{}_{;\alpha} T_{i_1\cdots i_p} B_{\gamma}^{i_1} \underset{n+2}{n}{}^{i_2} \cdots \underset{n+p}{n}{}^{i_p} \\ &\quad + \underset{E}{b}{}^{\alpha\gamma} T_{i_1\cdots i_p;i} B_{\alpha}^{i} B_{\gamma}^{i_1} \underset{n+2}{n}{}^{i_2} \cdots \underset{n+p}{n}{}^{i_p} \\ &\quad + f \underset{E}{b}_{\alpha\gamma} \underset{E}{b}{}^{\alpha\gamma} \\ &\quad - \sum_{h=2}^{p} \underset{E}{b}_{\alpha}{}^{\gamma} \underset{n+h}{b}{}^{\alpha\delta} T_{i_1\cdots i_h\cdots i_p} B_{\gamma}^{i_1} \underset{n+2}{n}{}^{i_2} \cdots B_{\delta}^{i_h} \cdots \underset{n+p}{n}{}^{i_p} . \end{aligned}$$

The above equation turns to

$$\begin{aligned} w^{\alpha}{}_{;\alpha} &= \frac{1}{2} \underset{E}{b}{}^{\alpha\gamma} (T_{i_1\cdots i_p;i} + T_{ii_2\cdots i_p;i_1}) B_{\alpha}^{i} B_{\gamma}^{i_1} \underset{n+2}{n}{}^{i_2} \cdots \underset{n+p}{n}{}^{i_p} \\ &\quad + f \underset{E}{b}_{\alpha\gamma} \underset{E}{b}{}^{\alpha\gamma} \\ &\quad - \sum_{h=2}^{p} \underset{E}{b}{}^{\alpha\gamma} \underset{n+h}{b}_{\alpha}{}^{\delta} T_{i_1\cdots i_h\cdots i_p} B_{\gamma}^{i_1} \underset{n+2}{n}{}^{i_2} \cdots B_{\delta}^{i_h} \cdots \underset{n+p}{n}{}^{i_p} \\ &= \frac{1}{2} \underset{E}{b}{}^{\alpha\gamma} \left\{ 2\rho_{i_2\cdots i_p} g_{i_1 i} - \sum_{h=2}^{p} (-1)^h (\rho_{i\cdots \hat{i}_h \cdots i_p} g_{i_h i} + \rho_{ii_2\cdots \hat{i}_h \cdots i_p} g_{i_h i_1}) \right\} \\ &\quad \cdot B_{\alpha}^{i} B_{\gamma}^{i_1} \underset{n+2}{n}{}^{i_2} \cdots \underset{n+p}{n}{}^{i_p} + f \underset{E}{b}_{\alpha\gamma} \underset{E}{b}{}^{\alpha\gamma} , \end{aligned}$$

by virtue of our assumption, (1. 19), (2. 4) and the skew symmetry of $T_{i_1\cdots i_p}$.
Thus we have

$$w^{\alpha}{}_{;\alpha} = f \underset{E}{b}_{\alpha\beta} \underset{E}{b}{}^{\alpha\beta} + nH_1 \rho_{i_2\cdots i_p} \underset{n+2}{n}{}^{i_2} \cdots \underset{n+p}{n}{}^{i_p} .$$

Therefore by means of Green's theorem we get the following integral formula:

(II) $$\int_{V^n} (f \underset{E}{b}_{\alpha\beta} \underset{E}{b}{}^{\alpha\beta} + nH_1 \rho_{i_2\cdots i_p} \underset{n+2}{n}{}^{i_2} \cdots \underset{n+p}{n}{}^{i_p}) dV = 0$$

From (II)–(I)$\times nH_1$, we obtain

(4. 3) $$\int_{V^n} f (\underset{E}{b}_{\alpha\beta} \underset{E}{b}{}^{\alpha\beta} - nH_1^2) dV = 0 .$$

This result is analogous to Morohashi's result [36].
Hence we have the following theorem:

THEOREM 4. 2. *Let M^{n+p} be a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a vector field ξ^i generating a continuous one-parameter group of conformal transformations in M^{n+p} and V^n*

a closed orientable submanifold in M^{n+p} *such that*

(i) *the mean curvature vector field* H^i *of* V^n *is parallel with respect to the connection induced on the normal bundle,*

(ii) *the scalar function* f *has fixed sign on* V^n.

Then the submanifold V^n *is umbilical with respect to Euler-Schouten unit vector* $\underset{E}{n}^i$.

PROOF. From (4.3) and our assumptions we have

$$\underset{E}{b}_{\alpha\beta}\underset{E}{b}^{\alpha\beta}-nH_1^2=0,$$

because $\underset{E}{b}_{\alpha\beta}\underset{E}{b}^{\alpha\beta}-nH_1^2$ is non negative. Thus this equation shows that V^n is umbilical with respect to Euler-Schouten unit vector $\underset{E}{n}^i$ by means of Lemma 1.6.

REMARK. When $p=1$, Theorem 4.2 coincides with Theorem 0.1 due to Y. Katsurada.

Department of Mathematics,
Hokkaido University

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