H-projective-recurrent Kählerian manifolds and Bochner-recurrent Kählerian manifolds

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Introduction.

T. Adati and T. Miyazawa [1] investigated the conformal-recurrent Riemannian manifolds and M. Matsumoto [2] the projective-recurrent Riemannian manifolds. In their paper, they concerned with the more general Riemannian manifolds, that is, the Riemannian metric g is not necessarily positive definite.

Recently, L. R. Ahuja and R. Behari [3] studied the H-projettive-recurrent Kählerian manifolds.

The purpose of the present paper is to make researches in the H-projective-recurrent Kählerian manifolds and the Bochner-recurrent Kählerian manifolds.

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§ 1. Preliminaries.

Let M be an $n(=2m)$ dimensional Kählerian manifold with Kählerian structure (g, J) satisfying

$$J^i{}_a J^a{}_j = -\delta^i{}_j, \quad J_{ij} = -J_{ji}, \quad \nabla_h J^i{}_j = 0, \tag{1.1}$$

where $J_{ij} = g_{ia} J^a{}_j$.

It is well known that the tensor

$$P_{hijk} = R_{hijk} - \frac{1}{n+2}(R_{ij}g_{hk} - R_{hj}g_{ik} + H_{ij}J_{hk} - H_{hj}J_{ik} - 2H_{hi}J_{jk}), \tag{1.2}$$

where $H_{ij} = R_{ia}J^a{}_j$, is called the holomorphically projective (for brevity, H-projective) curvature tensor of M, and the tensor

$$\begin{aligned} B_{hijk} = R_{hijk} - \frac{1}{n+4}(&R_{ij}g_{hk} - R_{hj}g_{ik} + H_{ij}J_{hk} - H_{hj}J_{ik} - 2H_{hi}J_{jk} \\ &+ R_{hk}g_{ij} - R_{ik}g_{hj} + H_{hk}J_{ij} - H_{ik}J_{hj} - 2H_{jk}J_{hi}) \\ &+ \frac{R}{(n+2)(n+4)}(g_{ij}g_{hk} - g_{hj}g_{ik} + J_{ij}J_{hk} - J_{hj}J_{ik} - 2J_{hi}J_{jk}) \end{aligned} \tag{1.3}$$

the Bochner curvature tensor of M.

We concider a tensor U_{hijk} given by

$$(1.4)\qquad U_{hijk}=R_{hijk}-\frac{R}{n(n+2)}(g_{ij}g_{hk}-g_{hj}g_{ik}+J_{ij}J_{hk}-J_{hj}J_{ik}-2J_{hi}J_{jk}).$$

Hence we call this tensor the H-concircular curvature tensor of M. The H-projective curvature tensor and the Bochner curvature tensor coincide with the H-concircular curvature tensor of M if and only if M is an Einstein space.

We call that a Kählerian manifold M is H-projective-recurrent if $\nabla_l P_{hijk}=\kappa_l P_{hijk}$ where κ_l is the vector of H-projective-recurrence, Bochner-recurrent $\nabla_l B_{hijk}=\kappa_l B_{hijk}$ where κ_l is the vector of Bochner-recurrence and H-concircular-recurrent if $\nabla_l U_{hijk}=\kappa_l U_{hijk}$ where κ_l is the vector of H-concircular-recurrence.

We call that a Kählerian manifold M is H-projective-symmetric if the H-projective curvature tensor is parallel, that is, $\nabla_l P_{hijk}=0$. Similarly, we define the Bochner-symmetric Kählerian manifold and H-concircular-symmetric Kählerian manifold.

We have well known the following identities:

$$(1.5)\qquad \begin{aligned}
&g_{ab}J^a{}_iJ^b{}_j=g_{ij},\\
&R_{ab}J^a{}_iJ^b{}_j=R_{ij},\qquad R_{ia}J^a{}_j=-R_{ja}J^a{}_i,\\
&\nabla^a R_{aijk}=\nabla_k R_{ij}-\nabla_j R_{ik},\qquad \nabla_k R=2\nabla_a R^a{}_k,\\
&H_{ij}=-H_{ji},\qquad H_{ab}J^{ab}=R,\\
&H_{ia}J^a{}_j=H_{ja}J^a{}_i=-R_{ij},\\
&H_{ij}=-(1/2)R_{abij}J^{ab}=R_{aijb}J^{ab},\\
&\nabla_a H_{kj}J^a{}_i=\nabla_k R_{ij}-\nabla_j R_{ik},\qquad \nabla_a RJ^a{}_k=2\nabla_a H^a{}_k.
\end{aligned}$$

§ 2. H-projective-recurrent Kählerian manifolds.

THEOREM 1. *A necessary and sufficient condition for a Kählerian manifold M to be H-projective-recurrent is that M be H-concircular-recurrent.*

PROOF. We assum that a Kälerian manifold M is H-concircular-recurrent, i.e.

$$(2.1)\qquad \nabla_l U_{hijk}=\kappa_l U_{hijk}.$$

From (1.4), we can write (2.1) as

$$(2.1)^*\qquad \nabla_l R_{hijk}=\kappa_l R_{hijk}+\frac{1}{n(n+2)}(\nabla_l R-\kappa_l R)\mathscr{A}_{hijk},$$

where $\mathscr{A}_{hijk}=g_{ij}g_{hk}-g_{hj}g_{ik}+J_{ij}J_{hk}-J_{hj}J_{ik}-2J_{hi}J_{jk}$.

Contracting (2. 1)* with g^{hk}, we get

$$\nabla_l R_{ij}=\kappa_l R_{ij}+\frac{1}{n}(\nabla_l R-\kappa_l R)g_{ij}. \tag{2. 2}$$

Substituting (2. 1)* and (2. 2) in $\nabla_l P_{hijk}$, we have

$$\nabla_l P_{hijk}=\kappa_l P_{hijk}, \tag{2. 3}$$

that is, M is H-projective-recurrent.

Conversely, we assume that M is H-projective-recurrent, than we have

$$\begin{aligned}\nabla_l R_{hijk}=\kappa_l R_{hijk}+\frac{1}{n+2}\Big\{&(\nabla_l R_{ij}g_{hk}-\nabla_l R_{hj}g_{ik}+\nabla_l H_{ij}J_{hk}-\nabla_l H_{hj}J_{ik}\\ &-2\nabla_l H_{hi}J_{jk})-\kappa_l(R_{ij}g_{hk}-R_{hj}g_{ik}+H_{ij}J_{hk}-H_{hj}J_{ik}-2H_{hi}J_{jk})\Big\}.\end{aligned} \tag{2. 3}^*$$

Trancevecting (2. 3)* with g^{ij}, we get

$$\nabla_l R_{hk}=\kappa_l R_{hk}+\frac{1}{n}(\nabla_l R-\kappa_l R)g_{hk}.$$

Substituting this in (2. 3)*, we obtain (2. 1)*, i. e. (2. 1). Q.E.D.

From THEOREM 1, we have the following corollaries.

COROLLARY 1. *If a H-projective-recurrent Kählerian manifold M satisfies* $\nabla_l R=\kappa_l R$, *where* κ_l *is the vector of H-projective-recurrence, then M is recurrent.*

COROLLARY 2. *A necessary and sufficient condition for a Kählerian manifold M to be H-projective-spmmetric is that M be H-concircular-symmetric.*

COROLLARY 3. *If a H-projective-symmetric Kählerian manifold M has the constant scalar curvature, then M is symmetric.*

PROPOSITION 2. *If a Kählerian manifold M is H-projective-recurrent, then M satisfies the identity*

$$(n-2)\nabla_k R=2n\kappa_a R^a{}_k-2\kappa_k R, \tag{2. 4}$$

where κ_k *is the vector of H-projective-recurrence.*

PROOF. Contracting (2. 1)* with g^{lh}, we get

$$\nabla^a R_{aijk}=\kappa^a R_{aijk}+\frac{1}{n(n+2)}(\nabla^a R-\kappa^a R)\mathscr{A}_{aijk}, \tag{2. 5}$$

where $\kappa^a=g^{ab}\kappa_b$.

Using (1.5) in the left side of (2.5), we obtain

$$\nabla_a H_{kj} J^a{}_i = \kappa^a R_{aijk} + \frac{1}{n(n+2)}(\nabla^a R - \kappa^a R)\mathscr{A}_{aijk}\,. \tag{2.6}$$

Transvecting this with $J^i{}_l J^j{}_m$, we get

$$\nabla_l R_{km} = \kappa^a R_{abck} J^b{}_l J^c{}_m + \frac{1}{n(n+2)}(\nabla^a R - \kappa^a R)(g_{lm} g_{ak} + g_{am} g_{lk} + 2g_{al} g_{mk} + J_{lm} J_{ak} + J_{am} J_{lk})\,. \tag{2.7}$$

Moreover contracting this with g^{km}, we obtain

$$\nabla_l R = 2\kappa^a R_{al} + \frac{2}{n}(\nabla_l R - \kappa_l R)\,,$$

whence (2.4) follows. Q.E.D.

As an immediate consequence of this proposition and COROLLARY 3, we have the following

COROLLARY 4. *In a H-projective-symmetric Kählerian manifold M, the scalar curvature R is constant. Therefore M is symmetric.*

Now, we assume that a Kählerian manifold M is H-projective-recurrent and M is not of constant holomorphic sectional curvature. We have

$$\nabla_l (P_{hijk} P^{hijk}) = 2\kappa_l (P_{hijk} P^{hijk})\,, \tag{2.8}$$

whence it follows that κ_l is gradient.

Using the Ricci identity and THEOREM 1, we have the following

PROPOSITION 3. *A H-projective-recurrent Kählerian manifold M satisfies the condition* $\nabla_m \nabla_l R_{hijk} = \nabla_l \nabla_m R_{hijk}$.

Next, we have the following

THEOREM 4. *If a Kählerian manifold M is H-projective-recurrent, then M is recurrent.*

PROOF. We have the following two cases: (a) M is of constant holomorphic sectional curvature, (b) the vector of H-projective-recurrence κ_l is gradient. In the case (a), M is symmetric, whence it follows that M is recurrent.

Now, we shall concider with the case (b).

We consider a tensor U_{ij} given by

$$U_{ij} = R_{ij} - \frac{R}{n} g_{ij}\,. \tag{2.9}$$

In a H-projective-recurrent Kählerian manifold M, from THEOREM 1, we have (2. 1), whence we obtain

$$\nabla_l U_{ij} = \kappa_l U_{ij} . \tag{2.10}$$

Since κ_l is gradient, we have

$$\nabla_m \nabla_l U_{ij} - \nabla_l \nabla_m U_{ij} = 0 . \tag{2.11}$$

Applying the Ricci identity to (2. 11), we obtain

$$\begin{aligned} 0 &= R_{mli}{}^a U_{aj} + R_{mlj}{}^a U_{ia} \\ &= U_{mli}{}^a U_{aj} + U_{mlj}{}^a U_{ia} + \frac{R}{n(n+2)} (\mathscr{A}_{mli}{}^a U_{aj} + \mathscr{A}_{mlj}{}^a U_{ia}) . \end{aligned} \tag{2.12}$$

Differentiating this covariantly, we get

$$\begin{aligned} 0 &= 2\kappa_p (U_{mli}{}^a U_{aj} + U_{mlj}{}^a U_{ia}) \\ &+ \frac{1}{n(n+2)} (\nabla_p R + \kappa_p R)(\mathscr{A}_{mli}{}^a U_{aj} + \mathscr{A}_{mlj}{}^a U_{ia}) . \end{aligned} \tag{2.13}$$

It follows from (2. 12) and (2. 13) that

$$(\nabla_p R - \kappa_p R)(\mathscr{A}_{mli}{}^a U_{aj} + \mathscr{A}_{mlj}{}^a U_{ia}) = 0 . \tag{2.14}$$

Contracting this with g^{li}, we obtain

$$(\nabla_p R - \kappa_p R) U_{mj} = 0 . \tag{2.15}$$

Thus we find either $\nabla_p R = \kappa_p R$ or $U_{mj} = 0$.

In the case $\nabla_p R = \kappa_p R$, from COROLLARY 1, M is recurrent. In the case $U_{ij} = 0$, M is symmetric. (see § 3. THEOREM 6 or [3]) Q.E.D.

§ 3. Bochner-recurrent Kählerian manifolds.

It is clear that a Bochner-recurrent Kählerian manifold satisfying the condition $\nabla_k R_{ij} = \kappa_k R_{ij}$, where κ_k isthe vector of Bochner-recurrence, is recurrent.

In this section, first, we shall prove the following

THEOREM 5. *In order that a Bochner-recurrent Kählerian manifold M is H-projective-recurrent, it is necessary and sufficient to be* $\nabla_k R_{ij} = \kappa_k R_{ij} + \frac{1}{n} (\nabla_k R - \kappa_k R) g_{ij}$, *where* κ_k *is the vector of Bochner-recurrence.*

PROOF. We assum that a Kählerian manifold M is H-projective-recurrent, then from the proof of THEOREM 1 we have (2. 1)* and (2. 2). Substituting (2. 1)* and (2. 2) in $\nabla_l B_{hijk}$, we have

$$\nabla_l B_{hijk} = \kappa_l B_{hijk} . \tag{3.1}$$

Conversely, we assume that a Bochner-recurrent Kählerian manifold M satisfies the condition (2.2) where κ_l is the vector of Bochner-recurrence, then we have

$$
\begin{aligned}
\nabla_l R_{hijk} = \kappa_l R_{hijj} + \frac{1}{n+4}\Big\{&\nabla_l(\mathscr{B}_{hijk}+\mathscr{B}_{ihkj}-2H_{hi}J_{jk}-2H_{jk}J_{hi}) \\
&-\kappa_l(\mathscr{B}_{hijk}+\mathscr{B}_{ihkj}-2H_{hi}J_{jk}-2H_{jk}J_{hi})\Big\} \\
&-\frac{1}{(n+2)(n+4)}(\nabla_l R-\kappa_l R)\mathscr{A}_{hijk},
\end{aligned} \tag{3.1*}
$$

where $\mathscr{B}_{hijk}=R_{ij}g_{hk}-R_{hj}g_{ik}+H_{ij}J_{hk}-H_{hj}J_{ik}$.
Substituting (2.2) in (3.1)*, we have (2.1)*, that is, (2.1). From THEOREM 1, M is H-projective-recurrent. Q.E.D.

THEOREM 6.[1] *If a Bochner-recurrent Kählerian manifold M is Ricci-symmetric, then either the Bochner curvature tensor vanishes or the vector of Bochner-recurrence is zero. Consequently M is symmetric.*

PROOF. If a Bochner-recurrent Kählerian manifold M is Ricci-symmetric, We have

$$\nabla_l R_{hijk} = \kappa_l B_{hijk}. \tag{3.2}$$

From the Bianchi's identity and (3.2), we get

$$\kappa_l B_{hijk} + \kappa_h B_{iljk} + \kappa_i B_{lhjk} = 0. \tag{3.3}$$

Transvecting (3.3) with κ^l, we have

$$\kappa_l\kappa^l B_{hijk} + \kappa_h\kappa^l B_{iljk} + \kappa_i\kappa^l B_{lhjk} = 0. \tag{3.4}$$

Since $\nabla^a R_{aijk}=\nabla_k R_{ij}-\nabla_j R_{ik}=0$, we have $\kappa^l B_{iljk}=0$ and $\kappa^l B_{lhjk}=0$.

Now, we obtain $(\kappa_l\kappa^l)B_{hijk}=0$. Consequently, κ_l is zero or the Bochner curvature tenson vanishes. Therefore M is symmetric. Q.E.D.

THEOREM 7. *If a Kählerian manifold M is Bochner-recurrent and Ricci-recurrent, then M is recurrent.*

PROOF. We assume that the Bochner curvature tensor does not vanish in a Bochner-recurrent Kählerian manifold M. Then the vector of Bochner-recurrence κ_l is gradient.

We put κ_l^* the vector of Ricci-recuarence and

$$\mathscr{C}_{hijk} = R_{hijk} - B_{hijk}, \tag{3.5}$$

whence it follows that

$$\nabla_l R_{hijk} = \kappa_l B_{hijk} + \kappa_l^* \mathscr{C}_{hijk}. \tag{3.6}$$

1) This theorem was proved by T. Yamada.

Since either $B_{hijk}=0$ or κ_l is gradient, we have

$$\nabla_m \nabla_l B_{hijk} - \nabla_l \nabla_m B_{hijk} = 0 . \tag{3.7}$$

Using the Ricci identity to (3.7), we obtain

$$\begin{aligned} 0 &= R_{mlh}{}^a B_{aijk} + R_{mli}{}^a B_{hajk} + R_{mlj}{}^a B_{hiak} + R_{mlk}{}^a B_{hija} \\ &= B_{mlh}{}^a B_{aijk} + B_{mli}{}^a B_{hajk} + B_{mlj}{}^a B_{hiak} + B_{mlk}{}^a B_{hija} \\ &\quad + \mathscr{C}_{mlh}{}^a B_{aijk} + \mathscr{C}_{mli}{}^a B_{hajk} + \mathscr{C}_{mlj}{}^a B_{hiak} + \mathscr{C}_{mlk}{}^a B_{hija} . \end{aligned} \tag{3.8}$$

The covariant differentiation of (3.8) gives

$$\begin{aligned} 0 &= 2\kappa_p (B_{mlh}{}^a B_{aijk} + B_{mli}{}^a B_{hajk} + B_{mlj}{}^a B_{hiak} + B_{mlk}{}^a B_{hija}) \\ &\quad + (\kappa_p + \kappa_p^*)(\mathscr{C}_{mlh}{}^a B_{aijk} + \mathscr{C}_{mli}{}^a B_{hajk} + \mathscr{C}_{mlj}{}^a B_{hiak} + \mathscr{C}_{mlk}{}^a B_{hija}) . \end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9), that

$$(\kappa_p - \kappa_p^*)(\mathscr{C}_{mlh}{}^a B_{aijk} + \mathscr{C}_{mli}{}^a B_{hajk} + \mathscr{C}_{mlj}{}^a B_{hiak} + \mathscr{C}_{mlk}{}^a B_{hija}) = 0 . \tag{3.10}$$

In the case $\kappa_p = \kappa_p^*$, clearly, M is recurrent.

Next, we assume that

$$\mathscr{C}_{mlh}{}^a B_{aijk} + \mathscr{C}_{mli}{}^a B_{hajk} + \mathscr{C}_{mlj}{}^a B_{hiak} + \mathscr{C}_{mlk}{}^a B_{hija} = 0 . \tag{3.11}$$

Contracting this with g^{lh}, we have

$$R_m{}^a B_{aijk} = 0 . \tag{3.12}$$

Transvecting this with κ^{*m}, we obtain

$$\begin{aligned} 0 &= \kappa^{*b} R_b{}^a B_{aijk} \\ &= \frac{R}{2} \kappa^{*a} B_{aijk} . \end{aligned} \tag{3.13}$$

Thus, we find either

$$\kappa^{*a} B_{aijk} = 0 \tag{3.14}$$

or $R=0$.

In the case (3.14), transvecting (3.11) with $\kappa^{*l}\kappa^{*h}$, we obtain $\kappa^{*l}\kappa^{*h}\mathscr{C}_{mlh}{}^a B_{aijk}=0$, whence it follows that

$$R(\kappa^*{}_a \kappa^{*a}) B_{mijk} = 0 . \tag{3.15}$$

In the case $B_{hijk}=0$, we have

$$\begin{aligned} \nabla_l R_{hijk} &= \nabla_l \mathscr{C}_{hijk} \\ &= \kappa^*{}_l \mathscr{C}_{hijk} \\ &= \kappa^*{}_l R_{hijk} , \end{aligned}$$

that is, M is recurrent.

Next, we shall consider the base $\kappa^*{}_k=0$.
In this case, from THEOREM 6, M is symmetric.

Finally, we shall consider the case $R=0$.
Transvecting (3.11) with R^{lh} and using (3.12), we have

$$(3.16)\qquad \begin{aligned} 0&=(R^{ab}R_{ab})B_{mijk}+R^{dc}R_{cb}J^b{}_dJ_m{}^aB_{aijk}\\ &=(R^{ab}R_{ab})B_{mijk}\,. \end{aligned}$$

Consequently M is recurrent. Q.E.D.

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