Examples of the manifolds $f^{-1}(0)\cap S^{2n+1}$, $f(Z)=Z_0^{a_0}+Z_1^{a_1}+\cdots+Z_n^{a_n}$

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Consider the polynomials $f(z)=Z_0^{a_0}+Z_1^{a_1}+\cdots+Z_n^{a_n}$, $a_i\geqq 2$, $z_i\in\boldsymbol{C}$ $(i=0,1,2,\cdots,n)$ and closed differentiable manifolds of dim $(2n-1)$, $K_a=f^{-1}(0)\cap S^{2n+1}$, where S^{2n+1} denotes the unit sphere in $\boldsymbol{C}^{n+1}$. The purpose of this paper is to give examples which shows what manifolds K_a are when $(a_0, a_1, \cdots, a_n)=(2,2,\cdots,2,p,q)$, $q\equiv 0(p)$ and $n\geqq 3$. This paper is a continuation of [1], so we will use the same notations as them in [1]. Let $q\equiv 0(p)$ be satisfied. Then $K_{a'}$, $a'=(2,2,\cdots,2,p,q-1)$ is a homotopy sphere which is denoted by Σ in the sequel if and only if n is odd or both p and $q-1$ are odd in case of n being even. This is an easy consequence of [3, §14]. In the sequel we assume that a and a' are as stated above. Unless otherwise stated, a manifold means a smooth manifold.

THEOREM 1. *Let* $n\geqq 3$ *and* $q\equiv 0(p)$.

(i) *If* n *is odd, then* K_a *is diffeomorphic to* $(S^{n-1}\times S^n)_1\#(S^{n-1}\times S^n)_2\#\cdots\#(S^{n-1}\times S^n)_{p-1}\#\Sigma$ *when* p *is odd or both* p *and* q/p *are even, and to* $\partial D(\tau_{S^n})_1\#\cdots\#\partial D(\tau_{S^n})_{p/2}\#(S^{n-1}\times S^n)_{p/2+1}\#\cdots\#(S^{n-1}\times S^n)_{p-1}\#\Sigma$ *when* p *is even and* q/p *is odd.*

(ii) *If* n *is even,* $p=3$, *and* $q\equiv 0(6)$, *then* K_a *is diffeomorphic to* $(S^{n-1}\times S^n)\#(S^{n-1}\times S^n)\#\Sigma$.

At first we consider ths case when n is odd. Let F_a be a fiber of Milnor fibering associated to the polynomial f and $\bar{F}_a$ the closure of F_a in S^{2n+1} [5]. Now we recall the exact esquence $0\to H_n(K_a)\to H_n(\bar{F}_a)\xrightarrow{\Psi} H_n(\bar{F}_a, K_a)\xrightarrow{\partial} H_{n-1}(K_a)\to 0$. [5]

To know the modules $H_n(K_a)$ and $H_{n-1}(K_a)$ we must examine the matrix

$$\Psi=\begin{pmatrix} A-{}^tA, & A, & \cdots & A \\ -{}^tA, & A-{}^tA, & \ddots & \vdots \\ \vdots & \ddots & \ddots & A \\ -{}^tA, & \cdots & -{}^tA, & A-{}^tA \end{pmatrix}$$

, where $A=\begin{pmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix}$ and degree of A is $(q-1)$

[1, Theorem 1. 6]. Let E be a unit matrix, $S=\begin{pmatrix} 1, & -1, & & 0 \\ & \ddots & \ddots & \\ & & 1, & -1 \\ 0 & & & 1 \end{pmatrix}$,

$$C=\begin{pmatrix} 1, & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1, & 1 \\ -1,\cdots, & & -1, & 0 \end{pmatrix},\quad D=\begin{pmatrix} 0, & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0, & 1 \\ -1,\cdots\cdots, & & -1, & -1 \end{pmatrix}$$ and degrees

of them are $(q-1)$. Then

$$\begin{pmatrix} E, & -E & & 0 \\ & \ddots & \ddots & \\ & & E, & -E \\ 0 & & & E \end{pmatrix}\begin{pmatrix} S & & 0 \\ & \ddots & \\ 0 & & S \end{pmatrix}\quad \Psi=\begin{pmatrix} C-D, & E-C & & 0 \\ & C-D, & \ddots & \\ 0 & & C-D, & E-C \\ D, & D,\cdots\cdots, & D, & C \end{pmatrix}.$$

Since $C-D=E$, this matrix is $\begin{pmatrix} E, & -D & & 0 \\ & \ddots & \ddots & \\ 0 & & E, & -D \\ D, & D, & ,D, & C \end{pmatrix}$. Hence,

$$\begin{pmatrix} E & & & 0 \\ & \ddots & & \\ 0 & & E, & 0 \\ -D, & -(D+D^2),\cdots, & -\sum_{i=1}^{p-2} D^i, & E \end{pmatrix}\begin{pmatrix} E, & -D & & 0 \\ & \ddots & \ddots & \\ 0 & & E, & -D \\ D, & D,\cdots, & D, & C \end{pmatrix}=\begin{pmatrix} E, & -D & & 0 \\ & \ddots & \ddots & \\ 0 & & E, & -D, \\ & & 0, & C+\sum_{i=2}^{p-1} D^i \end{pmatrix}$$

Since $H_n(\bar{F}_a)$ is a free module and provided with a canonical basis $x_1, x_2, \cdots, x_{(p-1)(q-1)}$ [1, §1], we can represent an element x of $H_n(\bar{F}_a)$ by an integer vector $(x_1, x_2, \cdots, x_{p-1})$, where each x_i is a $(q-1)$ tuple of integers, $(x_{i1}, x_{i2}, \cdots, x_{iq-1})$. Then a vector x so that $\Psi^t x=0$ satisfies the equations ${}^t x_1=D^t x_2$, ${}^t x_2=D^t x_3$, $\cdots$, ${}^t x_{p-2}=D^t x_{p-1}$ and $\left(C+\sum_{i=2}^{p-1} D^i\right)\ {}^t x_{p-1}=0$. It follows from the direct computations that

$$D^i = \begin{pmatrix} & \mathbf{0} & & 1. & & \\ & & & & \ddots & \\ & & & & & 1 \\ -1 & \cdots & -1 & -1 & \cdots & -1 \\ & 1 \ddots & & & & \\ & & 10 & & \mathbf{0} & \end{pmatrix} \begin{matrix} 1 \\ \vdots \\ q-i-1 \\ q-i \\ \vdots \\ q-1 \end{matrix}$$

and therefore

$$C+\sum_{i=2}^{p-1} D^i = \begin{matrix} 1 & 2 & \cdots & p & \cdots & & & q-1 \end{matrix}$$

$$\begin{pmatrix} 1, & 1, & \cdots\cdots, & 1, & 0, & & & \\ 0, & 1, & \cdots\cdots, & 1, & 1, & \cdots & \mathbf{0} & \\ & \mathbf{0} & & \ddots & 1, & \cdots\cdots, & 1, & 1 \\ -1, & -1, & \cdots\cdots, & -1, & 0, & \cdots\cdots, & 0, & 0 \\ 0, & -1, & \cdots\cdots, & -1, & -1, & & & 0 \\ \vdots & & \ddots & & & \ddots & \ddots & \vdots \\ 0 & \cdots\cdots & 0, & -1, & \cdots\cdots, & -1, & -1, & 0 \end{pmatrix} \begin{matrix} 1 \\ \vdots \\ q-p \\ q-p+1 \\ q-p+2 \\ \vdots \\ q-1 \end{matrix}$$

Since $q-p\equiv 0(p)$, all vectors x_{p-1} which satisfy $\left(C+\sum_{i=2}^{p-1} D^i\right)\,{}^t x_{p-1}=0$ are generated, for example, by vectors $x^i_{p-1}=(0,\cdots,0,1,0\cdots 0,-1\,;\,0,\cdots,0,1,0\cdots$

$$(*)\quad \begin{pmatrix} x^i_1 \\ x^i_2 \\ \vdots \\ x^i_{p-i} \\ \vdots \\ x^i_{p-1} \end{pmatrix} = \left(\begin{matrix} 0-10\cdots\cdots 010\cdots \\ \ddots \quad\quad \ddots \\ \\ 10\cdots\cdots 0-10\cdots\cdots 0 \\ \ddots \quad\quad \ddots \\ 0\cdots\cdots 010\cdots\cdots 0-1 \end{matrix}\ \vdots\ \begin{matrix} 0-10\cdots\cdots 010\cdots \\ \ddots \quad\quad \ddots \\ 1 \\ 10\cdots\cdots 0-10\cdots\cdots 0 \\ \ddots \quad\quad \ddots \\ 0\cdots\cdots 010\cdots\cdots 0-1 \end{matrix}\ \vdots\ \cdots\cdots\right.$$

$$\cdots\cdots \left.\vdots\ \begin{matrix} 0-10\cdots\cdots 010\cdots \\ \ddots \quad\quad \ddots \\ 1 \\ 0 \\ 10 \quad 0-10 \quad 0 \\ \ddots \quad\quad -1 \\ 0\cdots\cdots 010\cdots\cdots 0 \end{matrix}\right)$$

$0,-1;\ \cdots;\ 0,\cdots,0,1,0\cdots0)$ where $(jp+i)-$th components are 1 and $(j+1)p-$th components are -1 ($j=0,1,2,\cdots$ and $i=1,2,\cdots,p-1$). By direct computations we have (*).

Let α be a map; $H_n(F_a)\to\Pi_{n-1}(SO_n)$ in [1, (1. 6)].

PROPOSITON 2. *Let* $n\geqq 3$. *If* n *is odd, then* $H_{n-1}(K_a)\cong H_n(K_a)\cong \boldsymbol{Z}\oplus\cdots\oplus\boldsymbol{Z}$ (($p-1$) *sum of* $\boldsymbol{Z}$) *and the module Ker* Ψ *is generated by the above elements* $(x_1^i, x_2^i,\cdots,x_{p-1}^i)$ $(i=1,2,\cdots,p-1)$. *Moreover* $\alpha((x_1^i,x_2^i,\cdots,x_{p-1}^i))$ *is equal to* $i(p-1)(q/p)$ *modulo* 2. *If* n *is even,* $p=3$ *and* $q=0\,(6)$, *then* $H_n(K_a)\cong H_{n-1}(K_a)\cong\boldsymbol{Z}\oplus\boldsymbol{Z}$.

(PROOF) We have already proved the first part. By [7, Lemma 2]

$$\alpha((x_1^i,x_2^i,\cdots,x_{p-1}^i))$$
$$=\sum_{j,k}x_{jk}^i+(x_1^i,x_2^i,\cdots,x_{p-1}^i)\begin{pmatrix}A-E, & A,\cdots\cdots & A\\ & A-E & \ddots & \vdots\\ & & \ddots & A\\ 0 & & & A-E\end{pmatrix}\begin{pmatrix}{}^tx_1^i\\ {}^tx_2^i\\ \vdots\\ {}^tx_{p-1}^i\end{pmatrix}$$
$$\equiv\sum_{j\leqq k}x_j^iA\,{}^tx_k^i\qquad(2).$$

By using (1), (2) and (3),

$$\sum_{j\leqq k}x_j^iA\,{}^tx_k^i=\sum_{j=1}^{p-1}jx_j^iA\,{}^tx_{p-1}^i$$
$$=\frac{1}{2}(q/p)\left\{p(p-1)-i(i-1)-(p-i)(p-i-1)\right\}$$
$$=(q/p)(i)(p-i)$$
$$\equiv(q/p)(i)(p-1)\qquad(2).$$

(1) $x_j^iA\,{}^tx_k^i=x_{j+1}^iA\,x_{k+1}^i$. This follows from the fact that ${}^tDAD=A$.

(2) $x_{p-1}^i\,{}^tA=(1,1,\cdots,1,0,\cdots,0;\ 1,1,\cdots,1,0,\cdots,0;\ \cdots;\ 1,1,\cdots,1,0,\cdots,0)$, where $(jp+k)-$th components are 1 when $1\leqq k\leqq i$ and 0 when $i<k\leqq p$ $(j=0,1,2,\cdots)$

(3) It is easily shown using (*) that if $i\leqq p-i$, then $x_j^iA\,{}^tx_{p-1}^i$ is equal to $-q/p$ when $j<i$, 0 when $i\leqq j\leqq p-i-1$ and q/p when $p-i\leqq j\leqq p-1$, and that if $i>p-i$, then $x_j^iA\,{}^tx_{p-1}^i$ is equal to $-q/p$ when $j\leqq p-i-1$, 0 when $p-i\leqq j<i$ and q/p when $i\leqq j$.

In case of n being odd the matrix is transformed into $\left(\begin{array}{c|cc}0 & 0 & 0\\ & 0 & 0\\ \hline E & & 0\end{array}\right)$ by a unimodular integer matrix. (Q. E. D.)

PROOF OF THEOREM 1. Consider the two manifolds $\bar{F}_a$ and $\bar{F}_{a'}$. The $\bar{F}_a$ is constructed as follows. We take a $2n$-disk D^{2n}, embeddings $\varphi_i: \partial D_i^n \times D^n \to \partial D^{2n}$. Then we attach n-handles $D_i^n \times D^n$ to D^{2n} using φ_i and round the corners. The attaching maps φ_i $(i=1, 2, \cdots, (p-1)(q-1))$ are determined in [1, Theorem 1. 6]. It follows from Theorem A in [1] that $\bar{F}_a$ is constructed from $\bar{F}_{a'}$ by attaching $(p-1)$ n-handles on the boundary $\partial \bar{F}_{a'}$. By removing the interior of $\bar{F}_{a'}$ from $\bar{F}_a$ we have a manifold N with $\partial N = (-K_{a'}) \cup K_a$ which is diffeomorphic to $(K_{a'} \times I) \cup$(the above $(p-1)$ n-handles). Since $q \equiv 0 (p)$ and n odd, $K_{a'}$ is a homotopy sphere Σ. Therefore we have a diffeomorphism d: $S^{2n-2} \to S^{2n-2}$ so that Σ is diffeomorphic to $D^{2n-1} \cup_d D^{2n-1}$. We embed the interval $I=[0, 1]$ smoothly in N so that $I \cap \partial N = I$ and the embedded path intersects transversely with ∂N. Then we remove its open tubular neighbourhood which is diffeomorphic to $\mathring{D}^{2n-1} \times I$. Then we attach $D^{2n-1} \times I$ again to $N - \mathring{L}^{2n-1} \times I$ by the diffeomorphism $d \times id$ of $\partial D^{2n-1} \times I$. We denote this manifold by M'. It is clear that $\partial M' = -(\Sigma \# (-\Sigma)) \cup K_a \# (-\Sigma)$. Since $\Sigma \# (-\Sigma)$ is a standard sphere, we finally have a manifold M by attaching a $2n$-disk on $\Sigma \# (-\Sigma)$ to M'. From the construction we know that M comes from D^{2n} by attaching $(p-1)$ n-handles. Here we take another handle decomposition of M by representing each generator $(x_1^i, x_2^i, \cdots, x_{p-1}^i)$ of $H_n(K_a)$ by an embedded sphere in M which intersects transversely with other embedded spheres. On the other hand we have an exact sequence $0 \to H_n(K_a) \to H_n(M) \to H_n(M, K_a) \xrightarrow{\partial} H_{n-1}(K_a) \to 0$. Since $H_n(K_a) \cong H_{n-1}(K_a) \cong H_n(M) \cong H_n(M, K_a) \cong \boldsymbol{Z} \oplus \cdots \oplus \boldsymbol{Z}$ ($(p-1$ sum of $\boldsymbol{Z}$), the homomorphism, $H_n(M) \to H_n(M, K_a)$ is a zero mpa. Hence the intersection pairing of $H_n(M)$ is a zero bilinear form. It follows from [7] that M is diffeomorphic to $T_1 \# T_2 \# \cdots \# T_{p-1}$ where T_i is $D^n \times S^n$ or $D(\tau_{S^n})$ according as $\alpha((x_1^i, x_2^i, \cdots, x_{p-1}^i))=0$ or 1. By proposition 2, the number of $\{i\}$ so that $\alpha((x_1^i, x_2^i, \cdots, x_{p-1}^i))=1$ is equal to $p/2$ when p is even and q/p odd. In other cases its number is 0.

Now we proceed to the case when n is even. Similarly we can construct a manifold M so that it comes from D^{2n} by attaching two n-handles and that $\partial M = K_a$. Since $q \equiv 0 (6)$, we again have an exact sequence $0 \to H_n(K_a) \to H_n(M) \to H_n(M, K_a) \xrightarrow{\partial} H_{n-1}(K_a) \to 0$, where these modules are isomorphic to $\boldsymbol{Z} \oplus \boldsymbol{Z}$. Hence the intersection pairing is a zero quadratic form. And the attaching maps of the two n-handles corresponds to the trivial element of $\Pi_{n-1}(SO_n)$ [4, p. 51].

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References

[1] Y. ANDO: A relation between the fibers of Milnor fiberings associated to polynomials $f(Z)=Z_0^{a_0}+\cdots+Z_n^{a_n}$, Hokkaido Math. Journal, 2 (1973), 252–258.

[2] E. BRIESKORN; Beispiele zur Differential topologie von Singularitäten, Inventions Math., 2 (1966), 1–14.

[3] F. HIRZEBRUCH and K. H. MAYER: $O(n)$-Mannigfaltigkeiten, exotische Sphären und Singularitäten, Springer-Verlag, Berlin, 1968.

[4] J. MILNOR: A procedure for killing the homotopy groups of differentiable manifolds, Symposia in pure Math. A. M. S., Vol. III (1961), 39–55.

[5] J. MILNOR: Singular points of complex hypersurfaces, Annals of Mathematics Studies, Princeton University Press, 1968.

[6] F. PHAM: Formules de Picard-Lefshetz généralisées et ramifications des integrales, Bull. Soc. Math. France, 93 (1965), 333–367.

[7] C. T. C. WALL: Classification of $(n-1)$ connected $2n$-manifolds, Annals of Math., 75 (1962), No. 1, 163–189.

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