On a problem of D. G. Higman

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Dedicated to Professor Kiiti Morita on his 60th birthday

In his paper [3], D. G. Higman gave a characterization of (projective) symplectic groups $PS_p(4, q)$ of dimension 4 over the field F_q ([3], Theorem 2) and proposed the similar characterization for higher dimensional case. In this note, we will give a characterization of higher dimensional symplectic groups by adopting Kantor's idea in [5].

For notation we follow that of Higman [3] mostly. Given a group G of permutations of a finite set Ω we denote by a^g the image of $a\in\Omega$ under $g\in G$, and by G_a the stabilizer of a, $G_a=\{g\in G|a^g=a\}$. For a subgroup H of G and a subset X of Ω we let $a^H=\{a^g|g\in H\}$, $X^g=\{a^g|a\in X\}$ and $G_X=\bigcap_{a\in X} G_a$. We call the number of orbits of G_a, $a\in\Omega$, the rank of G and we call the lengths of these orbits the subdegrees of G. Our theorem is the following.

THEOREM. *Let G be a transitive rank 3 permutation group on a finite set Ω whose subdegrees are 1, $(q^{n-1}-q)/(q-1)$, q^{n-1} where q is a power of a prime number p and $n\geqq 4$. Assume that there are at least q elements of G_a, $a\in\Omega$, fixing a G_a-orbit of length $(q^{n-1}-q)/(q-1)$ pointwise. Then n is even and G contains a normal subgroup isomorphic to the projective symplectic group $PS_p(n, q)$ which is generated by all the symplectic elations.*

Proof. For $a\in\Omega$, we denote G_a-orbits by $\{a\}$, $\Delta(a)$, $\Gamma(a)$ with $\Delta(a)^g=\Delta(a^g)$, $\Gamma(a)^g=\Gamma(a^g)\,(g\in G)$ and $|\Delta(a)|=(q^{n-1}-q)/(q-1)$, $|\Gamma(a)|=q^{n-1}$. The intersection numbers λ, μ of G are defined by

$$|\Delta(a)\cap\Delta(b)|=\begin{cases}\lambda & \text{if } b\in\Delta(a)\\ \mu & \text{if } b\in\Gamma(a).\end{cases}$$

Aecording to Lemma 5 in [3], we have

$$\mu q^{n-1}=\frac{q^{n-1}-q}{q-1}\left(\frac{q^{n-1}-q}{q-1}-\lambda-1\right).$$

Hence $\mu=1+q+\cdots+q^{n-3}$ and $\lambda=-1+q+\cdots+q^{n-3}$. Thus, by Lemma 8 in [3], a block design $\mathscr{D}$ whose points are the elements of Ω and whose blocks are the symbols $b^\perp$, one for each $b\in\Omega$, and whose incidence $a\in b^\perp$

is defined by $a\in b^{\cup}\Delta(b)$, is symmetric with parameters

$$\left(\frac{q^n-1}{q-1},\quad \frac{q^{n-1}-1}{q-1},\quad \frac{q^{n-2}-1}{q-1}\right),$$

and G is a automorphism group of $\mathscr{D}$ and primitive on Ω.

Now we prove that $\mathscr{D}$ is the projective space $\mathscr{P}(n-1, q)$, namely, $\mathscr{D}$ is isomorphic to the design of points and hyperplanes of the desarguesian projective space $\mathscr{P}(n-1, q)$ of dimension $n-1$ over F_q. For two distinct points $a, b\in\Omega$, we define a line by

$$a+b=\bigcap_{a,b\in x^{\perp}} x^{\perp}.$$

$a+b$ is called a line of singular type or a line of hyperbolic type according as $a\in b^{\perp}$ or $a\notin b^{\perp}$. Then we have that

(1) *If* $x\in a+b$, $x\neq a$, *then* $a+x=a+b$, *and* $a\in b^{\perp}$ *if and only if* $x\in b^{\perp}$, *and so* $a+b$ *and the type of* $a+b$ *are uniquely determined by any two distinct points in* $a+b$ ([3], § 7, ii)).

(2) $G_{a^{\cup}\Delta(a)}$ *fixes all lines through* a ([3], § 7, v)).

(3) $|G_{a^{\cup}\Delta(a)}|$ *divides* $h-1$, *where* h *is the number of points on a line of hyperbolic type* ([3], § 7, viii)).

Let $a+b$, a and $b\in\Omega$, be a singular line and put $|a+b|=1+m_1$. Then there are $(|a^{\perp}|-1)/m_1$ lines in $a^{\perp}$ through a and $(|a^{\perp}\cap b^{\perp}|-1)/m_1$ lines in $a^{\perp}\cap b^{\perp}$ through a. Hence $m_1|q=(|a^{\perp}|-1, |a^{\perp}\cap b^{\perp}|-1)$. For $d\in\Delta(a)\cap\Gamma(b)$,

$$\begin{aligned}|G_{a,b}:G_{a,b,d}| &= \frac{|G_b:G_{b,d}|}{|G_b:G_{a,b}|}\cdot|G_{b,d}:G_{a,b,d}|\\ &= \frac{q^{n-2}}{1+q+\cdots+q^{n-3}}\cdot|G_{b,d}:G_{a,b,d}|\,.\end{aligned}$$

Hence $q^{n-2}||d^{G_{a,b}}|$. Tnen, since $\Delta(a)\cap\Gamma(b)$ is invariant by $G_{a,b}$ and $|\Delta(a)\cap\Gamma(b)|=q^{n-2}$, $G_{a,b}$ is transitive on $\Delta(a)\cap\Gamma(b)$. Therefore a Sylow p-subgroup P of $G_{a,b}$ is transitive on $\Delta(a)\cap\Gamma(b)$. Let us assume that $m_1<q$. Since $|a^{\perp}-(a+b)|=(q^{n-1}-q)/(q-1)-m_1$, $pm_1\nmid|a^{\perp}-(a+b)|$. Since P acts on $a^{\perp}-(a+b)$, there is a point $c\in a^{\perp}-(a+b)$ such that $|c^P|\leqq m_1$. Then for each point d of $\Delta(a)\cap\Gamma(b)$,

$$|d^{P_c}|=|P_c:P_{c,d}|=\frac{|P:P_{c,d}|}{|P:P_c|}\geqq\frac{|P:P_d|}{|P:P_c|}\geqq\frac{q^{n-2}}{m_1}.$$

Since $a+c\not\ni b$, we can choose $d^{\perp}$ such that $a, c\in d^{\perp}$ and $b\notin d^{\perp}$, namely, $d\in a^{\perp}\cap c^{\perp}$ and $d\notin b^{\perp}$. Then $|d^{P_c}|\leqq|d^{G_{a,c}}|\leqq\lambda=(q^{n-2}-1)/(q-1)-2$. Thus

$$\frac{q^{n-2}-1}{q-1}-2\geqq\frac{q^{n-2}}{m_1}\geqq\frac{pq^{n-2}}{q}=pq^{n-3},$$

which is impossible. Hence every singular line contains exactly $1+q$ points. Next let $a+b$, a and $b\in\Omega$, be a hyperbolic line and put $|a+b|=1+m_2$. Then we have

$$1+m_2\leqq\frac{(q^n-1)/(q-1)-(q^{n-2}-1)/(q-1)}{(q^{n-1}-1)/(q-1)-(q^{n-2}-1)/(q-1)}=1+q$$

([1], p. 65). On the other hand, from the assumption $q\leqq|G_{a\cup\Delta(a)}|$ and (3), we have $q\leqq m_2$. Hence $|a+b|=1+q$. Thus $\mathscr{D}$ is a symmetric block design with parameters $((q^n-1)/(q-1), (q^{n-1}-1)/(q-1), (q^{n-2}-1)/(q-1))$ and each line contains $1+q$ points.

According to a result of Dembowski-Wagner ([2]. Theorem), $\mathscr{D}$ is the block design of $\mathscr{P}(n-1, q)$. Since the coorespondence $a\leftrightarrow a^{\perp}$ defines a polarity δ of $\mathscr{D}$ and $a\in a^{\perp}$, δ is a symplectic polarity of $\mathscr{D}$ and the action of $g\in G$ commutes with δ. Since $G_{a\cup\Delta(a)}\neq 1$, G contains a $(a, a^{\perp})$-elation for each $a\in\Omega$. Then the conclusion of our theorem follows by a result of HigmaneMclaughlin ([4], Theorem 1).

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