On a generalization of F. M. Markel' theorem

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The purpose of this paper is to prove the following result.

THEOREM. *Let G be a non-identity finite group satisfyin the following conditions* (*a*) *and* (*b*): (*a*) *If the orders of centrlizers of two elements are equal, they are conjugate in G.* (*b*) *A Sylow 2-subgroup of G is abelian. Then G is isomorphic to the symmetric group of degree* 3.

The notation in this paper is standard. (See. D. Gorenstein [2])

LEMMA 1. *Let G be a finite group satisfying the condition* (*a*). Then x is cojugate to x^k, for any element x in G, where k is prime to the order of x.

PROOF. Obvious.

LEMMA 2. *Let G be a finite solvable group, S a Sylow 2-subgroup of G. If the order of x is odd and x is in $N_G(S)$, then x is a non-real element.*

PROOF. We prove by induction on the order of G. Let K be a minimal normal subgroup of G. If K does not contain x, the image of x in G/K is non-real by induction. So x is non-real in this case. If K contains x, we have that $[x, S]\subseteq S\cap K=1$. Since $C_G(x)$ contains a Sylow 2-subgroup of G, we have that the order of $N_G(<x>)/C_G(x)$ is odd. Hence x is not a real element in this case, too.

Now we separate the proof of the theorem into two parts; G is solvable and G is nonsolvable.

Part (1). G is solvable.

LEMMA 3. $G=O_{2',2}(G)$.

PROOF. By lemma 2, we have that $N_G(S)=S$. Since Sylow 2-subgroups of G are abelian, Burnside' transfer theorem implies the lemma.

LEMMA 4. *A Sylow 2-subgroup of G is elementary abelian.*

PROOF. Obviaus.

From now on we use the following notation throughout this paper; $H=O(G)$. We fix a prime p such that $O^p(H)\lneqq H$ and set $H_0=O^p(H)$.

LEMMA 5. $p=3$.

PROOF. Let U be the inverse image of the Frattini subgroup of H/H_0

in H and set $\bar{G}=G/U$. For the non-identity element $\bar{y}$ in $\bar{G}$, $\bar{y}$ is conjugate to $\bar{y}^k$, $k=1, 2, \cdots, p-1$ by lemma 1. By the structure of $\bar{G}$, we have that $p-1=2$.

If H is a 3-group, we have the theorem by F. M. Markel [3]. So we may assume that H is not a 3-group and will derive a contradiction. We fix a prime q such that $O^q(H_0)\lneqq H_0$. We note q is neither 2 nor 3. Let U_1 denote the inverse image of the Frattini subgroup of $H_0/O^q H_0)$ in H_0. Set $\bar{V}=H_0/U_1$.

LEMMA 6. *$q-1$ is divisible by 2 and 3.*

PROOF. Since q is neither 2 nor 3, we have the lemma by lemma 1.

LEMMA 7. *H is a 3-group.*

PROOF. We use the notation mentioned above. Suppose false. We set $\bar{G}=G/U_1$, $\bar{S}$ a Sylow 2-subgroup of $\bar{G}$.

Let $\bar{V}=\bar{V}_1\times\cdots\times\bar{V}_r$, where $\bar{V}_i$ is an irreducible component of $\bar{V}$, under the conjugate action of $\bar{S}$, $1\leq i\leq r$. Since $\bar{S}$ is an elementary abelian 2-group and $\bar{V}_1$ is $\bar{S}$-irreducible, we have that $|\bar{S}:C_{\bar{S}}(\bar{V}_1)|=2$. As the order of $\bar{V}_1$ is neither 2 nor 3, by lemma 1, there exists a 3-element $\bar{y}$ acting on $\bar{V}_1$. Since $\bar{V}_1$ is cyclic, we may assume that $\bar{y}$ normalizes $\bar{S}$. Then $\bar{y}$ is non-real by lemma 2, so the inverse image of $\bar{y}$ is non-real. This contradicts the condition (a).

Part (2). G is nonsolvable.

By lemma 1, we have that $O^{2'}(G)=G$. So H. Bender [1] imaplies that $G/O(G)$ is a direct product of an abelan 2-group and finite simple groups with abelian Sylow 2-subgroups; $G/O(G)\cong L_1\times\cdots\times L_s$, where L_i is isomorphic one of the following groups, $1\leq i\leq s$; (i) an abelian 2-group, (ii) $L_2(2^n)$, $2\leq n$, (iii) $L_2(q)$, $q\equiv 3$ or 5 (mod. 8), (iv) Janko' simple group of the order 175,560 or (v) Ree type. We may assume that one of the component is listed from (ii) to (v). If it is of (ii)-type, then the cyclic subgroup of the order 2^n+1 in it does not satisfy the conclusion of lemma 1. If it is of (iii)-type, by the same reason, the subgroup of the order $q+1/2$ or $q-1/2$ does not satisfy the conclusion of lemma 1. If it is of (iv)-type the subgroup of the order 5 in it does not satisfy it. Finally if it is of (v)-type non-identity 3-elements in it are non-real. In every case, we have a contradiction. The proof is complete.

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References

[1] H. BENDER: On groups with abelian Sylow 2-subgroups. Math. Zeit. 117 (1970).
[2] D. GORENSTEIN: Finite Groups. Harper Row (1968).
[3] F. M. MARKEL: Groups with many conjugate elemeuts. J. of Algebra 27 (1973).

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